Research Article

A Weak Limit Theorem for Galton-Watson Processes in Varying Environments

Zhenlong Gao and Yanhua Zhang

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

Correspondence should be addressed to Zhenlong Gao; gzlzyh@mail.qfnu.edu.cn

Received 28 March 2014; Accepted 22 July 2014; Published 12 August 2014

Academic Editor: Chong Lin

Copyright © 2014 Z. Gao and Y. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We extend Donsker's theorem and the central limit theorem of classical Galton-Watson process to the Galton-Watson processes in varying environment.

1. Introduction

There has been a lot of interesting works on Markov chains in random environments, which is mainly concentrated in branching processes in random environments and random walks in random environments (see [1]).

The study of branching processes in random environments dates back to late 60s or early 70s in the last century (see [2–5]). Our paper deals with a Galton-Watson branching process in the varying environment (GWVE) which is a special case of branching processes in random environments. The main concern is the weak convergence for a GWVE, which is an extension of Donsker's theorem (see [6, 7]).

In the following context, $\{X_{ni}, n \geq 0, i \geq 1\}$ is a double sequence of independent and nonnegative integer valued random variables, where for fixed n , $\{X_{ni}, i \geq 1\}$ have the same distribution $\{p_{ni}, i = 0, 1, 2, \dots\}$ with mean $\mu_n > 0$ and variance $\sigma_n^2 > 0$.

Definition 1. Assume $Z_0 \equiv 1$ and for any $n \geq 1$, define

$$Z_{n+1} = \begin{cases} \sum_{j=1}^{Z_n} X_{nj}, & \text{if } Z_n \neq 0; \\ 0, & \text{if } Z_n = 0, \end{cases} \quad (1)$$

then $\{Z_n, n \geq 0\}$ is said to be a GWVE.

Define $m_n = E(Z_n)$; it is well known that $m_n = \mu_0 \cdot \mu_1 \cdots \mu_{n-1}$ and there exists a nonnegative random variable V such that $Z_n/m_n \xrightarrow{\text{a.e.}} V$, as $n \rightarrow +\infty$ (see [8]).

For any fixed r , let $\xi_{nj}^{(r)} := X_{n,r}^{(j)}$ be the size of the r th generation of GWVE starting with the j th particle at time n ; then $\{\xi_{nj}^{(r)}, j \geq 1\}$ are i.i.d. with mean $m_{n,r}$ and variance $\sigma_{n,r}^2$ (see (4) and (5)). For each n , define

$$Y_n(t, \omega) = \frac{1}{\sigma_n \sqrt{Z_n}} \left(\sum_{j=1}^{[Z_n t]} \xi_{nj}^{(r)} - m_{n,r} [Z_n t] \right), \quad t \in [0, 1], \quad (2)$$

where $[x]$ is the largest integer that is less than x . Our main result is a weak limit theorem for GWVE, which is an extension of Donsker's theorem.

Theorem 2. Suppose that $m_n \rightarrow \infty$ and $P(V = 0) = 0$; then $Y_n \xrightarrow{d} B$, where B is the standard Brown motion on $[0, 1]$.

Let D be the space of functions defined on $[0, 1]$ and having discontinuities of at most the first kind. For any $\alpha \in R$, define $A_\alpha = \{x \in D : x(1) \leq \alpha\}$; it turns out that $W(\partial A_\alpha) = 0$, where W is the Wiener measure on D . Note that $Z_{n+r} = \sum_{j=1}^{Z_n} \xi_{nj}^{(r)}$; by Theorem 2 one has the following.

Corollary 3 (CLT). Suppose that $m_n \rightarrow \infty$ and $P(V = 0) = 0$; then for any fixed r ,

$$\frac{1}{\sqrt{Z_n \sigma_{n,r}^2}} (Z_{n+r} - m_{n,r} Z_n) \xrightarrow{d} N(0, 1), \quad (3)$$

where $N(0, 1)$ is the standard normal random variable.

So, Theorem 2 is an extension of the central limit theorem for classical Galton-Watson process (see [9, 10]).

2. Auxiliary Results

Let us begin with a result of ξ_{nj} .

Proposition 4. $\{\xi_{nj}, j \geq 1\}$ are independent and identically distributed with

$$m_{n,r} = E(\xi_{nj}) = \mu_n \mu_{n+1} \cdots \mu_{n+r-1}, \quad (4)$$

$$\sigma_{n,r}^2 = \text{Var}(\xi_{nj}) = (m_{n,r})^2 \sum_{j=n}^{n+r-1} \frac{\sigma_j^2}{\mu_j^2 m_{n,j-n}}. \quad (5)$$

Proof. According to the definition of definition of GWVE, $\{\xi_{nj}, j \geq 1\}$ are independent and identically distributed.

Denote the generating functions of X_{n1} and $\xi_{n,1}$ by $\phi_n(s)$ and $g_{n,r}(s)$, respectively; then it can be proved that

$$g_{n,r}(s) = \phi_n(\phi_{n+1}(\cdots \phi_{n+r-1}(s) \cdots)). \quad (6)$$

Therefore,

$$m_{n,r} = E(\xi_{n1}) = g'_{n,r}(1) = \prod_{j=n}^{n+r-1} \phi'_j(1) = \prod_{j=n}^{n+r-1} \mu_j. \quad (7)$$

So (4) is proved. In addition, the first and second derivatives of $g_{n,r}(s)$ are as follows:

$$\begin{aligned} g'_{n,r}(s) &= g'_{n,r-1}(\phi_{n+r-1}(s)) \phi'_{n+r-1}(s), \\ g''_{n,r}(s) &= g''_{n,r-1}(\phi_{n+r-1}(s)) (\phi'_{n+r-1}(s))^2 \\ &\quad + g'_{n,r-1}(\phi_{n+r-1}(s)) \phi''_{n+r-1}(s). \end{aligned} \quad (8)$$

By (8) one has

$$\begin{aligned} \text{Var}(X_{n,r}^{(1)}) &= (\text{Var}(X_{n,r-1}^{(1)}) - m_{n,r-1} + m_{n,r-1}^2) \mu_{n+r-1}^2 \\ &\quad + m_{n,r-1} (\sigma_{n+r-1}^2 - \mu_{n+r-1} + \mu_{n+r-1}^2) + m_{n,r} - m_{n,r}^2 \\ &= \text{Var}(X_{n,r-1}^{(1)}) \mu_{n+r-1}^2 + \sigma_{n+r-1}^2 m_{n,r-1}. \end{aligned} \quad (9)$$

Thus,

$$\frac{\text{Var}(X_{n,r}^{(1)})}{m_{n,r}^2} = \frac{\text{Var}(X_{n,r-1}^{(1)})}{m_{n,r-1}^2} + \frac{\sigma_{n+r-1}^2}{\mu_{n+r-1}^2 m_{n,r-1}}. \quad (10)$$

Since $m_{n,1} = \mu_n$, $\sigma_{n,1}^2 = \sigma_n^2$, $\xi_{n,1} = X_{n,1}$, we complete the proof of (5) by (10). $\square$

For any n , define

$$\eta_{nj} = \frac{\xi_{nj} - m_{n,r}}{\sigma_{n,r}}, \quad (11)$$

$$X_n(t, \omega) = \frac{1}{\sqrt{m_n}} \sum_{j=1}^{[m_n t]} \eta_{nj}, \quad t \in [0, 1].$$

The proof of Theorem 2 depends on the following proposition.

Proposition 5. $X_n \xrightarrow{d} B$, where B is standard Brown motion on $[0, 1]$.

Proof. It lose no generality if we assume that $\{m_n\}$ are integers. The proof is divided into two steps. We first show that the finite-dimensional distributions of the X_n are convergent to those of B . Consider first a single time point s . We must prove that $X_n(s, \cdot) \xrightarrow{d} W_s$.

Since $\{\eta_{nj}, j \geq 1\}$ have the same distribution, we can set

$$\varphi_n(t) = E(\exp(it\eta_{nj})). \quad (12)$$

Note $E(\eta_{nj}) \equiv 0$ and $\text{Var}(\eta_{nj}) \equiv 1$, according to (3.8) of [11] P101; one obtains

$$\begin{aligned} \varphi_n(s) &= \varphi_n(0) + \varphi'_n(0)s + \frac{\varphi''_n(0)}{2!} s^2 + o(s^2) \\ &= 1 - \frac{s^2}{2} + o\left(\frac{s^2}{2}\right), \quad (s \rightarrow 0). \end{aligned} \quad (13)$$

For any fixed t and k large enough,

$$\varphi_n\left(\frac{t}{\sqrt{k}}\right) = 1 - \frac{t^2}{2k} + o\left(\frac{1}{2k}\right). \quad (14)$$

Since $m_n \rightarrow \infty$, for n large enough, we have

$$\begin{aligned} E \exp(itX_n(s)) &= E\left(\exp\left(it \frac{1}{\sqrt{m_n}} \sum_{j=1}^{[m_n s]} \eta_{nj}\right)\right) \\ &= \left[1 - \frac{t^2}{2m_n} + o\left(\frac{1}{2m_n}\right)\right]^{[m_n s]} \\ &\rightarrow \exp\left\{-\frac{st^2}{2}\right\} \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (15)$$

This means that the characteristic function of $X_n(s)$ is convergent to that of B_s ; by Lévy continuous theorem we complete the proof of single point case.

Consider now two time points s and t with $s < t$; we are to prove

$$(X_n(s), X_n(t)) \xrightarrow{d} (B_s, B_t). \quad (16)$$

Note that

$$(X_n(s), X_n(t)) = (X_n(s), X_n(t) - X_n(s)) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (17)$$

By Corollary 1 to Theorem 5.1 in [12], it is only needed to prove

$$(X_n(s), X_n(t) - X_n(s)) \xrightarrow{d} (B_s, B_t - B_s). \quad (18)$$

Since the components on the left are independent by the independence of the $\{\xi_{ni}, i \geq 1\}$. Equation (16) follows from the case of one time point and Theorem 3.2 of [12].

A set of three or more time points can be treated in the same way, and hence the finite-dimensional distributions converge properly.

In the next step, we will show that $\{X_n\}$ is tight. According to Theorem 15.6 of [12], it is enough to establish the inequality

$$\begin{aligned} \Delta_n &:= E \left\{ |X_n(t) - X_n(t_1)|^2 |X_n(t_2) - X_n(t)|^2 \right\} \\ &\leq 4(t_2 - t_1)^2, \quad \forall 0 \leq t_1 \leq t \leq t_2 \leq 1. \end{aligned} \quad (19)$$

Since $\{\eta_{nj}, j \geq 1\}$ are i.i.d. with $E(\eta_{nj}) \equiv 0$ and $\text{Var}(\eta_{nj}) \equiv 1$; by the definition of X_n , we have

$$\begin{aligned} \Delta_n &= E \left\{ \left| \frac{1}{\sqrt{m_n}} \sum_{j=[m_n t_1]+1}^{[m_n t]} \eta_{nj} \right|^2 \cdot \left| \frac{1}{\sqrt{m_n}} \sum_{j=[m_n t]+1}^{[m_n t_2]} \eta_{nj} \right|^2 \right\} \\ &= \frac{([m_n t] - [m_n t_1]) \cdot ([m_n t_2] - [m_n t])}{m_n^2} \\ &\leq \frac{([m_n t_2] - [m_n t_1])^2}{m_n^2}. \end{aligned} \quad (20)$$

If $t_2 - t_1 \geq 1/m_n$, then there exist $k_1 < k_2$ such that

$$t_1 \in \left[\frac{k_1}{m_n}, \frac{k_1 + 1}{m_n} \right), \quad t_2 \in \left[\frac{k_2}{m_n}, \frac{k_2 + 1}{m_n} \right). \quad (21)$$

Hence,

$$\begin{aligned} \frac{[m_n t_2] - [m_n t_1]}{m_n} &= \frac{k_2 - k_1}{m_n} = \frac{k_2 - (k_1 + 1)}{m_n} + \frac{1}{m_n} \\ &\leq (t_2 - t_1) + (t_2 - t_1). \end{aligned} \quad (22)$$

So (19) is true when $t_2 - t_1 \geq 1/m_n$. Next, if $t_2 - t_1 < 1/m_n$, then either t_1 and t lie in the same subinterval $[k/m_n, (k+1)/m_n)$ or else t and t_2 do. In either of these cases $\Delta_n = 0$ by (20). This establishes (19) in general and proves the proposition. $\square$

3. The Proof of Theorem 2

We are now ready to prove Theorem 2.

Proof. Note that for each n ,

$$\begin{aligned} Y_n(t, \omega) &= \frac{1}{\sigma_{n,r} \sqrt{Z_n}} \left(\sum_{j=1}^{[Z_n t]} \xi_{nj} - \mu_n [Z_n t] \right) \\ &= \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{[Z_n t]} \eta_{nj}, \quad t \in [0, 1]. \end{aligned} \quad (23)$$

We assume at first that V is bounded, so that there exists a constant k such that $0 < V \leq k$ with probability 1. We may adjust the m_n so that they are integer and so that $k < 1$.

If we define

$$|\Phi_n(t) - tV| = \begin{cases} t \frac{Z_n}{m_n}, & \text{if } \frac{Z_n}{m_n} \leq 1; \\ tV & \text{otherwise.} \end{cases} \quad (24)$$

Since

$$|\Phi_n(t) - tV| \leq \left| \frac{Z_n}{m_n} - V \right| \xrightarrow{\text{a.e.}} 0, \quad (25)$$

Φ_n converges in probability in the sense of the Skorohod topology to the elements $\Phi(t) = Vt$ of D_0 , where D_0 consists of those elements φ of D that are nondecreasing and satisfy $0 \leq \varphi(t) \leq 1$ for all t . Define

$$X'_n(t, \omega) = \begin{cases} \frac{1}{\sqrt{m_n}} \sum_{l_n \leq i \leq m_n t} \eta_{ni}(\omega), & \text{if } t \geq \frac{l_n}{m_n}; \\ 0, & \text{otherwise,} \end{cases} \quad (26)$$

where $\{l_n, n \geq 0\}$ is a sequence of nonnegative integers going to infinity slowly enough that $l_n/\sqrt{m_n} \rightarrow 0$ as $n \rightarrow +\infty$. Define $\delta_n = \sup_t |X_n(t) - X'_n(t)|$; then

$$\delta_n \leq \frac{1}{\delta \sqrt{m_n}} \sum_{i=1}^{l_n} |\eta_{ni}|. \quad (27)$$

By Minkowski's inequality and the fact that $l_n/\sqrt{m_n} \rightarrow 0$, one has

$$E^{1/2} \{\delta_n^2\} \leq \frac{1}{\delta \sqrt{m_n}} \sum_{i=1}^{l_n} E^{1/2} \{\eta_{ni}^2\} = \frac{l_n}{\delta \sqrt{m_n}} \rightarrow 0. \quad (28)$$

So that by Chebyshev's inequality $\delta_n \xrightarrow{P} 0$. By Proposition 4, $X_n \xrightarrow{d} B$. Since $d(X_n, X'_n) \leq \delta_n$, where d is the metric in D which generates the Skorohod topology, it follows by Theorem 4.1 of [12] that $X'_n \xrightarrow{d} B$. So, if A is a W -continuity set in D , we have

$$P \{X'_n \in A\} \rightarrow W(A). \quad (29)$$

Let $\mathcal{B}_0$ be the field of cylinders sets; that is, $\mathcal{B}_0$ consists of the form

$$\{\omega; (\xi_{ij}(\omega), 0 \leq i \leq m, 1 \leq j \leq k) \in H\} \quad (30)$$

with $H \in \mathcal{B}(R^{mk})$, the Borel σ -field of R^{mk} .

If $E \in \mathcal{B}_0$, since $l_n \rightarrow \infty$ and $\{X_{ni}, n \geq 0, i \geq 1\}$ are independent, then for large n ,

$$P(\{X'_n \in A\} \cap E) = P\{X'_n \in A\} P(E). \quad (31)$$

It follows by (29) that

$$P(\{X'_n \in A\} \cap E) \rightarrow W(A) P(E). \quad (32)$$

Since $(\Phi_n, Z_n/m_n) \xrightarrow{P} (\Phi, V)$ in the sense of the product topology on $D_0 \times R$ and every X'_n is $\sigma(\mathcal{B}_0)$ measurable, it follows by Theorem 4.5 of [12] that

$$\left(X'_n, \Phi_n, \frac{Z_n}{m_n}\right) \xrightarrow{d} (B, \Phi_0, V_0) \quad (33)$$

is relative to the product topology in $D \times D_0 \times R^1$, where V_0 is independent of B and has the same distribution as V , $\Phi_0(t) = V_0 t$. By the fact that $\delta_n \xrightarrow{P} 0$,

$$\left(X_n, \Phi_n, \frac{Z_n}{m_n}\right) \xrightarrow{d} (B, \Phi_0, V_0). \quad (34)$$

Now the mapping that carries the point (x, ϕ, α) to $\alpha^{-1/2}(x \circ \phi)$ is continuous at that point $x \in C$, $\phi \in C \cap D_0$ and $\alpha > 0$. By Corollary 1 to Theorem 5.1 of [12],

$$\left(\frac{Z_n}{m_n}\right)^{-1/2} (X_n \circ \Phi_n) \xrightarrow{d} (V_0)^{-1/2} (B \circ \Phi_0). \quad (35)$$

Since V_0 and B are independent, $(V_0)^{-1/2}(B \circ \Phi_0)$ has the same distribution as B . Moreover $(Z_n/m_n)^{-1/2}(X_n \circ \Phi_n)$ coincides with Y_n if $Z_n/m_n < 1$, the probability of which goes to 1 since $k < 1$. Thus $Y_n \xrightarrow{d} B$ if V is bounded.

Suppose V is not bounded. For $u > 0$, define

$$\begin{aligned} V_u &= \begin{cases} V, & \text{if } V \leq u; \\ u, & \text{if } V > u, \end{cases} \\ Z_{un} &= \begin{cases} Z_n, & \text{if } V \leq u; \\ m_n u, & \text{if } V > u. \end{cases} \end{aligned} \quad (36)$$

Then for each u , $Z_{un}/m_n \xrightarrow{P} V_u$ and by the case already treated if

$$Y_{un}(t, \omega) = \frac{1}{\sigma_{n,r} \sqrt{Z_{un}}} \left(\sum_{j=1}^{[Z_{un}t]} \xi_{nj} - \mu_n [Z_{un}t] \right) \quad (37)$$

then $Y_{un} \xrightarrow{d} B$. Since $P(Y_{un} \neq Y_n) \leq P(V > u)$, $Y_n \xrightarrow{d} B$ follows Theorem 4.2 of [12]. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the referee for his (her) valuable suggestions. They also thank Professor Xiaoyu Hu for her help. This work is supported by the Youth Foundation and Doctor's Initial Foundation of Qufu Normal University (XJ201113, BSQD20110127).

References

- [1] O. Zeitouni, "Random walks in random environment," in *Lectures on Probability Theory and Statistics*, J. Picard, Ed., vol. 1837, pp. 189–312, 2004.
- [2] W. L. Smith, "Necessary conditions for almost sure extinction of a branching process with random environment," *Annals of Mathematical Statistics*, vol. 39, no. 6, pp. 2136–2140, 1968.
- [3] W. L. Smith and W. E. Wilkinson, "On branching processes in random environments," *Annals of Mathematical Statistics*, vol. 40, no. 3, pp. 814–827, 1969.
- [4] K. B. Athreya and S. Karlin, "On branching processes with random environments. I: extinction probabilities," *Annals of Mathematical Statistics*, vol. 42, pp. 1499–1520, 1971.
- [5] K. B. Athreya and S. Karlin, "Branching processes with random environments, II: limit theorems," *Annals of Mathematical Statistics*, vol. 42, no. 6, pp. 1843–1858, 1971.
- [6] M. D. Donsker, "An invariance principle for certain probability limit theorems," *Memoirs of the American Mathematical Society*, vol. 6, pp. 1–12, 1951.
- [7] D. H. Hu, "The invariance principle and its applications to branching process," *Journal of Peking University*, vol. 1, pp. 1–27, 1964.
- [8] D. H. Fearn, "Galton-Watson processes with generation dependence," in *Proceedings of the 6th Berkeley Symposium on Mathematical Statistics and Probability*, pp. 159–172.
- [9] C. C. Heyde, "A rate of convergence result for the super-critical Galton-Watson process," *Journal of Applied Probability*, vol. 7, no. 2, pp. 451–454, 1970.
- [10] C. C. Heyde, "Some central limit analogues for supercritical Galton-Watson processes," *Journal of Applied Probability*, vol. 8, no. 1, pp. 52–59, 1971.
- [11] R. Durrett, *Probability: Theory and Examples*, Thomson Brooks/Cole, 3rd edition, 2005.
- [12] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, NY, USA, 1968.