Research Article
Some New Generating Functions for $q$-Hahn Polynomials

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We obtain some new generating functions for $q$-Hahn polynomials and give their proofs based on the homogeneous $q$-difference operator.

1. Introduction
Throughout this paper we suppose that $q \in \mathbb{C}$, $|q| < 1$, and the $q$-shifted factorials are defined by

$$ (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (n \geq 1). $$

Clearly,

$$ (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}. $$

We also adopt the following compact notation for the multiple $q$-shifted factorials:

$$ (a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n, $$

$$ (a_1, a_2, \ldots, a_m; q)_{\infty} = (a_1; q)_\infty(a_2; q)_\infty \cdots (a_m; q)_\infty. $$

The basic hypergeometric series or $q$-series, $\phi_q$, are defined by

$$ \phi_q \left( \pmatrix{a_1, a_2, \ldots, a_r \\
 b_1, b_2, \ldots, b_i}; q, z \right) $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_i; q)_n}{(q, b_1, b_2, \ldots, b_i; q)_n} \left[ (-1)^n q^{\frac{1}{2}} \right]^{1+r} z^n. $$

Euler identity is as follows:

$$ \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} = \frac{1}{(t; q)_{\infty}}. $$

The $q$-binomial theorem is as follows:

$$ \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}. $$

The usual $q$-differential operator or $q$-derivative operator $D_q$ is defined by (see [1, Page 177, (2.1)])

$$ D_q \left\{ f(a) \right\} = \frac{f(a) - f(aq)}{a}, $$

$$ D_q^n \left\{ f(a) \right\} = D_q \left\{ D_q^{n-1} \left\{ f(a) \right\} \right\}. $$

In [1], Chen and Liu introduced the $q$-exponential $T(bD_q)$ operator as follows (see [1, Page 17, (2.5)]):

$$ T \left( bD_q \right) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}. $$
and they get the $q$-operator identity of $T(b D_q)$ (see [1, Page 178, Theorems 2.2 and 2.3]) as follows:

$$T(b D_q) \left\{ \frac{1}{(at; q)_\infty} \right\} = \frac{1}{(at, bt; q)_\infty} |bt| < 1,$$

$$T(b D_q) \left\{ \frac{1}{(as, at; q)_\infty} \right\} = \frac{(abst; q)_\infty}{(as, at, bs, bt; q)_\infty} |bt| < 1. \quad (9)$$

Recently Chen et al. [2] introduced the following homogeneous $q$-difference $D_{xy}$

$$D_{xy} [f(x, y)] = \frac{f(x, q^{-1} y) - f(x, y)}{x - q^{-1} y} \quad (10)$$

and the homogeneous $q$-difference operator $E(D_{xy})$:

$$E(D_{xy}) = \sum_{k=0}^{\infty} D_{xy}^k q^k. \quad (11)$$

They obtained some properties of $D_{xy}$ as follows:

$$D_{xy} \left\{ P_n(x, y) \right\} = (1 - q^n) P_{n-1}(x, y),$$

$$D_{xy} \left\{ \frac{y(t; q)_\infty}{(x, y | q)_\infty} \right\} = \frac{(yt; q)_\infty}{(xt; q)_\infty} \quad (12)$$

The classical Rogers-Szegö polynomial is defined by means of the generating function:

$$\sum_{n=0}^{\infty} h_n(x | q) \frac{t^n}{(q^n; q)_n} = \frac{1}{(t; q)_\infty}, |t| < 1; \quad (13)$$

obviously, we have

$$T(D_q) [x^n] = h_n(x | q) = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad (14)$$

The homogeneous Rogers-Szegö polynomial is defined by

$$h_n(\frac{x+y}{2}, \frac{x-y}{2} | q) = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad (15)$$

where $P_n(x, y) = (x - y)(x - yq) \cdots (x - yq^{n-1})$. Clearly, $h_n(x, y | q) = \Phi_{n/x}(x)$ are the Cauchy polynomials with the following generating function:

$$\sum_{k=0}^{\infty} P_k(x, y) \frac{z^k}{(q; q)_k} = \frac{(yz; q)_\infty}{(xz; q)_\infty}, \quad |xz| < 1. \quad (16)$$

From the above properties, we have

$$E(D_{xy}) \left\{ P_n(x, y) \right\} = h_n(x, y | q), \quad (17)$$

$$\sum_{n=0}^{\infty} h_n(x, y | q) \frac{t^n}{(q^n; q)_n} = \frac{(yt; q)_\infty}{(xt, y; q)_\infty}. \quad (18)$$

**Lemma 1** (see [3, Lemma 2.3]). For $|t|, |xt| < 1$,

$$E(D_{xy}) \left\{ \frac{y(t; q)_\infty}{(x, y | q)_\infty} P_n(x, y) \right\} \frac{t^n}{(q^n; q)_n} \quad (19)$$

where $P_n(x, q)$ is the $q$-Hahn polynomial defined by [4]

$$\sum_{n=0}^{\infty} \Phi_n^{(a)}(x) \frac{t^n}{(q, q)_n} = \frac{(xt; q)_\infty}{(t, xt; q)_\infty}. \quad (20)$$

We have

$$\Phi_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} a^k x^k. \quad (21)$$

Clearly, $\Phi_n^{(0)}(x) = h_n(x | q)$.

Recently, Chen et al. [3] gave some new proofs of the following results based on the method of homogeneous $q$-difference operator $E(D_{xy})$.

**Theorem 2.** Consider the following:

$$\sum_{n=0}^{\infty} \Phi_n^{(a)}(x) \Phi_n^{(b)}(y) \frac{t^n}{(q^n; q)_n} \frac{t^n}{(q^n; q)_n} \quad (22)$$

$$= \frac{(xt, yb; q)_\infty}{(t, xt; q)_\infty} \frac{\Phi_n^{(a)}(x) \Phi_n^{(b)}(y)}{s^n} x^n.$$  

**Theorem 3.** Consider the following:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \Phi_{m+n}^{(a)}(x) \Phi_n^{(b)}(y) \frac{t^n}{(q^n; q)_n} \frac{t^n}{(q^n; q)_n} \quad (23)$$

$$= \frac{(xas; q)_\infty}{(s, xs, xt, q)_\infty} \frac{1}{s^n} \frac{s^n}{\Phi_n^{(a)}(x) \Phi_n^{(b)}(y)} x^n.$$  

For more references on the $q$-difference operators, see [1, 5–16].

In the present paper, we obtains some new generating functions for $q$-Hahn polynomials and give their proofs based on the homogeneous $q$-difference operator.

### 2. Some New Generating Functions for $q$-Hahn Polynomial

In the present section we obtain the following new generating functions of $q$-Hahn polynomial.
Theorem 4. For $|z| < 1$,

\[
\sum_{k=0}^{\infty} \Phi_{nk}^{(a)}(x) z^k (q; q)_k^n = \frac{(axz; q)_{\infty}}{(z, xz; q)_{\infty}} \sum_{k=0}^{n} \left[ \frac{n}{} \right] (a, z; q)_k z^k.
\]

(24)

Proof. Let $x \mapsto y$ and $a \mapsto b$ in (21), we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Phi_{nk}^{(a)}(x) \Phi_{nk}^{(b)}(y) z^n (q; q)_k^n = \sum_{n=0}^{\infty} \frac{\Phi_n^{(a)}(x) \Phi_n^{(b)}(y)}{(q; q)_k^n} z^n (q; q)_k^n.
\]

(25)

By the $q$-binomial theorem (6) and noting that $(b q; q)_k = (q; q)_k$, we have

\[
(xaz, ybz; q)_{\infty} \sum_{k=0}^{\infty} \frac{(a, b, z, q)_k}{(z, xz, yz; q)_{\infty}} (xyz)^k = \frac{(xaz; q)_{\infty}}{(z, xz, q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a, b, z, q)_k}{(aq, q)_k} (yz)^k = \frac{(xaz; q)_{\infty}}{(z, xz, q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a, z, q)_k}{(q, q)_k} (b q; q)_k (xyz)^k.
\]

(26)

By (17), (25), and (26), we obtain

\[
\sum_{k=0}^{\infty} \frac{\Phi_{nk}^{(a)}(x) (b q; q)_k (xyz)^k}{(q, q)_k} = \frac{(xaz; q)_{\infty}}{(z, xz, q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a, z, q)_k}{(q, q)_k} (b q; q)_k (xyz)^k.
\]

(27)

Comparing the coefficients of $z^k (q; q)_k^n$ on both sides of (27), we obtain the formula (24) immediately. This proof is complete.

Theorem 5. For $|t| < 1$,

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Phi_{nk}^{(a)}(x) \Phi_{nk}^{(b)}(y) \frac{t^n}{(q; q)_k^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Phi_{nk}^{(a)}(x) \Phi_{nk}^{(b)}(y) t^n}{(q; q)_k^n}.
\]

(28)

Proof. By (17) and (19), we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{h_{vn}^{(a)}(x, y | q) h_n^{(a)}(u, v | q) t^n}{(q; q)_k^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{h_{vn}^{(a)}(x, y | q) h_n^{(a)}(u, v | q) t^n}{(q; q)_k^n}.
\]

(29)

Setting $y/x = a, v/u = b, u = y$ in the last sum, we obtain the formula (28) of Theorem 5. This proof is complete.

Theorem 6. For $|l| < 1, |s| < 1, |t| < 1$,

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Phi_{nk}^{(a)}(x) \Phi_{nk}^{(b)}(y) t^n}{(q; q)_k^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Phi_{nk}^{(a)}(x) \Phi_{nk}^{(b)}(y) t^n}{(q; q)_k^n}.
\]

(30)

Proof. By (17) and (19), we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{h_{vn}^{(a)}(x, y | q) h_n^{(a)}(u, v | q) t^n}{(q; q)_k^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{h_{vn}^{(a)}(x, y | q) h_n^{(a)}(u, v | q) t^n}{(q; q)_k^n}.
\]

(31)
Setting $y/x = a, v/u = b, u = y$ in the last sum, we obtain the formula (30) of Theorem 6. This proof is complete.

**Theorem 7.** For $|t| < 1$,

\[
\sum_{m,n,k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y) = \sum_{k=0}^\infty \left( \sum_{i,j=0}^k \frac{k!}{i! j!} (x; q)_i (y; q)_j \frac{(a; q)_i (b; q)_j}{(q; q)_i (q; q)_j} \right) t^k.
\]

\[
= \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

**Proof.** Applying (2) and the Euler identity (5) and noting (21), then the right-hand side is equal to (30) as follows:

\[
\frac{(x; q)_m (y; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y) = \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

\[
= \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

By (30) and (33), we have

\[
\sum_{m,n,k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y) = \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

Comparing the coefficients of $t^m s^n/(q; q)_m (q; q)_n$ on both sides of (34), we obtain the formula (32) immediately. 

**Theorem 8.** For $|t| < 1$,

\[
\sum_{m,n=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y) = \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

\[
= \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]

\[
= \sum_{k=0}^\infty \frac{t^k}{(q; q)_k} \frac{(a; q)_m (b; q)_n (q; q)_k}{(q; q)_k} \phi^{(a)}_{m+k} (x) \phi^{(b)}_{n+k} (y).
\]
Proof. Set $n = 0$ and then let $k \mapsto n$ in (32) and note that
\[ \Phi^{(b)}_0 (x) = 1; \text{by (21) and (22), we obtain} \]
\[
\sum_{n=0}^{\infty} \frac{\Phi^{(a)}_m (x)}{\Phi^{(b)}_m (y)} \frac{t^n}{(q; q)_n}
= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} \sum_{i,j=0}^{\infty} \frac{n}{i} \frac{n}{j} (x;q)_i 
\times (y;q)_j x^{n-i} y^{n-j} \Phi^{(a)}_m (xq^j) 
\]
\[
= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} \sum_{i,j=0}^{\infty} \frac{n}{i} \frac{n}{j} (x;q)_i (y;q)_j x^{n-i} y^{n-j} 
\times \sum_{s=0}^{m} m \binom{m}{s} (a;q)_s (xq^s)^s 
\]
\[
= \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} \sum_{s=0}^{m} m \binom{m}{s} (a;q)_s x^{s+n} y^n \Phi^{(a)}_n (x;q) \Phi^{(b)}_n (y) 
\times \sum_{s=0}^{m} m \binom{m}{s} (a;q)_s (x;xyt;q)_s \Phi^{(a)}_s (x;xyt,q) 
\times s \Phi_2 \left( \frac{xyt, xa, yb}{xyt, xt, yt;q}_s \right) 
\times \left( (x;yat, yb; q)_s \Phi_2 \left( \frac{xyt, xa, yb}{xyt, xt, yt; q}_s \right) \right) \times s \Phi_2 \left( \frac{xyt, xa, yb}{xyt, xt, yt; q}_s \right).
\]

This proof is complete. \hfill \blacksquare

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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