Research Article

The Construction of Type-2 Fuzzy Reasoning Relations for Type-2 Fuzzy Logic Systems

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Type-2 fuzzy reasoning relations are the type-2 fuzzy relations obtained from a group of type-2 fuzzy reasonings by using extended t-(co)norm, which are essential for implementing type-2 fuzzy logic systems. In this paper an algorithm is provided for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems. First, we give some properties of extended t-(co)norm and simplify the expression of type-2 fuzzy reasoning relations in accordance with different input subdomains under certain conditions. And then different techniques are discussed to solve the simplified expressions on the input subdomains by using the related methods on solving fuzzy relation equations. Besides, it is pointed out that the computation amount level of the proposed algorithm is the same as that of polynomials and the possibility of applying the proposed algorithm in the construction of type-2 fuzzy reasoning relations is illustrated on several examples. Finally, the calculation of an arbitrary extended continuous t-norm can be obtained as the special case of the proposed algorithm.

1. Introduction

Type-2 fuzzy sets first proposed by Zadeh in 1975 [1] are fuzzy sets equipped with ordinary fuzzy subsets of [0, 1] as membership grades, henceforth called fuzzy truth values. Then Mizumoto and Tanaka [2, 3] used Zadeh’s extension principle to extend minimum and maximum both based on minimum for calculating union and intersection on type-2 fuzzy sets, respectively, and showed that the results of the union and intersection keep the convexity and normality. Based on the theory of type-2 fuzzy sets, Karnik et al. [4] proposed a new fuzzy system called type-2 fuzzy system. Up to now, both the theory and application of type-2 fuzzy systems have been widely researched (see, e.g., [5–8]). What is more, type-2 fuzzy neural networks and type-2 fuzzy classification and pattern recognition have been also studied (see, e.g., [9, 10]). However, the computation process of the extended operations on the noninterval type-2 fuzzy sets is more complex than that of ordinary operations on type-1 fuzzy sets, which blocks the wide use of the noninterval type-2 fuzzy logic systems, type-2 fuzzy neural networks, and so on. In recent years, a heated wave of research about the operation on type-2 fuzzy sets has been set off. For example, Karnik and Mendel [11] further generalized these definitions of operations presented by Mizumoto and Tanaka and gave some analytical formulae for extensions of extended maximum and minimum based on minimum or product. Kawaguchi and Miyakoshi [12, 13] showed that extended continuous t-(co)norms based on arbitrary t-norm satisfy the definitions of type-2 t-(co)norms. C. L. Walker and E. A. Walker [14, 15] considered the algebras of fuzzy truth values equipped with extended maximum and minimum based on minimum. Coupland and John [16, 17] presented geometric methods for performing the operations of extended minimum and maximum based on minimum. Starczewski [18] provided analytical expressions for membership functions of five kinds of extended t-norms. Ling and Zhang [19] reconstructed the framework of set-theoretic operations on triangle type-2 fuzzy sets by presenting polygon type-2 fuzzy sets and gave manageable and simplified formulas for operations on triangle type-2 fuzzy sets. Hu and Kwong [20] discussed extended t-norm on a linearly ordered set with a unit interval and a real number set as special cases.

From the above it can be seen that these research works have well contributed to the properties of extended t-(co)norms and gave many useful results for the calculations
of some kinds of extended t-(co)norms. All of these promote
the structure of noninterval type-2 fuzzy logic systems
since extended t-(co)norms are the important tools in the
construction of type-2 fuzzy reasoning relations. Neverthe-
less, there are still many other extended t-norms whose
membership functions lack analytical expressions or feasible
algorithms. It hampers the attempt of the construction of
type-2 fuzzy reasoning relations by using these extended
t-norms. Besides, the work [18] leaves a key problem to us
that, except for extended minimum and maximum both
based on minimum, no theory guarantees that the results
of general extended t-(co)norms on two type-2 fuzzy sets
still satisfy the calculation conditions (e.g., convexity and
normality). Moreover, there are always more than two fuzzy
truth values in the calculation process of the construction of
type-2 fuzzy reasoning relations; it may be time-consuming
and laborious to proceed the calculation just on two fuzzy
truth values each time. It is a natural idea that we can solve
the computation in an integral and faster way. This paper
is devoted to deal with these problems we have mentioned
above. The following rows present our results: we show
that the results of extended continuous t-(co)norms based
on arbitrary t-norm keep the convexity and normality and
simplify the expression of type-2 fuzzy reasoning relations
of type-2 fuzzy logic systems with single input and single
output (SISO) in accordance with different input subdomains
under the condition that all the fuzzy truth values of type-
2 fuzzy sets participated in the calculation are required to
be convex and normal (Theorem 2). After that, we solve
the simplified expressions on three input subdomains (from
Theorem 3 to Theorem 9), which demonstrate an algorithm
to construct type-2 fuzzy reasoning relations. The complexity
of the algorithm is analyzed and it is pointed out that the
computation amount level of the proposed algorithm is the
same as that of polynomials. And then the possibility of
applying the proposed algorithm in the construction of type-
2 fuzzy reasoning relations is illustrated on several examples.
Besides, the calculation of a class of extended t-norms being
broader than those in [18] can be obtained as the special case
of the proposed algorithm.

This paper is organized in five sections. The follow-
ing section contains some preliminary knowledge and the
concrete expression of type-2 fuzzy reasoning relations of
SISO type-2 fuzzy logic systems. In Section 3 the method
for the construction of type-2 fuzzy reasoning relations is
investigated under certain conditions on the basis of the
properties of extended t-(co)norm and the related methods
on solving method of fuzzy relation equations. Section 4 gives
several examples by using the presented method. Conclusions
are given in Section 5.

2. Preliminaries

A type-2 fuzzy set \( \widetilde{A} \) on the domain \( X \) is characterized by a
membership function \( \mu_{\widetilde{A}} : X \rightarrow [0,1] \), \( x \mapsto \mu_{\widetilde{A}}(x) \),
where \( \mathcal{F}([0,1]) = \{ f \mid f : [0,1] \rightarrow [0,1] \} \), and \( f \in \mathcal{F}([0,1]) \) is called a fuzzy truth value. Convenience to the
following writing, we denote \( \mu_{\widetilde{A}}(x) \) by \( \mu_{\widetilde{A}(x)} \). Moreover, \( f \) is normal if there exists an \( x \in [0,1] \) such that \( f(x) = 1 \) and
cvx if, for any \( x_1, x_2 \in [0,1] \) and each \( \lambda \in [0,1] \), \( f(\lambda x_1 + (1 - \lambda) x_2) \geq f(x_1) \land f(x_2) \). Let \( \mathcal{F}_{CN}([0,1]) \) be the set of
both convex and normal fuzzy truth values. Assume that
\( \widetilde{A}, \widetilde{B} \in \mathcal{F}(X) \). Let * and *' be t-norm and t-conorm. Union
and intersection on type-2 fuzzy sets are given as follows. For
\( \forall x \in X, \forall w \in [0,1] \),
\[
\widetilde{A} \cup \widetilde{B} = \bigcup_{\mu_{\widetilde{A}}(u) \cup \mu_{\widetilde{B}}(v) = \mu_{\widetilde{A}}(u) \land \mu_{\widetilde{B}}(v) \land \mu_{\widetilde{A}}(u) \leq 1 \land \mu_{\widetilde{B}}(v) \leq 1} \left( \mu_{\widetilde{A}}(u) \lor \mu_{\widetilde{B}}(v) \right)
\]
\[
\widetilde{A} \cap \widetilde{B} = \bigcap_{\mu_{\widetilde{A}}(u) \cap \mu_{\widetilde{B}}(v) = \mu_{\widetilde{A}}(u) \land \mu_{\widetilde{B}}(v) \land \mu_{\widetilde{A}}(u) \leq 1 \land \mu_{\widetilde{B}}(v) \leq 1} \left( \mu_{\widetilde{A}}(u) \lor \mu_{\widetilde{B}}(v) \right)
\]
where \( \mu^{(\lor,\land)} \) and \( \mu^{(\lor,\lor)} \) are called extended t-conorm
and extended t-norm, respectively. Let \( X \times Y \) be a new domain
constructed by two domains \( X, Y \). A type-2 fuzzy set \( \widetilde{R} \in \mathcal{F}(X \times Y) \) is called a type-2 fuzzy relation between \( X \) and \( Y \),
where
\[
\mu_{\widetilde{R}} : X \times Y \rightarrow [0,1], \quad (x, y) \mapsto \mu_{\widetilde{R}}(x, y) \equiv \mu_{\widetilde{R}(x,y)}
\]
In the following, we will give the expression of type-2 fuzzy
relation from a group of type-2 fuzzy reasoning. This type-2
fuzzy relation is called a type-2 fuzzy reasoning relation. Let
\( \{ \widetilde{A}_i \}_{1 \leq i \leq N} \) and \( \{ \widetilde{B}_i \}_{1 \leq i \leq N} \) be, respectively, type-2 fuzzy sets on
input domain \( X \) and output domain \( Y \). For a group of type-2
fuzzy reasonings in a SISO type-2 fuzzy logic system
\[
\text{if } x \text{ is } \widetilde{A}_i \text{ then } y \text{ is } \widetilde{B}_i, \quad i = 1, \ldots, N,
\]
which can be rewritten as \( \{ \widetilde{A}_i \rightarrow \widetilde{B}_i, i = 1, \ldots, N \} \) and
induce the total type-2 fuzzy reasoning relation as follows:
\[
\widetilde{R} = \bigcup_{i=1}^{N} \widetilde{R}_i = \bigcup_{i=1}^{N} \left( \widetilde{A}_i \rightarrow \widetilde{B}_i \right).
\]
By choosing the suitable \( \mu^{(\lor,\land)} \) and \( \mu^{(\lor,\lor)} \) we can obtain that
\[
\mu_{\widetilde{R}(x,y)}(w) = \mu_{\bigcup_{i=1}^{N} \widetilde{R}_i(x,y)}(w)
\]
\[
= \left( \bigcup_{i=1}^{N} \left( \mu_{\widetilde{A}_i(x)} \rightarrow \mu_{\widetilde{B}_i(y)} \right) \right)(w)
\]
\[
= \sup_{\forall i=1, \ldots, N} \left( \mu_{\widetilde{A}_i(x)}(u_i) \rightarrow \mu_{\widetilde{B}_i(y)}(v_i) \right)
\]
where \( \mathcal{F}' \) and *' indicate the same t-norm. It is clear that the
difficulty on the calculation of type-2 fuzzy reasoning relation
is to solve the expression (5). For convenience, we first fix \( x \) and \( y \) and
denote
\[
u = (v_1, \ldots, v_N)
\]
\[
u = (v_1, \ldots, v_N)
\]
\[
F(w) = \mu_{\widetilde{R}(x,y)}(w)
\]
Define $\inf \emptyset = 1$. Moreover, some necessary interpretations about the two operations are presented in the following.

(1) $\mathcal{F}_* (a, b) \geq b$ since $a \star b \leq 1 \star b = b$.

(2) If $a \star c \leq b$ then $\mathcal{F}_* (a, b) \geq c$; if $a \star c \geq b$, then $\mathcal{L}_* (a, b) \leq c$.

(3) Both $\mathcal{F}_* (a, b)$ and $\mathcal{L}_* (a, b)$ are monotone decreasing about the first variable, that is,

$$a_1 \leq a_2 \Rightarrow \mathcal{F}_* (a_1, b) \geq \mathcal{F}_* (a_2, b),$$

$$\mathcal{L}_* (a_1, b) \geq \mathcal{L}_* (a_2, b),$$

since $\{x \in [0, 1] | a_1 \star x \leq b\} \supseteq \{x \in [0, 1] | a_2 \star x \leq b\}$ and $\{x \in [0, 1] | a_1 \star x \geq b\} \subseteq \{x \in [0, 1] | a_2 \star x \geq b\}$.

Let

$$G_b = \{ i \in \{ 1, \ldots, N \} | a_i \geq b \} = \{ k_j, j = 1, \ldots, |G_b| \}.$$  

In this work it is assumed that $\star$ is continuous and the following results presented in [27] are fitted for (10) on $[0, 1]$.

**Lemma 1.** Let $\star$ be a continuous t-norm. Then the following items are equivalent.

(1) $\mathcal{X}_* \neq \emptyset$ if and only if $G_b \neq \emptyset$; that is, there exists $i \in \{ 1, \ldots, N \}$, such that $a_i \geq b$ if and only if $\mathcal{X}$ has the greatest solution $x^* = (x_1^0, \ldots, x_N^0)$.

(2) If $\mathcal{X}_* \neq \emptyset$, then (10) has the minimum solution where the jth minimum solution $x_j^0 = (x_{j1}^0, \ldots, x_{jN}^0)$ ($1 \leq j \leq |G_b|$) is

$$x_j^0 = \begin{cases} \mathcal{L}_* (a_k, b), & i = k_j, \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, the solution set of (10) can be written as

$$\mathcal{X}_* = \bigcup_{j=1}^{\left|G_b\right|} [x_j^0, x^*].$$

### 3. The Construction of Type-2 Fuzzy Reasoning Relations

In this section, we will demonstrate the solving process for the expression (7) gradually. First, we will simplify the expression (7) in accordance with three subdomains of $w$. Importantly, for two of these subdomains we will, respectively, reduce $P_w$ into its subdomains $P_{w1}$ and $P_{w2}$ but keeping the values of $F(w)$ without change (Theorem 2). Then all the elements in $P_{w1}$ and $P_{w2}$ will be found out (Theorem 3). Following it, $P_{w1}$ and $P_{w2}$ will be further reduced into smaller subsets $X_1$ and $X_2$ still keeping the values of $F(w)$ without change, respectively (Theorem 5). Finally, some theorems about how to get the exact value of $F(w)$ will be presented on the basis of the characteristics of the $f(u, v)$ on $X_1$ and $X_2$ (Theorems 7 and 9).
It needs to be stated that the proposed method to solve $F(w)$ differs from the native algorithm which is just finding the maximal number of $f(u, v)$ from all the elements in $P_w$ (or $P_{w,1}$ and $P_{w,2}$). The native algorithm is impractical due to its huge computation. But what form of the elements in $P_w$ is the key to solving the problem (7). Let $g \in \mathcal{F}_{CN}([0, 1])$. Denote $[g]_1 = \{u \in [0, 1] \mid g(u) = 1\}$.

**Theorem 2.** Let $w \in [0, 1], g_1, \ldots, g_N, h_1, \ldots, h_N \in \mathcal{F}_{CN}([0, 1])$, where $[g]_1 = [m_{g_1}, n_{g_1}, \ldots, [g_N]_1 = [m_{g_N}, n_{g_N}], [h_1]_1 = [m_{h_1}, n_{h_1}], \ldots, [h_N]_1 = [m_{h_N}, n_{h_N}]$. Denote

$$
\alpha = \bigvee_{i=1}^{N} (m_{g_i} \star m_{h_i}), \quad \beta = \bigvee_{i=1}^{N} (n_{g_i} \star n_{h_i}),
$$
$$
m_g = (m_{g_1}, \ldots, m_{g_N}), \quad m_h = (m_{h_1}, \ldots, m_{h_N}),
$$
$$
n_g = (n_{g_1}, \ldots, n_{g_N}), \quad n_h = (n_{h_1}, \ldots, n_{h_N}),
$$
$$
P_{w,1} = \left\{ (u, v) \in [0, 1]^{2N} \bigg| \bigvee_{i=1}^{N} (u_i \star v_i) = w, \quad u \leq m_g \ star v \leq m_h \Bigg\}, \quad (16)
$$
$$
P_{w,2} = \left\{ (u, v) \in [0, 1]^{2N} \bigg| \bigvee_{i=1}^{N} (u_i \star v_i) = w, \quad u \geq n_g \ star v \geq n_h \Bigg\}.
$$

Then the following items hold.

1. If $w \in [0, a]$, then $F(w) = \sup\{f(u, v) \mid (u, v) \in P_{w,1}\}$.
2. If $w \in [a, b]$, then $F(w) = 1$.
3. If $w \in [b, 1]$, then $F(w) = \sup\{f(u, v) \mid (u, v) \in P_{w,2}\}$.

Before the proof of Theorem 2, several conclusions and their proofs will be given in the following and the conclusion (a) is from [18].

(a) Let $w \in [0, 1], f, g \in \mathcal{F}_{CN}([0, 1])$, where $[f]_1 = [m_f, n_f]$ and $[g]_1 = [m_g, n_g]$. Assume that $*$ is continuous. Denote

$$
L = \{(u, v) \in [0, 1]^2 \mid u \star v = w\},
$$
$$
L_1 = \{(u, v) \in [0, 1]^2 \mid u \star v = w, u \leq m_f, v \leq m_g\}, \quad (17)
$$
$$
L_2 = \{(u, v) \in [0, 1]^2 \mid u \star v = w, n_f \leq u, n_g \leq v\}.
$$

Then the following items hold.

1. If $w \in [0, m_f \ star m_g]$, then $(f \triangleright (\star, \star)) \triangleright (g)(w) = \sup\{f(u) \star g(v) \mid (u, v) \in L_1\}$.
2. If $w \in [m_f \ star m_g, n_f \ star n_g]$, then $(f \triangleright (\star, \star)) \triangleright (g)(w) = 1$.
3. If $w \in [n_f \ star n_g, 1]$, then $(f \triangleright (\star, \star)) \triangleright (g)(w) = \sup\{f(u) \star g(v) \mid (u, v) \in L_2\}$.
4. (b) Suppose that the conditions is the same as that of (a).

Denote

$$
\mathcal{C}_1 = \{(u, v) \in [0, 1]^2 \mid u \lor v = w, u \leq m_f, v \leq m_g\},
$$
$$
\mathcal{C}_2 = \{(u, v) \in [0, 1]^2 \mid u \lor v = w, n_f \leq u, n_g \leq v\}.
$$

Then the following items hold.

1. If $w \in [0, m_f \lor m_g]$, then $(f \lor (\triangleright, \triangleright)) \triangleright (g)(w) = \sup\{f(u) \lor g(v) \mid (u, v) \in \mathcal{C}_1\}$.
2. If $w \in [m_f \lor m_g, n_f \lor n_g]$, then $(f \lor (\triangleright, \triangleright)) \triangleright (g)(w) = 1$.
3. If $w \in [n_f \lor n_g, 1]$, then $(f \lor (\triangleright, \triangleright)) \triangleright (g)(w) = \sup\{f(u) \lor g(v) \mid (u, v) \in \mathcal{C}_2\}$.

Proof. This proof is similar as that of (a) in [18] since $\lor$ is also monotone increasing in the first and second variables.

(c) Let $w_1, w_2, \tau_1, \tau_2 \in [0, 1]$, where $w_1 < w_2$. Assume that $\ast$ is continuous. Denote

$$
\mathcal{M}_1 = \{(a, b) \in [0, 1]^2 \mid a \ast b = w_1, a \leq \tau_1, b \leq \tau_2\},
$$
$$
\mathcal{M}_2 = \{(c, d) \in [0, 1]^2 \mid c \ast d = w_2, c \leq \tau_1, d \leq \tau_2\}, \quad (19)
$$
$$
\mathcal{M}_3 = \{(a, b) \in [0, 1]^2 \mid a \ast b = w_1, \tau_1 \leq a, \tau_2 \leq b\},
$$
$$
\mathcal{M}_4 = \{(c, d) \in [0, 1]^2 \mid c \ast d = w_2, \tau_1 \leq c, \tau_2 \leq d\}.
$$

Then for every $(a, b) \in \mathcal{M}_1$ [resp. $\mathcal{M}_3$], there exists $(c, d) \in \mathcal{M}_2$ [resp. $\mathcal{M}_4$] such that $a \leq c$ and $b \leq d$.

Proof. Let $(a, b) \in \mathcal{M}_1$ and $(u, v) \in \mathcal{M}_2$. Since $w_1 < w_2$, by the monotonicity of $\ast$, we have $a \leq u$ or $b \leq v$. Assume that $a \leq u$. If $b \leq v$, then the conclusion is obvious. For the case of $b > v$, there is

$$
a \ast b \leq u \ast v \leq u \ast b,
$$

that is,

$$
a \ast b \leq w_2 \leq u \ast b. \quad (20)
$$

By the continuity of $\ast$, it can be inferred that there exists $z \in [a, u]$ such that $w_2 = z \ast b$. Let $(c, d) = (z, b)$. Then there are $c \ast d = w_2, a \leq c \leq \tau_1$, and $b \leq d \leq \tau_2$. Clearly, $(c, d) \in \mathcal{M}_2$.

Similarly, we can prove that if $b \leq v$ and $a > u$, there exists $x \in [b, v]$ such that $w_2 = a \ast x$. Let $(c, d) = (a, x)$. Then there are $c \ast d = w_2, a \leq c \leq \tau_1$, and $b \leq d \leq \tau_2$. To sum up, we can conclude that for every $(a, b) \in \mathcal{M}_1$, there exists $(c, d) \in \mathcal{M}_2$ such that $a \leq c, b \leq d$. In a similar way, we can prove that for every $(a, b) \in \mathcal{M}_3$, there exists $(c, d) \in \mathcal{M}_4$ such that $a \leq c$ and $b \leq d$.

(d) Let $f, g \in \mathcal{F}_{CN}([0, 1])$, where $[f]_1 = [m_f, n_f]$ and $[g]_1 = [m_g, n_g]$. Assume that $\ast$ and $\lor$ are continuous.

Then $(f \triangleright (\ast, \ast)) \triangleright (g) \in \mathcal{F}_{CN}([0, 1])$. Furthermore, $(f \triangleright (\ast, \ast))_1 = [m_f \ star m_g, n_f \ star n_g]$ and $(f \lor (\ast, \ast))_1 = [m_f \lor m_g, n_f \lor n_g]$. 

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Proof. Let \( [f_1] = [m_f, n_f], [g_1] = [m_g, n_g], w_1, w_2, w_3 \in [0, 1] \), where \( w_1 \leq w_2 \leq w_3 \). By the continuity of \( * \) and conclusion (a), we obtain that if \( w_2 \in [m_f * m_g, n_f * n_g] \), then \((f \triangledown_R^{(\ast, \ast)} g)(w_2) = 1\). For the converse, let \( w \in [0, m_f * m_g] \).

If there exists \( w_0 \in [0, m_f * m_g] \) such that \((f \triangledown_R^{(\ast, \ast)} g)(w_0) = 1\), that is, there exists \((u_0, v_0) \in [0, 1]^2\) such that \( u_0 = w_0 < m_f * m_g \) and \( f(u_0) = F(v_0) = 1\). By the monotonicity of \( * \), we have \( u_0 < m_f \) or \( v_0 < m_g \).

Without loss of generality, we can assume that \( u_0 < m_f \). Thus \( f(u_0) < f(m_f) = 1 \), which leads to a contradiction.

Therefore, \((f \triangledown_R^{(\ast, \ast)} g)(w) < 1\). In a similar way, we can prove that if \( w \in (n_f * n_g, 1] \), then \((f \triangledown_R^{(\ast, \ast)} g)(w) < 1\).

To sum up, we have \([f \triangledown_R^{(\ast, \ast)} g][1] = [m_f * m_g, n_f * n_g]\).

Next we will give the proof of Theorem 2.

Proof. It is known that \( F(w) = (\bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i))(w) \).

From conclusion (d) it can be obtained that \([f_i \triangledown_R^{(\ast, \ast)} g_i][i] = [m_f, m_g, n_f, n_g], i = 1, \ldots, N\). Moreover, denote

\[
F_{\text{C}(1)}(0, 1) \text{ and } [f_i \triangledown_R^{(\ast, \ast)} g_i][1] = [m_f, m_g, n_f, n_g], i = 1, \ldots, N \text{.}
\]

From conclusion (a), we obtain that if \( z_i \in [0, m_f * m_g], \) then \((f_i \triangledown_R^{(\ast, \ast)} g_i)(z_i) = \sup(\{\langle u_i, v_i \rangle \mid \langle u_i, v_i \rangle \in G_i\}) \) if \( z_i \in [n_f * n_g, 1] \), then \((f_i \triangledown_R^{(\ast, \ast)} g_i)(z_i) = \sup(\{\langle u_i, v_i \rangle \mid \langle u_i, v_i \rangle \in G_i\}) \). From conclusion (d) we have \([f_i \triangledown_R^{(\ast, \ast)} g_i][i] = [\bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i)][i] = [\bigcup_{i=1}^N (m_f, m_g), \bigcup_{i=1}^N (n_f, n_g)] = [\alpha, \beta] \). Denote \( z = (z_1, \ldots, z_N) \) and

\[
E_1 = \left\{ z \in [0, 1]^N \mid \sum_{i=1}^N z_i = w, z_i \in [m_f * m_g, 1] \right\},
\]

\[
E_2 = \left\{ z \in [0, 1]^N \mid \sum_{i=1}^N z_i = w, z_i \in [n_f * n_g, 1] \right\}.
\]

From the above discussion and conclusion (b), we have that if \( w \in [0, \alpha] \), then

\[
F(w) = \sup \left\{ \bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i)(z_i) \mid z \in E_1 \right\} = \sup \left\{ \bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i)(z_i) \mid z \in E_1 \right\} = \sup \left\{ \bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i)(z_i) \mid z \in E_2 \right\} \]

which implies that (22) holds. If \( w_2 \in (n_f * n_g, 1] \), then \( n_f * n_g \leq w_2 \leq w_3 \). In a similar way, we can prove that \((f \triangledown_R^{(\ast, \ast)} g)(w_2) \leq (f \triangledown_R^{(\ast, \ast)} g)(w_2)\). Thus (22) holds.

To sum up, there is \((f \triangledown_R^{(\ast, \ast)} g)(w) \in F_{\text{C}(1)}([0, 1])\). It is easy to prove that the conclusion (c) is valid if \( * \) is replaced with \( \diamond \) since \( \diamond \) is also monotone increasing in the first and second variables. Therefore, in a similar way, we can give the proof of \( f \cup_R^{(\ast, \ast)} g \in \text{OCN}([0, 1]) \).

Next we will give the proof of Theorem 2.

Proof. It is known that \( F(w) = (\bigcup_{i=1}^N (f_i \triangledown_R^{(\ast, \ast)} g_i))(w) \).

From conclusion (d) it can be obtained that \([f_i \triangledown_R^{(\ast, \ast)} g_i][i] = [m_f, m_g, \alpha, \alpha] \) for \( w \in [0, \alpha] \) and \( w \in [\beta, 1] \) and assume that...
\( g_t, h_t \in \mathcal{F}_c([0, 1]) \), where \([g_t]_i = [m_{g_i}, n_{g_i}], [h_t]_i = [m_{h_i}, n_{h_i}], i = 1, \ldots, N \). Denote
\[
P_{o1} = \{ u \in [0, 1]^N \mid u \leq m_g \},
\]
\[
P'_{o1} = \{ v \in [0, 1]^N \mid v \leq m_h \},
\]
(30)
\[
P_{o2} = \{ u \in [0, 1]^N \mid u \geq n_g \},
\]
\[
P'_{o2} = \{ v \in [0, 1]^N \mid v \geq n_h \}.
\]
The idea about how to find the elements in \( P_{o1} \mid P_{o2} \), resp. is to solve the fuzzy relation equation
\[
(u_1 \cdot x_1) \lor (u_2 \cdot x_2) \lor \cdots \lor (u_N \cdot x_N) = w,
\]
by taking \( u \in P_{o1} \), resp. \( P_{o2} \). Thus \((u, x) \in P_{o1} \mid P_{o2} \), resp. In this way, all the elements in \( P_{o1} \mid P_{o2} \), resp. can be found. Denote
\[
G_w = \{ i \in \{1, \ldots, N\} \mid u_i \geq w \} \tag{32}
\]
Now we will provide all the elements of \( P_{o1} \) and \( P_{o2} \).

**Theorem 3.** Assume that \(*\) is continuous. For every \( u \in P_{o1} \) or \( P_{o2} \) denote the greatest solution of (31) in \([0, 1]^N\) as \( x_u^* \) and minimal solution of (31) in \([0, 1]^N\) as \( x_u^0 \), \( i = 1, \ldots, N \) if any.

The following items hold.

1. Suppose that \( w \in [0, \alpha) \). Then for every \( u \in P_{o1} \) the solution set of (31) in \( P'_{o1} \) is \( \bigcup_{j=1}^{N} [x_j^0, x_u^* \land m_h] \) denoted by \( Y_u^{(1)} \) and
\[
P_{o1} = \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(1)}, v \leq m_g \} \tag{33}
\]
2. Suppose that \( w \in (\beta, 1) \). Then for every \( u \in P_{o2} \) the solution set of (31) in \( P'_{o2} \) is \( \bigcup_{j=1}^{N} [x_j^0, n_h, x_u^* \land m_h] \) denoted by \( Y_u^{(2)} \) and
\[
P_{o2} = \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(2)}, n_g \leq v \} \tag{34}
\]
**Proof.** (1) From Lemma 1 it is obvious that \( Y_u^{(1)} \) is the solution set of (31) in \( P'_{o1} \). For every \((u, v) \in \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(1)}, u \leq m_g \} \), we have \( u \in P_{o1} \) and \( v \in Y_u^{(1)} \subseteq P'_{o1} \). Then from Lemma 1 it can be inferred that \( \bigcup_{j=1}^{N} (u_j \lor v_j) = w \). Thus \((u, v) \in P_{o1} \); that is, \( \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(1)}, u \leq m_g \} \subseteq P_{o1} \). For the converse case, let \((u, v) \in P_{o1} \). Then \( \bigcup_{j=1}^{N} (u_j \lor v_j) = w \). Obviously \( v \leq m_g \) and \( v \) is a solution of (31) with the coefficient vector \( u \). Denote the solution set of (31) in \([0, 1]^N\) as \( \bigcup_{j=1}^{N} [x_j^0, x_u^* \land m_h] \). Clearly there exists \( j \in \{1, \ldots, |G_u|\} \) such that \( v \in [x_j^0, x_u^* \land m_h] \). Thereby \( \forall j \in \{1, \ldots, |G_u|\} \) and \( [x_j^0, x_u^* \land m_h] \subseteq Y_u^{(1)} \), that is, \((u, v) \in \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(1)}, u \leq m_g \}; that is, \((u, v) \in \{ (u, v) \in [0, 1]^N \mid v \in Y_u^{(1)}, u \leq m_g \} \supseteq P_{o1} \). To sum up, the conclusion (1) holds. In a similar way, we can prove the case (2).

**Corollary 4.** Assume that \(*\) is continuous. Let \( u \in P_{o1} \) or \( P_{o2} \). Denote the greatest solution of (31) in \([0, 1]^N\) as \( x_u^* \) and minimal solution of (31) in \([0, 1]^N\) as \( x_u^0 \), \( i = 1, \ldots, N \) if any.

The following hold.

1. Let \( u \in P_{o1} \). Equation (31) has a solution in \( P'_{o1} \) if and only if there exists \( j \in G_u \) such that \( x_j^0 \leq m_h \).
2. Let \( u \in P_{o2} \). Equation (31) has a solution in \( P'_{o2} \) if and only if \( n_h \leq x_u^* \).

**Proof.** (1) Equation (31) has a solution in \( P'_{o1} \), if and only if \( T_u^{(1)} \neq 0 \), and if and only if there exists \( j \in G_u \) such that \( x_j^0 \leq m_h \).

(2) Equation (31) has a solution in \( P'_{o2} \), if and only if \( T_u^{(2)} \neq 0 \), and if and only if \( n_h \leq x_u^* \).

Next, on the basis of Theorem 3 we will further find sub-sets of \( P_{o1} \) and \( P_{o2} \) but keeping the values of \( F(u) \) without change.

**Theorem 5.** Assume that \(*\) is continuous. The following holds.

1. Suppose that \( w \in [0, \alpha) \). Then for every \( u \in P_{o1} \) the greatest solution of (31) in \([0, 1]^N\) is \( x_u^* \). Denote
\[
\mathcal{U}_1 = \{ u \in P_{o1} \mid \exists v \in P'_{o1}, s.t. (u, v) \in P_{o1} \},
\]
(35)
\[
\mathcal{X}_1 = \{ (u, v) \in [0, 1]^N \mid v = x_u^* \land m_h, u \in \mathcal{U}_1 \}
\]
Then
\[
F(u) = \sup \{ f(u, v) \mid (u, v) \in \mathcal{X}_1 \} \tag{36}
\]
2. Suppose that \( w \in (\beta, 1) \). Then for every \( u \in P_{o2} \) the minimal solutions of (31) in \([0, 1]^N\) are \( x_u^0 \), \( j = 1, \ldots, |G_u| \). Denote
\[
\mathcal{U}_2 = \{ u \in P_{o2} \mid \exists v \in P'_{o2}, s.t. (u, v) \in P_{o2} \},
\]
(37)
\[
\mathcal{X}_2 = \{ (u, v) \in [0, 1]^N \mid v \in \bigcup_{j=1}^{N} [x_j^0, n_h], j = 1, \ldots, |G_u| \}
\]
Then
\[
F(u) = \sup \{ f(u, v) \mid (u, v) \in \mathcal{X}_2 \} \tag{38}
\]
**Proof.** (1) Clearly, for every \( u \in \mathcal{U}_1 \), (31) has a solution in \( P'_{o1} \). From Theorem 3, there is \( (u, x_u^* \land m_h) \in P_{o1} \); that is, \( \mathcal{X}_1 \subseteq P_{o1} \). Then
\[
\sup \{ f(u, v) \mid (u, v) \in \mathcal{X}_1 \}
\]
\[
\leq \sup \{ f(u, v) \mid (u, v) \in P_{o1} \} = F(u) \tag{39}
\]
For the converse case, let \((u, v) \in P_{o1} \). Then \( u \in \mathcal{U}_1 \) and \( v \in Y_u^{(1)} \). We have \( v \notin x_u^* \land m_h \). Denote \( v_u = x_u^* \land m_h \). From the convexity of \( g_t \) and \( h_t \), and the monotonicity of t-norm, there is
\[
f(u, v) = T_u^{(1)} \left( g_t(u_t) \land h_t(v_t) \right)
\]
\[
\leq T_u^{(1)} \left( g_t(u_t) \land h_t(v_u) \right) = f(u, v_u) \tag{40}
\]
That is, for every \((u, v) \in P_{w1}\) there exists \((u, v_u) \in X_1\), such that \(f(u, v) \leq f(u, v_u)\). Thus
\[
F(w) = \sup \{ f(u, v) \mid (u, v) \in P_{w1} \} = \sup \{ f(u, v) \mid (u, v) \in X_1 \}.
\]
(41)
Combined with (39) and (41), it can be shown that the conclusion (1) holds.

(2) Clearly, for every \(u \in U_2\), (31) has a solution in \(P'_{o2}\).
From Theorem 3, there is \((u, x_u) \in P_{w2}\), where \(x_u = \{x_{u,j}^i \mid j = 1, \ldots, |G_u|\}\); that is, \(x_2 \subseteq P_{w2}\). Thus
\[
\sup \{ f(u, v) \mid (u, v) \in X_2 \} = \sup \{ f(u, v) \mid (u, v) \in P_{w2} \} = F(w).
\]
(42)
For the converse case, let \((u, v) \in P_{w2}\). Then \(u \in U_2\) and \(v \in \mathcal{V}_2\). There exists \(x_u \in \{x_{u,j}^i \mid i = 1, \ldots, |G_u|\}\), such that \(x_u \vee n_u = v\). Denote \(v_u = x_u \vee n_u\). From the convexity of \(g_i\) and \(h_i\) and the monotonicity of \(\tau\)-norm, there is
\[
f(u, v) = \tau_{i=1}^N (g_i(u) * h_i(v)) \\
\leq \tau_{i=1}^N (g_i(v_u) * h_i(v_u)) = f(u, v_u).
\]
(43)
That is, for every \((u, v) \in P_{w2}\), there exists \((u, v_u) \in X_2\), such that \(f(u, v) \leq f(u, v_u)\). Thus
\[
F(w) = \sup \{ f(u, v) \mid (u, v) \in P_{w2} \} = \sup \{ f(u, v) \mid (u, v) \in X_2 \}.
\]
(44)
Combined with (42) and (44), conclusion (2) holds.

If \(w \in [0, \alpha]\), from Theorem 5 it can be seen that all of the elements in \(X_1\) can be obtained when all of the elements in \(U_j\) and the greatest solutions of the corresponding equation (31) in \([0, a]_N\) are obtained. The following lemma describes the characteristics of the elements in \(U_1\). Denote
\[
J = \{ i \mid m_{g_i} + m_{h_i} \geq w \},
\]
(45)
\[
\hat{u}_j = \inf \{ x \in [0, m_{g_j}] \mid \mathcal{L}_*(x, w) \leq m_{h_j} \}, \quad j \in J.
\]

Lemma 6. Let \(w \in [0, \alpha]\) and \(u \in P_j\). Assume that \(\ast\) is continuous. Then (31) has a solution in \(P'_{o1}\) if and only if there exists \(j \in J\) such that
\[
(0, \ldots, 0, \hat{u}_j \vee w, 0, \ldots, 0) \leq u \leq m_{g_j}.
\]
(46)
Proof. For the first, we will prove that \(w \leq \hat{u}_j \vee w \leq m_{g_j}\), \(j \in J\). It can be seen that \(\mathcal{L}_*(m_{g_j}, w) \leq m_{h_j}\) since \(m_{g_j} \ast m_{h_j} \geq w\) for every \(j \in J\). Therefore, \(m_{g_j} \in \{x \in [0, m_{g_j}] \mid \mathcal{L}_*(x, w) \leq m_{h_j}\}\); that is, \(\hat{u}_j = \inf \{ x \in [0, m_{g_j}] \mid \mathcal{L}_*(x, w) \leq m_{h_j} \}\). Obviously \(w \leq \hat{u}_j \vee w \leq m_{g_j}\).

Let \(u\) satisfy (46). Since \(w \leq \hat{u}_j \vee w \leq u_j\), it can be inferred that (31) is solvable in \([0, 1]_N\) and \(x_u^0\) is a minimal solution in \([0, 1]_N\) from Lemma 1, where \(x_u^0 = (0, \ldots, 0, \mathcal{L}_*(u_j, w), 0, \ldots, 0)\). Then we have \(\mathcal{L}_*(u_j, w) \leq \mathcal{L}_*(\hat{u}_j, w) \leq m_{h_j}\) since \(\hat{u}_j \leq u_j \vee w \leq u_j \leq m_{g_j}\). That is, \(x_u^0 \leq m_{h_j}\), which verifies that (31) has a solution in \(P'_{o1}\) by Corollary 4.

For the converse case, let \(v \in P'_{o1}\) be a solution of (31). Then \((u_1 + v_1) \vee \cdots \vee (u_N + v_N) = w\). There exists \(j \in \{1, \ldots, N\}\) such that \(u_j + v_j = w\). Therefore \(m_{g_j} \geq u_j \geq w, m_{h_j} \geq v_j \geq w\) and \(m_{g_j} \ast m_{h_j} \geq u_j \ast v_j = w\). Thus \(j \in J\). It can be seen that equation \(u_j \ast x = w\) is solvable and its minimal solution is \(\mathcal{L}_*(u_j, w)\) by Lemma 1. Because \(v_j\) is also a solution, we have \(\mathcal{L}_*(u_j, w) \leq v_j \leq m_{h_j}\). So \(u_j \in \{x \in [0, m_{g_j}] \mid \mathcal{L}_*(x, w) \leq m_{h_j}\}\). Thereby, \(\hat{u}_j \leq u_j \leq m_{g_j}\). To sum up, \(\hat{u}_j \vee w \leq u_j \leq m_{g_j}\); that is, \(u\) satisfies (46).

Now we will solve the formula (5) with the situation of \(w \in [0, \alpha]\). For every \(u \in [0, 1]_N\) denote the greatest solution of (31) in \([0, 1]_N\) by \(x_u^*\) (if any), where \(x_u^* = (\mathcal{L}_*, (u_1, w), \ldots, \mathcal{L}_*, (u_N, w))\). From Theorem 5 and Lemma 6 it can be inferred that
\[
X_1 = \{(u, v) \in [0, 1]_N \mid \mathcal{V}(0, \ldots, 0, \hat{u}_j \vee w, 0, \ldots, 0) \leq u \leq m_{g_j} \}, \quad j \in J.
\]
(47)
Denote
\[
H_1 = \left\{ g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(v_1) \ast \cdots \ast h_N(v_N) \mid (u, v) \in X_1 \right\}.
\]
(48)
Obviously \(H_1\) can be viewed as a union of \(|J|\) subsets, where the \(j\)th subset is as follows:
\[
H_j = \left\{ g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(I_*(u_1, w) \wedge m_{h_1}) \ast \cdots \ast h_N(I_*(u_N, w) \wedge m_{h_N}) \mid (0, \ldots, 0, \hat{u}_j \vee w, 0, \ldots, 0) \leq u \leq m_{g_j} \}, \quad j \in J.
\]
(49)
That is, \(H_1 = \bigcup_{1 \leq j \leq |J|} H_j\). Notice that, for any \(j_1, j_2 \in J\) and \(j_1 \neq j_2\), it may appear that \(H_{j_1} \cap H_{j_2} \neq \emptyset\). However, it will not affect our final results. The following theorem provides a method to obtain \(F(w)\) when \(w \in [0, \alpha]\).

Theorem 7. Let \(w \in [0, \alpha]\). Assume that \(\ast\) is continuous. Denote
\[
F^1_i = \sup \{ g_i(u_i) \ast h_1(I_*(u_i, w) \wedge m_{h_1}) \mid u_i \in [0, m_{g_i}] \}, \quad i \in \{1, \ldots, N\},
\]
\[ F^1_j = \sup \left\{ g_j(u_j) \ast \hat{h}_j(\mathcal{F}_*(u_j, w) \wedge m_{h_j}) \right\} \]
\[ | u_j \in \left[ \hat{u}_j \lor w, m_{g_j} \right] \}
\[ j \in J. \]  

(50)

Then the following items hold:

1. \( \sup \mathcal{F}_j = F^1_j \ast F^2_j \ast \cdots \ast F^N_j \ast F^1_j, \quad j \in J. \)
2. \( F(w) = \sup \{ \sup \mathcal{F}_j, j \in J \}. \)

Proof. (1) Without loss of the generality, we prove the case of \( \mathcal{F}_1. \) Clearly, \( g_i(u_i) \ast \hat{h}_i(\mathcal{F}_*(u_i, w) \wedge m_{h_i}) \) is bounded in \([0, m_{g_i}]\). Then there exists \( u_i^\prime \in [0, m_{g_i}] \) such that \( g_i(u_i) \ast \hat{h}_i(\mathcal{F}_*(u_i, w) \wedge m_{h_i}) \) reaches the maximum \( F^1_i, j \in \{1, \ldots , N\}. \) Similarly, there exists \( u_i'' \in [\hat{u}_i \lor w, m_{g_i}] \) such that \( g_i(u_i) \ast \hat{h}_i(\mathcal{F}_*(u_i, w) \wedge m_{h_i}) \) reaches the maximum \( F^1_i, j \in J. \)

From Lemma 6, it is easy to see that

\[ \left( u_1^\prime, u_2^\prime, \ldots , u_N^\prime, \mathcal{F}_*(u_1^\prime, w) \wedge m_{h_1}, \mathcal{F}_*(u_2^\prime, w) \wedge m_{h_2}, \ldots , \mathcal{F}_*(u_N^\prime, w) \wedge m_{h_N} \right) \in \mathcal{X}_1. \]

(51)

Let \( u \) satisfy (46). Then there are

\[ g_1(u_1) \ast \hat{h}_1(\mathcal{F}_*(u_1, w) \wedge m_{h_1}) \]
\[ \leq g_1(u_1'') \ast \hat{h}_1(\mathcal{F}_*(u_1'', w) \wedge m_{h_1}) = F^1_i, \]
\[ g_i(u_i) \ast \hat{h}_i(\mathcal{F}_*(u_i, w) \wedge m_{h_i}) \]
\[ \leq g_i(u_i') \ast \hat{h}_i(\mathcal{F}_*(u_i', w) \wedge m_{h_i}) = F^1_i, \]
\[ i = 2, \ldots , N. \]

Thus

\[ g_1(u_1) \ast g_2(u_2) \ast \cdots \ast g_N(u_N) \]
\[ \ast \hat{h}_1(\mathcal{F}_*(u_1, w) \wedge m_{h_1}) \]
\[ \ast \hat{h}_2(\mathcal{F}_*(u_2, w) \wedge m_{h_2}) \]
\[ \ast \cdots \ast \hat{h}_N(\mathcal{F}_*(u_N, w) \wedge m_{h_N}) \]
\[ \leq g_1(u_1') \ast g_2(u_2') \ast \cdots \ast g_N(u_N') \]
\[ \ast \hat{h}_1(\mathcal{F}_*(u_1'', w) \wedge m_{h_1}) \]
\[ \ast \hat{h}_2(\mathcal{F}_*(u_2', w) \wedge m_{h_2}) \]
\[ \ast \cdots \ast \hat{h}_N(\mathcal{F}_*(u_N', w) \wedge m_{h_N}) \]
\[ = F^1_1 \ast F^1_2 \ast \cdots \ast F^1_N. \]

(52)

Therefore, \( \sup \mathcal{F}_j = F^1_1 \ast F^1_2 \ast \cdots \ast F^1_N. \)

(2) Clearly, \( F(w) = \sup \{ \sup \mathcal{F}_j, j \in J \} \) since \( \mathcal{F}_1 = \bigcup_{j \in J} \mathcal{F}_j \) and \( F(w) = \sup \mathcal{F}_1. \)

If \( w \in [\beta, 1] \), then by Theorem 5 all elements in \( \mathcal{X}_2 \) can be obtained when all of the elements in \( \mathcal{U}_2 \) and minimal solutions of the corresponding equation (31) in \([0, 1]^N\) are obtained. The following lemma describes the characteristics of the elements in \( \mathcal{U}_2 \). Denote

\[ \hat{u}_i = \sup \left\{ x \in [n_{g_i}, 1] \mid \mathcal{F}_*(x, w) \geq n_{h_i} \right\}, \]
\[ i = 1, \ldots , N. \]

(53)

**Lemma 8.** Let \( w \in [\beta, 1] \) and \( u \in \mathcal{P}_2. \) Assume that \( * \) is continuous. Then (31) has a solution in \( \mathcal{P}_2 \) and if and only if there exists \( i \in \{1, \ldots , N\} \) such that

\[ (n_{g_i}, \ldots , n_{g_i}, n_{g_i} \lor w, n_{g_i}, \ldots , n_{g_i}) \]
\[ \leq u \leq (\hat{u}_1, \ldots , \hat{u}_i, \hat{u}_{i+1}, \ldots , \hat{u}_N) \]
\[ \leq u \leq (\hat{u}_1, \ldots , \hat{u}_i, \hat{u}_{i+1}, \ldots , \hat{u}_N). \]

(54)

**Proof.** The first, we will give the proof of \( w \leq n_{g_i} \lor w \leq \hat{u}_i \) for every \( i \in \{1, \ldots , N\}. \) Note that \( n_{g_i} \land \hat{u}_i \leq \hat{u}_i \) since \( \forall_{\beta \leq 1} (n_{g_i} \ast \hat{u}_i) \leq \hat{u}_i. \) Then \( \mathcal{F}_*(n_{g_i}, w) \geq n_{h_i}. \) Therefore, \( \mathcal{F}_*(n_{g_i} \lor w, w) \geq n_{h_i} \) since \( \mathcal{F}_*(w, w) = 1 \geq n_{h_i}. \) We obtain that \( n_{g_i} \lor w \in [x \in [n_{g_i}, 1] \mid \mathcal{F}_*(x, w) \geq n_{h_i}] \), which indicates that \( \hat{u}_i \) always exists and \( \hat{u}_i \geq n_{g_i} \lor w. \) Obviously \( w \leq n_{g_i} \lor w \leq \hat{u}_i. \)

Let \( u \) satisfy (55). Because \( w \leq n_{g_i} \lor w \leq u_i \), from Lemma 1 it can be inferred that (31) has a solution in \([0, 1]^N\) and the greatest solution is

\[ x_u^* = (\mathcal{F}_*(u_1, w), \ldots , \mathcal{F}_*(u_N, w)). \]

(55)

Furthermore, for every \( j \in \{1, \ldots , N\} \) we have \( \mathcal{F}_*(u_j, w) \geq \mathcal{F}_*(\hat{u}_j, w) \geq n_{h_i} \) since \( u \leq (\hat{u}_1, \ldots , \hat{u}_N). \) Then \( x_u^* \geq n_{h_i} \) which indicates that (31) has a solution in \( \mathcal{P}_2 \) by Corollary 4.

For the conversion, assume that (31) has a solution in \( \mathcal{P}_2. \) There exists \( n_0 \in \{1, \ldots , N\} \) such that \( u_0 \geq w. \) Then \( u_0 \geq n_{g_0} \lor w \) since \( u_0 \geq n_{g_0} \geq n_{g_0}. \) That is to say, \( u \geq (n_{g_1}, \ldots , n_{g_{n_0}}, n_{g_0} \lor w, n_{g_{n_0}}, \ldots , n_{g_N}). \) On the other hand, there is \( x_u^* \geq n_{h_i} \) by Corollary 4. Thus for every \( j \in \{1, \ldots , N\} \), we have \( u_j \in [x \in [n_{g_j}, 1] \mid \mathcal{F}_*(x, w) \geq n_{h_j}] \), which indicates that \( u_j \leq \hat{u}_i. \) Therefore, \( u \leq (\hat{u}_1, \ldots , \hat{u}_N). \)

Next we will solve the formula (35) with the situation of \( w \in (\beta, 1]. \) For every \( u \in [0, 1]^N \), denote minimal solutions of (31) in \([0, 1]^N\) by \( x_u^0, j = 1, \ldots , |\mathbb{G}_2| \) (if any). From Theorem 5 and Lemma 8, it can be seen that

\[ \mathcal{X}_2 = \left\{ (u, v) \in [0, 1]^2N \mid v \in \left( x_u^0 \lor w, n_{g_i}, \ldots , n_{g_i} \lor v, w \right) \right\}, \]
\[ i = 1, \ldots , |\mathbb{G}_2| \].

(56)

For every \( u \) satisfying (55) there must exist \( x_u^0 \in \left( x_u^0 \lor w, n_{g_i}, \ldots , n_{g_i} \lor v, w \right) \) such that it has the following form:

\[ x_u^0 = (0, \ldots , 0, \mathcal{F}_*(u_1, w), 0, 0, \ldots , 0). \]

(57)
Then
\[ v = x_0^* \lor \mathbf{n}_h = (n_{h_1}, \ldots, n_{h_m}, \mathcal{L}_*(u_1, w) \lor n_{h_L}, \ldots, n_{h_N}). \]  

Denote
\[ \mathcal{F}_2 = \{ g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(\nu_1) \ast \cdots \ast h_N(\nu_N) \mid (u, \nu) \in \mathcal{F}_2 \}, \]

\[ \mathcal{F}_2 = \{ h_1(n_{h_1}) \ast \cdots \ast h_N(n_{h_N}) \mid i = 1, \ldots, N \}. \]

Thus \( \mathcal{F}_2 \) can be viewed as a union of \( N \) subsets, where the \( i \)th subset is as follows:
\[ \mathcal{F}_i = \{ g_1(u_1) \ast \cdots \ast g_i(u_i) \ast \cdots \ast g_N(u_N) \]
\[ \ast h_i(\mathcal{L}_*(u_i, w) \lor n_{h_i}) \]
\[ \ast \delta_i | (n_{g_1}, \ldots, n_{g_{i-1}}, w, n_{g_{i+1}}, \ldots, n_{g_N}) \leq u \leq (u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_N) \} \]
\[ i = 1, 2, \ldots, N, \]

that is, \( \mathcal{F}_2 = \bigcup_{i=1}^N \mathcal{F}_i \). In fact, it is natural that \( \mathcal{F}_2 \supseteq \bigcup_{i=1}^N \mathcal{F}_i \) since \( \mathcal{F}_2 \supseteq \mathcal{F}_i, i = 1, \ldots, N \). For the converse, take a \( g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(\nu_1) \ast \cdots \ast h_N(\nu_N) \in \mathcal{F}_2 \). From the form of the elements in \( \mathcal{F}_2 \), there exists \( i_0 \in \{1, \ldots, N\} \) such that
\[ v = (n_{h_1}, \ldots, n_{h_{i_0}}, \mathcal{L}_*(u_{i_0}, w) \lor n_{h_{i_0}}, n_{h_{i_0+1}}, \ldots, n_{h_N}), \]

which indicates that \( u_{i_0} \geq w \) by Lemma 1. Clearly \( u_{i_0} \geq n_{g_{i_0}} \).
Therefore,
\[ (n_{g_1}, \ldots, n_{g_{i_0}}, w, n_{g_{i_0+1}}, \ldots, n_{g_N}) \leq u \leq (u_1, \ldots, u_{i_0-1}, u_{i_0}, u_{i_0+1}, \ldots, u_N). \]

From the above, we have \( g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(\nu_1) \ast \cdots \ast h_N(\nu_N) \in \mathcal{F}_{i_0} \); that is,
\( g_1(u_1) \ast \cdots \ast g_N(u_N) \ast h_1(\nu_1) \ast \cdots \ast h_N(\nu_N) \in \bigcup_{i=1}^N \mathcal{F}_i \). Thus \( \mathcal{F}_2 \subseteq \bigcup_{i=1}^N \mathcal{F}_i \). To sum up, we obtain that \( \mathcal{F}_2 = \bigcup_{i=1}^N \mathcal{F}_i \).

Notice that, for any \( i_1, i_2 \in \{1, 2, \ldots, N\} \) and \( i_1 \neq i_2 \), it may appear that \( \mathcal{F}_{i_1} \cap \mathcal{F}_{i_2} \neq 0 \). However, it will not affect our final results. The following theorem provides a method to obtain \( F(w) \) when \( w \in (\beta, 1] \).

**Theorem 9.** Let \( w \in (\beta, 1] \). Assume that \( * \) is continuous. Then the following items hold:

1. \( \sup \mathcal{F}_i = \sup \{ g_1(u_1) \ast h_1(\mathcal{L}_*(u_i, w) \lor n_{h_i}) \mid u_i \in [n_{g_i} \lor w, \tilde{u}_i] \} \), \( i = 1, \ldots, N \),
2. \( F(w) = \sup \{ \sup \mathcal{F}_i | i = 1, \ldots, N \} \).

**Proof.** (1) Similar to the proof of Theorem 7, it can be obtained that there exists \( u_i \in [n_{g_i} \lor w, \tilde{u}_i] \) such that
\[ g_1(u_1) \ast \ast h_1(\mathcal{L}_*(u_i, w) \lor n_{h_i}) \]
\[ = \sup \{ g_1(u_i) \ast h_1(\mathcal{L}_*(u_i, w) \lor n_{h_i}) \}
\[ \lor n_{h_i} \} \mid u_i \in [n_{g_i} \lor w, \tilde{u}_i] \}, \]
\[ i \in \{1, \ldots, N\} \].

Without loss of the generality, we prove the situation of \( \mathcal{F}_1 \).

**Lemma 8.** It is easy to see that
\[ (u_1, n_{g_1}, \ldots, n_{g_N}, \mathcal{L}_*(u_1, w) \lor n_{h_1}, \ldots, n_{h_N}) \in \mathcal{F}_2. \]

Denote
\[ \delta_1 = h_2(n_{h_1}) \ast \cdots \ast h_N(n_{h_N}), \]
\[ \sigma_1 = g_2(n_{g_2}) \ast \cdots \ast g_N(n_{g_N}). \]

It is easy to see \( \sigma_1 = \delta_1 = 1 \). Here we shall prove that \( g_1(u_1) \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}) \ast \delta_1 \) is the greatest element in \( \mathcal{F}_1 \). Take \( g_1(u_1) \ast g_2(u_2) \ast \cdots \ast g_N(u_N) \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}) \ast \delta_1 \in \mathcal{F}_1 \). By the convexity of \( g_1 \), we have
\[ g_2(u_2) \ast \cdots \ast g_N(u_N) \subseteq g_2(n_{g_2}) \ast \cdots \ast g_N(n_{g_N}). \]

Moreover, from the assumptions, there is
\[ g_1(u_1) \ast h_1(\mathcal{L}_*(u_1, w)) \leq g_1(u_1) \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}). \]

Thus
\[ g_1(u_1) \ast g_2(u_2) \ast \cdots \ast g_N(u_N) \]
\[ \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}) \ast \delta_1 \]
\[ \leq g_1(u_1) \ast g_2(n_{g_2}) \ast \cdots \ast g_N(n_{g_N}) \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}) \ast \delta_1. \]

It can be obtained that
\[ \sup \mathcal{F}_1 = g_1(u_1) \ast g_2(n_{g_2}) \ast \cdots \ast \]
\[ g_N(n_{g_N}) \ast h_1(\mathcal{L}_*(u_1, w) \lor n_{h_1}) \ast \delta_1 \]
\[ = g_1(u_1) \ast h_1(\mathcal{L}_*(u_1, w)) \ast \delta_1 \ast \delta_1 \]
\[ = g_1(u_1) \ast h_1(\mathcal{L}_*(u_1, w)) \ast \delta_1 \]
\[ = \mathcal{F}_1(\mathcal{L}_*(u_1, w)). \]

(2) It is clear that \( \sup \mathcal{F}_2 = \sup \{ \sup \mathcal{F}_i, i = 1, \ldots, N \} \)
\[ \sup \mathcal{F}_2 = \bigcup_{i=1}^N \mathcal{F}_i \] and \( F(w) = \sup \mathcal{F}_2. \]

Up to now, we can get all the values of expression (5). That is to say, for every fixed \( x \in X \) and \( y \in Y \), we can calculate the function \( \mu_{\mathcal{R}_{x,y}}(w) \) when \( * \) is continuous and \( \mu_{\mathcal{R}_{x,y}} \) and \( \mu_{\mathcal{R}_{x,y}} \) are both convex and normal. On the basis of Theorems 2, 7, and 9, we will give the implementation procedures in the following.
Algorithm 10. Consider the following (Figure 1):

Step 1. Calculate \( \alpha = \bigvee_{i=1}^{N} (m_{g_i} * m_{h_i}) \), \( \beta = \bigvee_{i=1}^{N} (n_{g_i} * n_{h_i}) \). For every \( w \in [0,1] \) employ Step 2–Step 4.

Step 2. When \( w \in [0, \alpha) \), step size \( \Delta \) and calculate the variables

\[
J = \{ i \in \{1, \ldots, N\} | m_{g_i} * m_{h_i} \geq w \},
\]

\[
\hat{u}_j = \inf \left\{ x \in [0, m_{g_j}] | \mathcal{L}_s (x, w) \leq m_{h_j} \right\}, \quad j \in J.
\]

Find the greatest value of \( g_j(u_i) * h_j(\mathcal{J}_s (u_i, w) \lor m_{h_i}) \) in \([0, m_{g_i}]\), denoted by \( F^1_i, i \in \{1, \ldots, N\} \); find the greatest value of \( g_j(u_j) * h_j(\mathcal{J}_s (u_j, w) \lor m_{h_i}) \) in \([\hat{u}_j \lor w, m_{g_j}]\), denoted by \( \hat{F}^1_j, j \in J \). Let

\[
\lambda_j = \hat{F}^1_j \star \hat{F}^1_{j-1} \star \cdots \star \hat{F}^1_j \star \cdots \star F^1_N, \quad j \in J.
\]

Then \( F(w) = \sup \{ \lambda_j, j \in J \} \).

Step 3. When \( w \in [\alpha, \beta] \), \( F(w) = 1 \).

Step 4. When \( w \in (\beta, 1] \), step size \( \Delta \) and calculate the variables

\[
\tilde{u}_i = \sup \left\{ x \in [n_{g_i}, 1] | \mathcal{J}_s (x, w) \geq n_{h_i} \right\}, \quad i = 1, \ldots, N.
\]

For every \( i \in \{1, \ldots, N\} \), find the greatest value of \( g_j(u_i) * h_i(\mathcal{J}_s (u_i, w) \lor n_{h_i}) \) in \([n_{g_i} \lor w, \tilde{u}_i]\), denoted by \( F^2_i, i \in \{1, \ldots, N\} \). Then \( F(w) = \sup \{ F^2_i, i = 1, \ldots, N \} \).

Remark 11. The above algorithm can be applied in calculating extended continuous t-norm based on arbitrary t-norm on two type-2 fuzzy sets once setting \( N = 1 \) and extended maximum based on arbitrary t-norm on \( N \) type-2 fuzzy sets once setting \( h_i(u) = 1 \), \( i = 1, \ldots, N \). Hence the type-2 fuzzy reasoning relations of type-2 fuzzy logic systems with multiple input and single output can be calculated.

Remark 12. It can be seen from the operation steps above that the presented method to calculate the formula (5) is much simpler than the native algorithm (i.e., finding the maximum of \( f(u, v) \) from all of the combination \((u, v) \) in \( P_{g} \) (or \( P_{u1} \) and \( P_{u2} \))) which is a huge operation process undoubtedly. Take \( w \) from \([0,1] \) with step size \( \Delta_0 \). Then the amount of computation is no more than

\[
4N + \frac{\alpha}{\Delta_0} \cdot \left( \sum_{i=1}^{N} \frac{4 m_{g_i}}{\Delta} + N^2 \right) + \frac{1 - \beta}{\Delta_0} \cdot \left( \sum_{i=1}^{N} \frac{(1 - \tilde{u}_i) + 3 (\tilde{u}_i - n_{g_i} \lor w)}{\Delta} + N \right),
\]
where \( \mathcal{T}_*(a, b), \mathcal{L}_*(a, b), * \) or \( *' \) is considered one computation and the step size \( \Delta \) is small enough. According to above analysis, we can draw the following conclusions: the computation amount level of the proposed algorithm is the same as that of polynomials.

4. Examples

In this section some concrete examples for the construction of type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems on the proposed algorithm will be given. All of them are realized by using MATLAB2010 (b).

**Example 1.** Let input domain \( X = \{ x \} \) and output domain \( Y = \{ y \} \). Then each type-2 fuzzy reasoning relation \( \tilde{R}_i \) \( (i \in \{ 1, \ldots, N \}) \) and the total type-2 fuzzy reasoning relation \( \tilde{R} \) are only defined on \( X \times Y = \{(x, y)\} \). In the group of type-2 fuzzy reasonings (3) we choose \( \tilde{A}_i, \tilde{B}_i, \) \( i = 1, \ldots, 7 \) as follows:

\[
\begin{align*}
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.3)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.34)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.36)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.4)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.5)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.55)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(u - 0.6)^2}{2 \times 0.2^2} \right), \\
\mu_{\tilde{A}_i}(x) (u) &= \exp \left( \frac{(v - 0.5)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.55)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.6)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.65)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.68)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.7)^2}{2 \times 0.1^2} \right), \\
\mu_{\tilde{B}_i}(y) (v) &= \exp \left( \frac{(v - 0.75)^2}{2 \times 0.1^2} \right).
\end{align*}
\]

\( i = 1, \ldots, 7 \)

Choose \( \bigcup_{i=1}^{K}(\mathcal{V},*') = \bigcup_{i=1}^{K}(\mathcal{V},\oplus) \) and \( \cap_{i=1}^{K}(\mathcal{V},*') = \cap_{i=1}^{K}(\mathcal{V},\otimes) \), where \( a \oplus b = 0 \vee (a + b - 1) \), \( \tilde{A}_i \), and \( \tilde{B}_i \) as stated in Example 1, \( i = 1, \ldots, 7 \). Then expression (5) is reduced as

\[
\mu_{\tilde{R}_{i}(x,y)} (w) = \sup_{\bigvee_{i=1}^{K}(\mathcal{V},\oplus) (u_i) = w} \left( \bigoplus_{i=1}^{K} \left( \mu_{\tilde{A}_{i}(x)} (u_i) \bigotimes \mu_{\tilde{B}_{i}(y)} (v_i) - 1 \right) \right),
\]

where \( \oplus \) and \( \otimes \) indicate the same t-norm. Here we shall calculate (77) by using our method. Clearly \( \mathcal{F}_*(a, b) = \mathcal{F}_*(a, b) = 1 \land (b - a + 1) \) and \( \alpha = \beta = 0.35 \). The function graph of \( \mu_{\tilde{R}_{i}(x,y)} (w) \) in (77) is shown in Figure 2.

**Example 2.** Choose \( \bigcup_{i=1}^{K}(\mathcal{V},*') = \bigcup_{i=1}^{K}(\mathcal{V},\oplus) \) and \( \cap_{i=1}^{K}(\mathcal{V},*') = \cap_{i=1}^{K}(\mathcal{V},\otimes) \), where

\[
a \otimes b = \begin{cases} 
    a \land b, & a \lor b = 1, \\
    0, & a \lor b < 1.
\end{cases}
\]

Then expression (5) is reduced as

\[
\mu_{\tilde{R}_{i}(x,y)} (w) = \sup_{\bigvee_{i=1}^{K}(\mathcal{V},\otimes) (v_i) = w} \left( \bigotimes_{i=1}^{K} \left( \mu_{\tilde{A}_{i}(x)} (u_i) \bigotimes \mu_{\tilde{B}_{i}(y)} (v_i) \right) \right),
\]

where \( \otimes \) and \( \otimes \) indicate the same t-norm. Here we will calculate (79) by using our method. Clearly \( \alpha = \beta = 0.35 \). The function graph of \( \mu_{\tilde{R}_{i}(x,y)} (w) \) in (79) is shown in Figure 3.
and normality, which guarantees the operation conditions of extended t-(co)norms for the next turn. It can be seen that the proposed algorithm deals with the antecedents and consequents of the group of type-2 fuzzy reasoning in an integral way and the computation amount level of the proposed algorithm is the same as that of polynomials, which indicates that the proposed algorithm may be well applied in type-2 fuzzy logic systems. Besides, it can be seen that the calculations of an extended continuous t-norm based on arbitrary t-norms can be obtained as the special case of the proposed algorithm, which is a new idea to calculate the membership functions of a class of extended t-norm. However, all the fuzzy truth values of type-2 fuzzy sets that participated in the calculation are required to be convex and normal. Obviously, by using our proposed algorithm more applications about noninterval type-2 fuzzy logic system and type-2 fuzzy neural network could be attempted.

5. Conclusions

In this paper, an algorithm for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems has been given under certain conditions. The results may serve to establish many new type-2 fuzzy logic systems by using different extended t-(co)norms. An important conclusion has been given that the results of extended continuous t-(co)norms based on arbitrary t-norm keep the convexity and normality, which guarantees the operation conditions of extended t-(co)norms for the next turn. It can be seen that the proposed algorithm deals with the antecedents and consequents of the group of type-2 fuzzy reasoning in an integral way and the computation amount level of the proposed algorithm is the same as that of polynomials, which indicates that the proposed algorithm may be well applied in type-2 fuzzy logic systems. Besides, it can be seen that the calculations of an extended continuous t-norm based on arbitrary t-norms can be obtained as the special case of the proposed algorithm, which is a new idea to calculate the membership functions of a class of extended t-norm. However, all the fuzzy truth values of type-2 fuzzy sets that participated in the calculation are required to be convex and normal. Obviously, by using our proposed algorithm more applications about noninterval type-2 fuzzy logic system and type-2 fuzzy neural network could be attempted.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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