Iterative Computation for Solving the Variational Inequality and the Generalized Equilibrium Problem

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An iterative algorithm for solving the variational inequality and the generalized equilibrium problem has been introduced. Convergence result is given.
introduced an iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [16] used the iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. In 1998, Haubruege et al. [17] studied the convergence analysis of the iterative schemes of Glowinski and Le Tallec and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions.

Our main purpose in the present paper is to solve the following equilibrium problem and variational inequality problem: finding a point $x^*$ such that

$$x^* \in C, \quad \phi(x^*) \in D$$

(4)

has received much attention. For related works, please refer to [18–20]. However, we observe that the involved operator $\phi$ in (4) is a bounded liner operator. In this paper, we devote to study the problem (3), where the transformation $\phi$ is a nonlinear mapping. For this purpose, we introduce a new iterative algorithm. Consequently, strong convergence analysis is demonstrated.

2. Preliminaries

In this section, we recall some useful lemmas.

Recall that the metric projection $\text{proj}_C : \mathbb{H} \to C$ satisfies $\|u - \text{proj}_C u\| = \inf \{\|u - v\| : v \in C\}$. The metric projection $\text{proj}_C$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is $\langle u - \text{proj}_C u, v - \text{proj}_C u \rangle \leq 0$, for all $u \in H, v \in C$.

Assume that $\theta : C \times C \to \mathbb{R}$ is a bifunction which satisfies the following conditions:

(C1) $\theta(u, u) = 0$, for all $u \in C$;

(C2) $\theta$ is monotone; that is, $\theta(u, v) + \theta(v, u) \leq 0$, for all $u, v \in C$;

(C3) for each $u, v, w \in C$, $\lim_{t \to 0} \theta(tu + (1-t)v, w) \leq \theta(u, v)$;

(C4) for each $u \in C$, $v \mapsto \theta(u, v)$ is convex and lower semicontinuous.

Lemma 1 (see [2]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\theta : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (C1)–(C4). Let $\tau > 0$ and $u \in C$. Then, there exists $w \in C$ such that

$$\theta(w, v) + \frac{1}{\tau} \langle v - w, w - u \rangle \geq 0, \quad \forall v \in C.$$  

(5)

Further, if $S_\tau(u) = \{w \in C : \theta(w, v) + (1/\tau) \langle v - w, w - u \rangle \geq 0, \text{ for all } v \in C\}$, then the following hold:

(a) $S_\tau$ is single-valued and $S_\tau$ is firmly nonexpansive;

(b) $EP(\theta)$ is closed and convex and $EP(\theta) = \text{Fix}(S_\tau)$.

Lemma 2 (see [21]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\eta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \eta_n \leq \limsup_{n \to \infty} \eta_n < 1$. Suppose that $x_{m_n} = (1 - \eta_n) y_n + \eta_n x_{n+1}$, for all $n \geq 0$, and $\limsup_{n \to \infty} \|y_{n+1} - y_n\| \leq \limsup_{n \to \infty} \|x_{m_n} - x_n\| \leq 0$. Then, $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 3 (see [22]). Assume that the sequence $\{a_n\}$ satisfies $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^{\eta} \leq 1$, where $\{b_n\}$ is a sequence in $(0, 1)$ and $\{c_n\}$ is a sequence such that $\sum_{n=1}^{\infty} c_n^{\eta} = \infty$ and $\limsup_{n \to \infty} \|a_n|c_n \| < \infty$. Then, $\lim_{n \to \infty} a_n = 0$.

3. Main Results

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A : C \to H$ be an $\eta$-inverse strongly monotone mapping. Let $\phi : C \to C$ be a weakly continuous and $\eta$-strongly monotone mapping such that $R(\phi) = C$. Let $B : C \to H$ be a $\xi$-inverse strongly $\phi$-monotone mapping. Let $\rho : C \to H$ be an $L$-Lipschitz continuous mapping. Let $\theta : C \times C \to \mathbb{R}$ be a bifunction which satisfies (C1)–(C4) in the above section. Let $\{\varepsilon_n\} \subseteq [0, 1], \{\eta_n\} \subseteq [0, 1], \{a_n\} \subseteq (0, \infty)$, and $\{r_n\} \subseteq (0, \infty)$ be four real number sequences and let $\xi > 0$ be a constant.

We use $\Delta$ to denote the solution set of (3). In order to solve (3), we introduce the following three-step algorithm.

Algorithm 4. Let $x_0 \in C$ be an initial guess. Define the sequence $\{x_n\}$ as follows:

$$u_n = \text{proj}_C \phi(x_n) - \omega_n \beta x_n, \quad n \geq 0,$$

$$\theta(z_n, y) + \frac{1}{\tau_n} \langle y - z_n, z_n - u_n \rangle \geq 0, \quad \forall y \in C,$$

$$\phi(x_{n+1}) = \eta_n\phi(x_n) + (1 - \eta_n) \text{proj}_C [\xi_n \psi(x_n) + (1 - \xi_n) z_n], \quad n \geq 0.$$  

(6)

Theorem 5. Suppose that $\Delta \neq \emptyset$. Assume that the following conditions are satisfied:

(r1) $a_n \in (a_1, a_2) \subseteq (0, 2\xi)$ and $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$;

(r2) $\tau_n \in (a_3, a_4) \subseteq (0, 2\eta)$ and $\lim_{n \to \infty} (\tau_{n+1} - \tau_n) = 0$;

(r3) $\eta_n \in [a_5, a_6) \subseteq (0, 1)$;

(r4) $\lim_{n \to \infty} \xi_n = 0$ and $\sum_{n=1}^{\infty} \xi_n = \infty$;

(r5) $\nu \in (L\xi, 2\xi)$.

Then, the sequence $\{x_n\}$ generated by (6) converges strongly to $x^* \in \Delta$ which solves the following variational inequality:

$$\langle \psi(x^*) - \phi(x^*), \phi(x) - \phi(x^*) \rangle \leq 0, \quad \forall x \in \Delta.$$  

(7)
Proof. First of all, we prove that the solution of the variational inequality (7) is unique. In fact, if \( x^* \in \Delta \) also solves (7), then we get
\[
\langle \zeta (x^*) - \phi (x^*), \phi (x^*) - \phi (x^*) \rangle \leq 0,
\]
(8)
\[
(\zeta \varphi (x^*) - \phi (x^*) - \phi (x^*) \rangle \leq 0.
\]
(9)
It follows that
\[
(\zeta \varphi (x^*) - \phi (x^*), \phi (x^*) - \phi (x^*) \rangle \leq 0.
\]
(9)
So,
\[
\|\phi (x^*) - \phi (x^*)\|^2 \leq \zeta \|\phi (x^*) - \phi (x^*)\| \leq \zeta \|\phi (x^*) - \phi (x^*)\|,
\]
which implies that
\[
\|\phi (x^*) - \phi (x^*)\| \leq \zeta \|\phi (x^*) - \phi (x^*)\|.
\]
(11)
Since \( \phi \) is \( \nu \)-strongly monotone, we have
\[
\nu \|x^* - x^*\|^2 \leq \langle \phi (x^*) - \phi (x^*), x^* - x^* \rangle \leq \|\phi (x^*) - \phi (x^*)\| \|x^* - x^*\|.
\]
(12)
Thus,
\[
\nu \|x^* - x^*\| \leq \|\phi (x^*) - \phi (x^*)\| \leq \zeta \|x^* - x^*\|.
\]
(13)
Since \( \zeta L < \nu \), we deduce the contradiction. Therefore, \( x^* = x^* \).

So, the solution of variational inequality (7) is unique.

Let \( x^* \in \Delta \). Hence, \( x^* \in VI(B, \phi, \nu) \) and \( \phi(x^*) \in EP(\theta, \lambda) \). Note that
\[
x^* \in VI(B, \phi, \nu) \iff \phi (x^*) = projC \left( \phi (x^*) - \nu x^* \right),
\]
\[\forall \gamma > 0.\]
(14)
Since \( \omega_n > 0 \), we have \( \phi(x^*) = projC [\phi(x^*) - \omega_n B x^*] \), for all \( n \geq 0 \). For \( u, v \in C \), we have
\[
\|\phi (u) - \phi (v) - \phi (v) - \omega_n B v\|^2 \\
= \|\phi (u) - \phi (v)\|^2 - 2 \omega_n \langle B u - B v, \phi (u) - \phi (v) \rangle \\
+ \omega_n^2 \|B u - B v\|^2 \\
\leq \|\phi (u) - \phi (v)\|^2 - 2 \omega_n \zeta \|B u - B v\|^2 + \omega_n^2 \|B u - B v\|^2 \\
\leq \|\phi (u) - \phi (v)\|^2 + \omega_n (\omega_n - 2 \zeta) \|B u - B v\|^2.
\]
(15)
Hence,
\[
\|\phi (u) - \phi (v) - \omega_n B v \| \leq \|\phi (u) - \phi (v)\|,
\]
(16)
An induction implies that
\[
\begin{align*}
\|\phi(x_n) - \phi(x^*)\| & \leq \max \left\{ \|\phi(x_0) - \phi(x^*)\|, \frac{\|\nabla \phi(x^*) - \phi(x^*)\|}{1 - \frac{\nabla \phi(x^*)}{1 - L/v}} \right\}.
\end{align*}
\] (19)

Hence, \(\{\phi(x_n)\}\) is bounded. Since \(\phi\) is \(\nu\)-strongly monotone, we can deduce \(\|x_n - x^*\| \leq \|\phi(x_n) - \phi(x^*)\|\). So,
\[
\begin{align*}
\|x_n - x^*\| & \leq \frac{1}{\nu} \|\phi(x_n) - \phi(x^*)\|
\leq \frac{1}{\nu} \max \left\{ \|\phi(x_0) - \phi(x^*)\|, \frac{\|\nabla \phi(x^*) - \phi(x^*)\|}{1 - \frac{\nabla \phi(x^*)}{1 - L/v}} \right\}.
\end{align*}
\] (20)

This implies that \(\{x_n\}\) is bounded.

From (6), we have
\[
\theta(z_n, y) + \frac{1}{\tau_n} \langle y - z_n, z_n - (u_n - \tau_n \beta u_n) \rangle \geq 0, \quad \forall y \in C.
\] (21)

So,
\[
\theta(z_n, z_{n+1}) + \frac{1}{\tau_n} \langle z_{n+1} - z_n, z_n - (u_n - \tau_n \beta u_n) \rangle \geq 0.
\] (22)

Similarly,
\[
\begin{align*}
\theta(z_{n+1}, z_n) + \frac{1}{\tau_{n+1}} \langle z_{n+1} - z_n, z_n - (u_{n+1} - \tau_{n+1} \beta u_{n+1}) \rangle & \geq 0.
\end{align*}
\] (23)

Hence,
\[
\begin{align*}
\theta(z_n, z_{n+1}) + \theta(z_{n+1}, z_n) + \langle \beta u_{n+1} - \beta u_n, z_{n+1} - z_n \rangle \\
+ \langle z_{n+1} - z_n, \frac{z_{n+1} - u_{n+1}}{\tau_{n+1}} - \frac{z_n - u_n}{\tau_n} \rangle & \geq 0.
\end{align*}
\] (24)

Since \(\theta\) is monotone, we have
\[
\theta(z_n, z_{n+1}) + \theta(z_{n+1}, z_n) \leq 0.
\] (25)

So,
\[
\begin{align*}
\langle \beta u_{n+1} - \beta u_n, z_{n+1} - z_n \rangle \\
+ \langle z_{n+1} - z_n, \frac{z_{n+1} - u_{n+1}}{\tau_{n+1}} - \frac{z_n - u_n}{\tau_n} \rangle & \geq 0.
\end{align*}
\] (26)

Thus,
\[
\begin{align*}
\tau_n \langle \beta u_{n+1} - \beta u_n, z_{n+1} - z_n \rangle \\
+ \langle z_{n+1} - z_n, \frac{z_{n+1} - z_{n+1} + z_{n+1} - u_{n+1} - \frac{\tau_n}{\tau_{n+1}} (z_{n+1} - u_{n+1})}{\tau_n} \rangle & \geq 0.
\end{align*}
\] (27)

It follows that
\[
\begin{align*}
\|z_{n+1} - z_n\|^2 & \leq \tau_n \langle \beta u_{n+1} - \beta u_n, z_{n+1} - z_n \rangle \\
& + \langle z_{n+1} - z_n, \frac{z_{n+1} - u_{n+1} + \frac{\tau_n}{\tau_{n+1}} (z_{n+1} - u_{n+1})}{\tau_n} \rangle \\
& \leq \|z_{n+1} - z_n\| \|z_{n+1} - u_{n+1}\| \\
& + \left\| \frac{1}{\tau_{n+1}} \|z_{n+1} - u_{n+1}\| \right\| \|z_{n+1} - u_{n+1}\| \\
& \leq \|z_{n+1} - z_n\| \left( \|u_{n+1} - u_n\| + \frac{\tau_n}{\tau_{n+1}} \right) \|z_{n+1} - u_{n+1}\|.
\end{align*}
\] (28)

and hence
\[
\begin{align*}
\|z_{n+1} - z_n\| & \leq \|u_{n+1} - u_n\| + \frac{\tau_n}{\tau_{n+1}} \|z_{n+1} - u_{n+1}\| \\
& \leq \|u_{n+1} - u_n\| + \frac{1}{\alpha_3} \|r_{n+1} - r_n\| \|z_{n+1} - u_{n+1}\|.
\end{align*}
\] (29)

By (6) and (16), we have
\[
\begin{align*}
\|u_{n+1} - u_n\| & = \|\text{proj}_C \left[ (x_{n+1}) - \omega_{n+1} \mathbb{B} x_{n+1} \right] - \text{proj}_C \left[ (x_n) - \omega_n \mathbb{B} x_n \right]\| \\
& \leq \left\| \phi(x_{n+1}) - \omega_{n+1} \mathbb{B} x_{n+1} \right\| - \left\| \phi(x_n) - \omega_n \mathbb{B} x_n \right\| \\
& \leq \left\| \phi(x_{n+1}) - \omega_{n+1} \mathbb{B} x_{n+1} \right\| - \left\| \phi(x_n) - \omega_n \mathbb{B} x_n \right\| \\
& + \|\omega_{n+1} - \omega_n\| \|\mathbb{B}(x_n)\| \\
& \leq \left\| \phi(x_{n+1}) - \phi(x_n) \right\| + \|\omega_{n+1} - \omega_n\| \|\mathbb{B}(x_n)\|.
\end{align*}
\] (30)

Therefore,
\[
\begin{align*}
\|z_{n+1} - z_n\| & \leq \|\phi(x_{n+1}) - \phi(x_n)\| \\
& + \|\omega_{n+1} - \omega_n\| \|\mathbb{B}(x_n)\| + \frac{1}{\alpha_3} \|r_{n+1} - r_n\| \|z_{n+1} - u_{n+1}\|.
\end{align*}
\] (31)

It follows that
\[
\begin{align*}
\|z_{n+1} - z_n\| - \|\phi(x_{n+1}) - \phi(x_n)\| & \leq \|\omega_{n+1} - \omega_n\| \|\mathbb{B}(x_n)\| + \frac{1}{\alpha_3} \|r_{n+1} - r_n\| \|z_{n+1} - u_{n+1}\|. 
\end{align*}
\] (32)
Since \( \lim_{n \to \infty} (\omega_{n+1} - \omega_n) = 0 \), \( \lim_{n \to \infty} (\tau_{n+1} - \tau_n) = 0 \), and the sequences \( \{\xi(x_n)\} \), \( \{\phi(x_n)\} \), \( \{z_n\} \), \( \{u_n\} \), and \( \{Bx_n\} \) are bounded, we have

\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|\phi(x_{n+1}) - \phi(x_n)\|) \leq 0. \tag{33}
\]

From Lemma 2, we obtain

\[
\lim_{n \to \infty} (\|z_n - \phi(x_n)\| = 0. \tag{34}
\]

Note that

\[
\|\phi(x_{n+1}) - \phi(x_n)\| \leq (1 - \eta_n) \xi_n \|\xi - \phi (x_n)\| + (1 - \eta_n) (1 - \xi_n) \|z_n - \phi(x_n)\|. \tag{35}
\]

Hence,

\[
\lim_{n \to \infty} \|\phi(x_{n+1}) - \phi(x_n)\| = 0. \tag{36}
\]

This together with the \( \nu \)-strong monotonicity of \( \phi \) implies that

\[
\lim_{n \to \infty} \|\phi(x_{n+1}) - x_n\| = 0. \tag{37}
\]

From (18), we have

\[
\|\phi(x_{n+1}) - \phi(x^*)\| \\
\leq \eta_n \|\phi(x_n) - \phi(x^*)\| + (1 - \eta_n) [\xi_n \|\phi(x_{n+1}) - \phi(x^*)\| + (1 - \xi_n) \|u_n - \phi(x^*)\|] \\
\leq \eta_n \|\phi(x_n) - \phi(x^*)\| + (1 - \eta_n) \left[ \xi_n \|\phi(x_{n+1}) - \phi(x^*)\| + (1 - \xi_n) \|u_n - \phi(x^*)\| \right] \\
\leq \left[ 1 - (1 - \eta_n) \left( 1 - \frac{\zeta L^*}{v} \right) \right] \|\phi(x_n) - \phi(x^*)\| \\
+ (1 - \xi_n) (1 - \eta_n) \|u_n - \phi(x^*)\| \\
+ (1 - \eta_n) \xi_n \left( 1 - \frac{\zeta L^*}{v} \right) \|\phi(x_{n+1}) - \phi(x^*)\| \\
+ (1 - \eta_n) \xi_n \left( 1 - \frac{\zeta L^*}{v} \right) \|\phi(x_{n+1}) - \phi(x^*)\|. \tag{38}
\]

By the convexity of the norm and (17), we have

\[
\|\phi(x_{n+1}) - \phi(x^*)\|^2 \\
\leq \left[ 1 - (1 - \eta_n) \left( 1 - \frac{\zeta L^*}{v} \right) \right] \|\phi(x_n) - \phi(x^*)\|^2 \\
+ (1 - \xi_n) (1 - \eta_n) \|u_n - \phi(x^*)\|^2 \\
+ (1 - \eta_n) \xi_n \left( 1 - \frac{\zeta L^*}{v} \right) \left( \frac{\|\phi(x^*) - \phi(x^*)\|}{1 - \zeta L^*/v} \right)^2 \\
\leq \left[ 1 - (1 - \eta_n) \left( 1 - \frac{\zeta L^*}{v} \right) \right] \|\phi(x_n) - \phi(x^*)\|^2 \\
+ (1 - \xi_n) (1 - \eta_n) \|u_n - \phi(x^*)\|^2 \\
\times \|\phi(x_{n+1}) - \phi(x^*)\| \\
\times \|\phi(x_{n+1}) - \phi(x^*)\| \\
\times \left( \|\phi(x_{n+1}) - \phi(x^*)\| + \|\phi(x_{n+1}) - \phi(x^*)\| \right) \\
\times \left( \|\phi(x_{n+1}) - \phi(x^*)\| \right)^2. \tag{39}
\]

So,

\[
(1 - \eta_n) (1 - \xi_n) \omega_n (2\zeta - \omega_n) \|Bx_n - Bx^*\|^2 \\
\leq \|\phi(x_{n+1}) - \phi(x^*)\|^2 - \|\phi(x_{n+1}) - \phi(x^*)\|^2 \\
+ (1 - \eta_n) \xi_n \left( 1 - \frac{\zeta L^*}{v} \right) \left( \frac{\|\phi(x^*) - \phi(x^*)\|}{1 - \zeta L^*/v} \right)^2 \\
\leq \left( \|\phi(x_{n+1}) - \phi(x^*)\| + \|\phi(x_{n+1}) - \phi(x^*)\| \right) \\
\times \|\phi(x_{n+1}) - \phi(x^*)\| \\
\times \left( \|\phi(x_{n+1}) - \phi(x^*)\| \right)^2. \tag{40}
\]
Since \( \xi_n \to 0 \), \( \|\phi(x_{n+1}) - \phi(x_n)\| \to 0 \), and \( \liminf_{n \to \infty} (1 - \eta_n)(1 - \xi_n)\alpha_n(2\xi_n - \alpha_n) > 0 \), we obtain
\[
\lim_{n \to \infty} \|Bx_n - Bx^*\| = 0. \tag{41}
\]

Note that
\[
\left\| u_n - \phi(x^*) \right\|^2
= \left\| \text{proj}_C [\phi(x_n) - \alpha_n Bx_n] - \text{proj}_C [\phi(x^*) - \alpha_n Bx^*] \right\|^2
\leq \langle \phi(x_n) - \alpha_n Bx_n - (\phi(x^*) - \alpha_n Bx^*), u_n - \phi(x^*) \rangle
= \frac{1}{2} \left\\left\| [\phi(x_n) - \alpha_n Bx_n - (\phi(x^*) - \alpha_n Bx^*)] - \alpha_n Bx_n - Bx^* \right\|^2
+ \left\| u_n - \phi(x^*) \right\|^2
- \left\| \phi(x_n) - u_n - \alpha_n Bx_n - Bx^* \right\|^2.
\]
\leq \frac{1}{2} \left\\left\| [\phi(x_n) - \phi(x^*)] - u_n \right\|^2
\leq \frac{1}{2} \left\\left\| [\phi(x_n) - \alpha_n Bx_n - (\phi(x^*) - \alpha_n Bx^*)] - \alpha_n Bx_n - Bx^* \right\|^2
+ \left\| u_n - \phi(x^*) \right\|^2
- \left\| \phi(x_n) - u_n - \alpha_n Bx_n - Bx^* \right\|^2.
\]
\leq \frac{1}{2} \left\\left\| [\phi(x_n) - \phi(x^*)] - u_n \right\|^2
- \left\| \phi(x_n) - u_n - \alpha_n Bx_n - Bx^* \right\|^2.
\]
\leq \frac{1}{2} \left\\left\| \phi(x_n) - u_n - \alpha_n Bx_n - Bx^* \right\|^2
+ 2\alpha_n \langle \phi(x_n) - u_n, Bx_n - Bx^* \rangle.
\tag{42}
\]

It follows that
\[
\left\| u_n - \phi(x^*) \right\|^2
\leq \left\| \phi(x_n) - u_n - \alpha_n Bx_n - Bx^* \right\|^2
+ 2\alpha_n \langle \phi(x_n) - u_n, Bx_n - Bx^* \rangle.
\tag{43}
\]

From (39) and (43), we have
\[
\left\| \phi(x_{n+1}) - \phi(x^*) \right\|^2
\leq \left[ 1 - (1 - \eta_n)(1 - \frac{\zeta_n L_n}{y}) \right] \left\| \phi(x_n) - \phi(x^*) \right\|^2
+ (1 - \xi_n)(1 - \eta_n) \left\| u_n - \phi(x^*) \right\|^2
+ (1 - \eta_n) \zeta_n \left( 1 - \frac{c L}{y} \right) \left( \left\| \phi(x^*) - \phi(x^*) \right\| \right)^2
\leq \left[ 1 - (1 - \eta_n)(1 - \frac{\zeta_n L_n}{y}) \right] \left\| \phi(x_n) - \phi(x^*) \right\|^2
+ (1 - \xi_n)(1 - \eta_n) \left\| \phi(x_n) - \phi(x^*) \right\|^2
- (1 - \xi_n)(1 - \eta_n) \left\| \phi(x_n) - u_n \right\|^2
+ 2(1 - \zeta_n)(1 - \eta_n) \alpha_n \langle \phi(x_n) - u_n, Bx_n - Bx^* \rangle
+ (1 - \eta_n) \zeta_n \left( 1 - \frac{c L}{y} \right) \left( \left\| \phi(x^*) - \phi(x^*) \right\| \right)^2.
\tag{44}
\]

Then, we obtain
\[
(1 - \xi_n)(1 - \eta_n) \left\| \phi(x_n) - u_n \right\|^2
\leq \left( \left\| \phi(x_n) - \phi(x^*) \right\| + \left\| \phi(x_{n+1}) - \phi(x^*) \right\| \right) \left( \left\| \phi(x_{n+1}) - \phi(x_n) \right\| \right)
+ 2\alpha_n \langle \phi(x_n) - u_n, Bx_n - Bx^* \rangle
+ (1 - \eta_n) \zeta_n \left( 1 - \frac{c L}{y} \right) \left( \left\| \phi(x^*) - \phi(x^*) \right\| \right)^2.
\tag{45}
\]

Since \( \lim_{n \to \infty} \xi_n = 0 \), \( \lim_{n \to \infty} \|\phi(x_{n+1}) - \phi(x_n)\| = 0 \), and \( \lim_{n \to \infty} \|Bx_n - Bx^*\| = 0 \), we have
\[
\lim_{n \to \infty} \phi(x_n) = u_n = 0.
\tag{46}
\]

Next, we prove that \( \limsup_{n \to \infty} \langle \phi(x^*) - \phi(x_n), u_n - \phi(x^*) \rangle \leq 0 \), where \( x^* \) is the unique solution of (7). Let \( \{u_n\} \) be a subsequence of \( \{u_n\} \) such that
\[
\limsup_{n \to \infty} \langle \phi(x^*) - \phi(x_n), u_n - \phi(x^*) \rangle
= \lim_{i \to \infty} \langle \phi(x^*) - \phi(x_n), u_n - \phi(x^*) \rangle
= \lim_{i \to \infty} \langle \phi(x^*) - \phi(x_n), \phi(x_n) - \phi(x^*) \rangle.
\tag{47}
\]

By the boundedness of \( \{x_n\} \), there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) which converges weakly to some point \( z \in C \). Without loss of generality, we may assume that \( x_{n_i} \to z \). From the weak continuity of \( \phi \), we deduce \( \phi(x_{n_i}) \to \phi(z) \). Next, we prove \( z \in \Delta \). We firstly show \( z \in EP(\Theta, \Lambda) \). Noting that \( z_n = S_{\tau_n}(u_n - \tau_n\Lambda u_n) \), for any \( y \in C \), we have
\[
\theta(z_n, y) + \frac{1}{\tau_n} \langle y - z_n, z_n - (u_n - \tau_n\Lambda u_n) \rangle \geq 0.
\tag{48}
\]
Since $\theta$ is monotone, we have
\[
\frac{1}{\tau_n} \left< y - z_n, z_n - (u_n - \tau_n A u_n) \right> \geq \theta(y, z_n), \quad \forall y \in C.
\]
Hence,
\[
\left< y - z_n, \frac{z_n - u_n}{\tau_n} + \lambda u_n \right> \geq \theta(y, z_n), \quad \forall y \in C. \tag{50}
\]
Let $v_t = ty + (1 - t)z$, for all $t \in (0, 1]$ and $y \in C$. We have $v_t \in C$. So, from (50) we have
\[
\left< v_t - z_n, \lambda v_t \right> \geq \left< v_t - z_n, \frac{z_n - u_n}{\tau_n} + \lambda u_n \right> + \theta(v_t, z_n).
\tag{51}
\]
Note that $\|\lambda z_n - \lambda u_n\| \leq (1/\eta)\|z_n - u_n\| \to 0$. Further, from monotonicity of $\lambda$, we have $\langle v_t - z_n, \lambda v_t - \lambda z_n \rangle \geq 0$. Letting $t \to 0$ in (51), we have $\langle v_t - z, \lambda v_t \rangle \geq \theta(v_t, z)$. This together with (C1) and (C4) implies that
\[
0 = \theta(v_t, v_t) \leq t \theta(v_t, y) + (1 - t) \theta(v_t, z) \leq t \theta(v_t, y) + (1 - t) \langle v_t - z, \lambda v_t \rangle \tag{52}
\]
and hence $0 \leq \theta(v_t, y) + (1 - t)\langle \lambda v_t, y - z \rangle$. Letting $t \to 0$, we have $0 \leq \theta(z, y) + \langle y - z, \lambda z \rangle$. This implies that $z \in EP(\theta, \lambda)$. Next, we prove $z \in VI(B, \phi, C)$. Set
\[
Rv = \begin{cases} 
Bv + N_C(v), & v \in C, \\
\emptyset, & v \notin C.
\end{cases} \tag{53}
\]
It is well known that $R$ is maximal $\phi$-monotone. Let $(v, w) \in G(R)$ (the graph of $R$). Since $w - Bv \in N_C(v)$ and $x_n \in C$, we have $\langle \phi(v) - \phi(x_n), w - Bv \rangle \geq 0$. Noting that $u_n = \text{proj}_C[\phi(x_n) - \lambda_n B x_n]$, we get
\[
\langle \phi(v) - u_n, u_n - [\phi(x_n) - \lambda_n B x_n] \rangle \geq 0. \tag{54}
\]
It follows that
\[
\langle \phi(v) - u_n, \frac{u_n - \phi(x_n)}{\lambda_n} + B x_n \rangle \geq 0. \tag{55}
\]
Then,
\[
\langle \phi(v) - \phi(x_n), w \rangle \\
\geq \langle \phi(v) - \phi(x_n), Bv \rangle \\
\geq \langle \phi(v) - \phi(x_n), Bv \rangle - \left< \phi(v) - u_n, \frac{u_n - \phi(x_n)}{\lambda_n} \right> \\
- \left< \phi(v) - u_n, Bx_n \right> \\
= \langle \phi(v) - \phi(x_n), Bv - Bx_n \rangle + \langle \phi(v) - \phi(x_n), Bx_n \rangle \\
- \left< \phi(v) - u_n, \frac{u_n - \phi(x_n)}{\lambda_n} \right> - \left< \phi(v) - u_n, Bx_n \right> \\
\geq - \left< \phi(v) - u_n, \frac{u_n - \phi(x_n)}{\lambda_n} \right> - \langle \phi(x_n) - u_n, Bx_n \rangle. \tag{56}
\]
Since $\|\phi(x_n) - u_n\| \to 0$ and $\phi(x_n) \to \phi(z)$, we deduce that $\langle \phi(v) - \phi(z), w \rangle \geq 0$ by taking $i \to \infty$ in (56). Thus, $z \in R^{-1}0$ by the maximal $\phi$-monotonicity of $R$. Hence, $z \in VI(B, \phi, C)$. Therefore, $z \in \Delta$. From (47), we obtain
\[
\limsup_{n \to \infty} \langle \zeta \phi (x^*) - \phi (x^*), u_n - \phi (x^*) \rangle \\
= \lim_{i \to \infty} \langle \zeta \phi (x^*) - \phi (x^*), \phi (x_n) - \phi (x^*) \rangle \tag{57}
= \langle \zeta \phi (x^*) - \phi (x^*), \phi (z) - \phi (x^*) \rangle \leq 0.
\]
Set $y_n = \text{proj}_C[\zeta \phi (x_n) + (1 - \zeta) z_n]$, for all $n \geq 0$. Then, we have
\[
\|y_n - \phi (x^*)\| \leq \langle \zeta \phi (x_n) + (1 - \zeta) z_n - \phi (x^*), y_n - \phi (x^*) \rangle \\
\leq \zeta \langle \phi (x_n) - \phi (x^*), y_n - \phi (x^*) \rangle + (1 - \zeta) \langle z_n - \phi (x^*), y_n - \phi (x^*) \rangle \tag{58}
+ \zeta \langle \phi (x^*) - \phi (x^*), y_n - \phi (x^*) \rangle + (1 - \zeta) \|z_n - \phi (x^*)\| \|y_n - \phi (x^*)\|
\leq \frac{\zeta}{\gamma} \|\phi (x_n) - \phi (x^*)\| \|y_n - \phi (x^*)\| \\
+ \zeta \langle \phi (x^*) - \phi (x^*), y_n - \phi (x^*) \rangle + (1 - \zeta) \|z_n - \phi (x^*)\| \|y_n - \phi (x^*)\|.
\]
\[ \leq \zeta_n \left( \frac{cL}{\nu} \right) \| \phi(x_n) - \phi(x^*) \| \| y_n - \phi(x^*) \| + \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle \\
+ (1 - \zeta_n) \| \phi(x_n) - \phi(x^*) \| \| y_n - \phi(x^*) \| \\
= \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \zeta_n \right] \| \phi(x_n) - \phi(x^*) \| \| y_n - \phi(x^*) \| \\
+ \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle \\
= \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \zeta_n \right] \| \phi(x_n) - \phi(x^*) \| \| y_n - \phi(x^*) \| \\
+ \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle . \tag{58} \]

It follows that

\[ \| y_n - \phi(x^*) \|^2 \leq \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \zeta_n \right] \| \phi(x_n) - \phi(x^*) \|^2 \\
+ 2 \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle. \tag{59} \]

Therefore,

\[ \| \phi(x_{n+1}) - \phi(x^*) \|^2 \leq \eta_n \| \phi(x_n) - \phi(x^*) \|^2 + (1 - \eta_n) \| y_n - \phi(x^*) \|^2 \\
+ (1 - \eta_n) \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \zeta_n \right] \| \phi(x_n) - \phi(x^*) \|^2 \\
+ 2 \left( 1 - \eta_n \right) \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle \\
= \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \right] \| \phi(x_n) - \phi(x^*) \|^2 \\
+ 2 \left( 1 - \eta_n \right) \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle \\
= \left[ 1 - \left( 1 - \frac{Lc}{\nu} \right) \right] \| \phi(x_n) - \phi(x^*) \|^2 \\
+ \zeta_n \langle \zeta \phi(x^*) - \phi(x^*), y_n - \phi(x^*) \rangle \\
\times \left( \frac{2}{1 - \zeta cL / \nu} \right) \| y_n - \phi(x^*) \|^2, \tag{60} \]

where \( \gamma_n = (1 - \zeta cL / \nu) \) and \( \gamma_n = (2(1 - \zeta cL / \nu)) \). It is easily seen that \( \sum \gamma_n = \infty \).

Since

\[ \| y_n - u_n \| \leq \| y_n - z_n \| + \| z_n - u_n \| \leq \zeta_n \| \phi(x_n) - z_n \| + \| z_n - u_n \| \rightarrow 0 \tag{61} \]

and by \( \limsup_{n \to \infty} \langle \zeta \phi(x^*) - \phi(x^*), u_n - \phi(x^*) \rangle \leq 0 \), we get \( \limsup_{n \to \infty} \zeta_n \leq 0 \). We can therefore apply Lemma 3 to conclude that \( \phi(x_n) \rightarrow \phi(x^*) \) and \( x_n \rightarrow x^* \). This completes the proof. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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