Research Article

Analytical Solutions for the Elastic Circular Rod Nonlinear Wave, Boussinesq, and Dispersive Long Wave Equations

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The solving processes of the homogeneous balance method, Jacobi elliptic function expansion method, fixed point method, and modified mapping method are introduced in this paper. By using four different methods, the exact solutions of nonlinear wave equation of a finite deformation elastic circular rod, Boussinesq equations and dispersive long wave equations are studied. In the discussion, the more physical specifications of these nonlinear equations, have been identified and the results indicated that these methods (especially the fixed point method) can be used to solve other similar nonlinear wave equations.

1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of sciences such as biology, chemistry, communication, and especially many branches of physics like condensed matter physics, field theory, fluid dynamics, plasma physics, and optics, and so forth. In this paper, we consider nonlinear wave equation of a finite deformation elastic circular rod, Boussinesq equations, and dispersive long wave equations with same physical behavior. The elastic circular rod is one important component in the structures, in which the dynamics of these components are governed by double nonlinearity and double dispersion wave equation [1]. The Boussinesq equation, which was first introduced in 1871, arises in several physical applications. The dynamics of shallow water waves, which are seen in various places like sea beaches, lakes, and rivers, are governed by the Boussinesq equation. In recent years there has been much interest in some variants of the Boussinesq systems [2–6]. These coupled Boussinesq equations [7] arise in shallow water waves for two-layered fluid flow. This situation occurs when there is an accidental oil spill from a ship which results in a layer of oil floating above the layer of water. The (2 + 1)-dimensional dispersive long wave equations [8, 9] were first derived by Boiti as a compatibility condition for a "weak"

Lax pair. A good understanding of the solutions for these equations is very helpful to coastal and civil engineers in applying the nonlinear water model to coastal harbor design.

In the field of nonlinear science to find exact solutions for a nonlinear system is one of the most fundamental and significant studies. The evaluation of exact solutions of nonlinear wave has complicated nonlinear defects; such equations are often very difficult to be solved. Although intensive investigations have made significant progress in recent years, many methods have been proposed to construct exact solutions, such as Weierstrass elliptic function method, the homogeneous balance method, sine-cosine method, the nonlinear transformation method, the hyperbolic tangent functions finite expansion, improved mapping approach, and further extended tanh method.

The homogeneous balance method is a powerful tool to find solitary wave solutions of nonlinear partial differential equations. Fan introduced the homogeneous balance method into the search for Bäcklund transformations and obtained more solutions [10].

The mapping method is a very effective direct method to construct exact solutions of nonlinear equations. Zhang et al. make use of the auxiliary equation and the expanded mapping methods to find the new exact periodic solutions for (2 + 1)-dimensional dispersive long wave equations [11].

The basic idea of the fixed point methods consists in finding an iteration function, which generates successive approximations to the solution [12].

Many periodic solutions have been recently expressed in terms of various Jacobi-elliptic functions for a wide class of nonlinear evolution equations, which have been obtained by means of Jacobi elliptic expansion method [13].

The longitudinal wave equation of a finite deformation elastic circular rod, Boussinesq equations, and dispersive long wave equations are nonlinear partial differential equations of different scientific field. We find that they have the same characteristics. In this paper, the analytical solutions of the differential equations for the elastic circular rod, Boussinesq equations, and dispersive long wave equations are solved by using homogeneous balance method, Jacobi elliptic function method, fixed point method, and modified mapping method. The more physical specifications of these nonlinear equations have been identified and the results indicated.

2. Three Types of Nonlinear Wave Equations

The longitudinal wave equation of a finite deformation elastic circular rod is [1]

$$
u_{tt} - c_0^2 u_{xx} = \left[\frac{3E}{2\rho} u^2 + \frac{E}{2\rho} u^3 + \frac{\nu^2 R^2}{2} \left(u_{tt} - c_1^2 u_{xx} \right) \right]_{xx},
$$
(1)

where $c_0 = \sqrt{E/\rho}$ is the longitudinal wave velocity for a linear elastic rod and $c_1 = \sqrt{\mu/\rho}$ is the shear wave velocity. ρ is the density of the material. ν is the Poisson ratio. R is the diameter of bar. E is Yong's modulus of material. μ is the elastic shear modulus of material. t is the time variable.

Make the traveling wave transformation

$$
u(x, t) = u(\xi), \qquad \xi = k(x - \lambda t).
$$
(2)

Substituting (2) into (1) and integrating it with respect to ξ twice, we can obtain

$$
k^2 u'' + \beta_1 u + \beta_2 u^2 + \beta_3 u^3 = N,
$$
(3)

where N is the second integral constant. The first integral constant is zero and

$$
\beta_1 = \frac{2\left(\lambda^2 - c_0^2\right)}{\nu^2 R^2 \left(\lambda^2 - c_1^2\right)}, \qquad \beta_2 = \frac{3c_0^2}{\nu^2 R^2 \left(\lambda^2 - c_1^2\right)},
$$

$$
\beta_3 = \frac{c_0^2}{\nu^2 R^2 \left(\lambda^2 - c_1^2\right)}.
$$
(4)

The variant Boussinesq equations were discussed in [7]. They are coupling wave equations which can be expressed as follows:

$$
H_t + (Hu)_x + u_{xxx} = 0,
$$

$$
u_t + H_x + uu_x = 0,
$$
(5)

where $u(x, t)$ is the velocity and $H(x, t)$ is the total depth. Make the traveling wave transformation

$$
H(x, t) = H(\xi), \qquad u(x, t) = u(\xi), \qquad \xi = k(x - \lambda t).
$$
(6)

Substituting (6) into (5) and integrating with respect to ξ, we have

$$
-\lambda H + Hu + k^2 u'' = a_1, \qquad H = \lambda u - \frac{1}{2} u^2 + b_1,
$$
(7)

where a_1, b_1 are integral constants.

Substituting the second formula of (7) into the first one and assuming $\beta_1 = b_1 - \lambda^2$, $\beta_2 = (3/2)\lambda$, $\beta_3 = -(1/2)$, $N = a_1 + \lambda b_1$, we can obtain (3). So the solutions of (7) are equivalent to that of (3) and the second formula of (7).

$(2+1)$-dimension dispersive long wave equations were discussed in [8, 9]. They can be written as follows:

$$
u_{ty} + h_{xx} + \frac{1}{2} \left(u^2 \right)_{xy} = 0,
$$

$$
h_t + \left(hu + u + u_{xy} \right)_x = 0.
$$
(8)

Suppose that the traveling wave solutions for (8) can be given in the following forms:

$$
u(x, y, t) = u(\xi), \qquad h(x, y, t) = h(\xi),
$$

$$
\xi = x + ny + \omega t.
$$
(9)

Substituting (9) into (8), we have

$$
nu'' + \left(1 - n\omega^2 \right) u - \frac{3}{2} n\omega u^2 - \frac{n}{2} u^3 = N_1,
$$

$$
h = -n\omega u - \frac{1}{2} n u^2,
$$
(10)

where N_1 is the second integral constant.

Obviously, the first formula of (10) and (3) have the same characteristics. On the other hand, the solutions of (10) and (7) are same with each other except for the coefficients. Therefore, in following parts, the solutions of (5) are presented.

3. Homogeneous Balance Method

In [14, 15] the homogeneous balance method is used to solve nonlinear wave equations. For example, the following equation was solved in [15]:

$$
u_t + 6uu_x + u_{xxx} = 0.
$$
(11)

The exact solution of the following corresponding equation was given by using the same method:

$$
u_t - Du_{xx} + \lambda \left(u^3 + \alpha u^2 + \beta u \right) = 0.
$$
(12)

To solve (5), the homogeneous balance method was improved in this paper.

Now, suppose that

$$u(x,t) = f'\omega_x + b, \qquad H(x,t) = g'\omega_x^2 + c, \qquad (13)$$

where $f(\omega)$, $g(\omega)$, and $\omega(x,t)$ are the undetermined functions and b, c are undetermined constants.

Substituting (13) into (5), the equations can be rewritten as follows:

$$\left(f''g' + f'g'' + f^{(4)}\right)\omega_x^4 + 6f'''\omega_x^2\omega_{2x}$$
$$+ f''\left(c\omega_x^2 + 3\omega_{2x}^2 + 4\omega_x\omega_{3x}\right) + g''\left(b\omega_x^3 + \omega_x^2\omega_t\right)$$
$$+ 3f'g'\omega_x^2\omega_{2x} + f'\left(c\omega_{2x} + \omega_{4x}\right)$$
$$+ 2g'\left(b\omega_x\omega_{2x} + \omega_x\omega_{xt}\right) = 0$$
$$\left(g'' + f'f''\right)\omega_x^3 + f''\left(\omega_x\omega_t + b\omega_x^2\right) + f'^2\omega_x\omega_{2x}$$
$$+ f'\left(\omega_{xt} + b\omega_{2x}\right) + 2g'\omega_x\omega_{2x} = 0, \qquad (14)$$

where $\omega_{2x} = \omega_{xx}$, $\omega_{3x} = \omega_{xxx}$, $\omega_{4x} = \omega_{xxxx}$. In (14), let

$$f''g' + f'g'' + f^{(4)} = 0, \qquad g'' + f'f'' = 0. \qquad (15)$$

Integrating the second formula of (15), we have

$$g' + \frac{1}{2}f'^2 = 0. \qquad (16)$$

Furthermore, we can obtain

$$g' = -\frac{1}{2}f'^2, \qquad g'' = -f'f''. \qquad (17)$$

Substituting (17) into the first formula of (15), we have

$$f^{(4)} - \frac{3}{2}f'^2f'' = 0. \qquad (18)$$

Integrating (18), we have

$$f''' - \frac{1}{2}f'^3 = 0. \qquad (19)$$

We know the solution of (19) is

$$f = 2\ln\omega. \qquad (20)$$

Obviously, we have

$$f'^2 = -2f'', \qquad f'f'' = -f''',$$
$$f'g' = -f''', \qquad f'^3 = 2f'''. \qquad (21)$$

Substituting (17) and (21) into (14), the equations can be rewritten as follows:

$$\omega_x^2\left(3\omega_{2x} + b\omega_x + \omega_t\right)f'''$$
$$+ \left(c\omega_x^2 + 3\omega_{2x}^2 + 4\omega_x\omega_{3x} + 2b\omega_x\omega_{2x} + 2\omega_x\omega_{xt}\right)f''$$
$$+ \left(c\omega_{2x} + \omega_{4x}\right)f' = 0,$$
$$\omega_x\left(\omega_t + b\omega_x\right)f'' + \left(\omega_{xt} + b\omega_{2x}\right)f' = 0. \qquad (22)$$

Letting all coefficients of f in (22) be zero, we have

$$3\omega_{2x} + b\omega_x + \omega_t = 0,$$
$$c\omega_x^2 + 3\omega_{2x}^2 + 4\omega_x\omega_{3x} + 2b\omega_x\omega_{2x} + 2\omega_x\omega_{xt} = 0,$$
$$c\omega_{2x} + \omega_{4x} = 0,$$
$$\omega_t + b\omega_x = 0, \qquad \omega_{xt} + b\omega_{2x} = 0. \qquad (23)$$

From first and fifth second formula of (23), it can be obtained that

$$\omega = \alpha(t)x + \beta(t), \qquad (24)$$

where $\alpha(t)$ is undetermined function and $\beta(t)$ is arbitrary function.

From second formula of (23), we have

$$\alpha(t) = e^{-(c/2)t+k}, \qquad (25)$$

where k is the integral constant.

Therefore, from (13), (20), (24), and (25), we know the solutions of (5) are

$$u_1(x,t) = \frac{2\alpha(t)}{\alpha(t)x + \beta(t)} + b,$$
$$H_1(x,t) = \frac{-2\alpha^2(t)}{\left[\alpha(t)x + \beta(t)\right]^2} + c. \qquad (26)$$

4. Jacobi Elliptic Function Method

In [16], the NLS equation and Zakharov equation were studied by using the Jacobi elliptic function method. In this paper, (1) and (5) are discussed by using this method.

4.1. Jacobi Elliptic Sine Function Method. Let

$$u = a_0 + a_1\operatorname{sn}\xi. \qquad (27)$$

We know

$$\frac{du}{d\xi} = a_1\operatorname{cn}\xi\operatorname{dn}\xi,$$
$$\frac{d^2u}{d\xi^2} = -\left(1 + m^2\right)a_1\operatorname{sn}\xi + 2m^2a_1\operatorname{sn}^3\xi,$$
$$u^2 = a_0^2 + 2a_0a_1\operatorname{sn}\xi + a_1^2\operatorname{sn}^2\xi,$$
$$u^3 = a_0^3 + 3a_0^2a_1\operatorname{sn}\xi + 3a_0a_1^2\operatorname{sn}^2\xi + a_1^3\operatorname{sn}^3\xi, \qquad (28)$$

where $\operatorname{cn}\xi$ and $\operatorname{dn}\xi$ are the Jacobi elliptic cosine function and the third type Jacobi elliptic function, respectively, and m $(0 < m < 1)$ is module.

Substituting (27) into (3), we have

$$\beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3 - N$$
$$+ a_1\left[-k^2\left(1 + m^2\right) + \beta_1 + 2\beta_2 a_0 + 3\beta_3 a_0^2\right]\operatorname{sn}\xi$$
$$+ a_1^2\left(\beta_2 + 3\beta_3 a_0\right)\operatorname{sn}^2\xi + a_1\left(2k^2m^2 + \beta_3 a_1^2\right)\operatorname{sn}^3\xi = 0.$$

$$(29)$$

Let the coefficients of all derivatives of $\operatorname{sn}\xi$ be zero and we have

$$a_0 = \frac{-\beta_2}{3\beta_3}, \qquad a_1 = \pm\frac{\sqrt{2}km}{\sqrt{-\beta_3}},$$

$$k^2 = \frac{1}{1+m^2}\left(\beta_1 + 2\beta_2 a_0 + 3\beta_3 a_0^2\right), \tag{30}$$

$$N = \beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3.$$

Therefore the solution of (3) is

$$u_2(x,t) = a_0 + a_1 \operatorname{sn} k(x - \lambda t), \tag{31}$$

where a_0, a_1, and k are denoted by (30), and λ is arbitrary constant.

When $m \to 1$, $\operatorname{sn}\xi \to \tanh\xi$,

$$u_2(x,t) = a_0 + a_1 \tanh k(x - \lambda t). \tag{32}$$

Thus, the solutions of (5) are (32) and the following formula:

$$H_2(x,t) = b_1 + \lambda u_2 - \frac{1}{2}u_2^2. \tag{33}$$

Assuming $\lambda = -1$, $b_1 = 1.5$, $\beta_1 = 0.5$, $\beta_2 = -1.5$, and $\beta_3 = -0.5$, it can be obtained that $a_0 = -1$, $a_1 = 2$, and $k = 1$. The solutions of (5) are $u(x,t) = -1 + 2\tanh(x+t)$ and $H(x,t) = 2 - 2\tanh^2(x+t)$. The solitary wave and behavior of the solutions $u(x,t) = -1 + 2\tanh(x+t)$ and $H(x,t) = 2 - 2\tanh^2(x+t)$ are shown in Figures 1 and 2, respectively, for $0 \le t \le 1$ and $-10 \le x \le 10$. The waveform is similar to the result in [17].

4.2. Jacobi Elliptic Cosine Function Method. Let

$$u = a_0 + a_1 \operatorname{cn}\xi. \tag{34}$$

Thus

$$u' = -a_1 \operatorname{sn}\xi \operatorname{dn}\xi, \qquad u'' = \left(2m^2 - 1\right)a_1 \operatorname{cn}\xi - 2m^2 \operatorname{cn}^3\xi,$$

$$u^2 = a_0^2 + 2a_0 a_1 \operatorname{cn}\xi + a_1^2 \operatorname{cn}^2\xi,$$

$$u^3 = a_0^3 + 3a_0^2 a_1 \operatorname{cn}\xi + 3a_0 a_1^2 \operatorname{cn}^2\xi + a_1^3 \operatorname{cn}^3\xi. \tag{35}$$

Substituting (35) into (3), we can obtain

$$\begin{aligned} &\beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3 - N \\ &+ a_1 \left[k^2\left(2m^2 - 1\right) + \beta_1 + 2\beta_2 a_0 + 3\beta_3 a_0^2\right] \operatorname{cn}\xi \\ &+ a_1^2 \left(\beta_2 + 3\beta_3 a_0\right) \operatorname{cn}^2\xi \\ &+ a_1 \left(-2k^2 m^2 + \beta_3 a_1^2\right) \operatorname{cn}^3\xi = 0. \end{aligned} \tag{36}$$
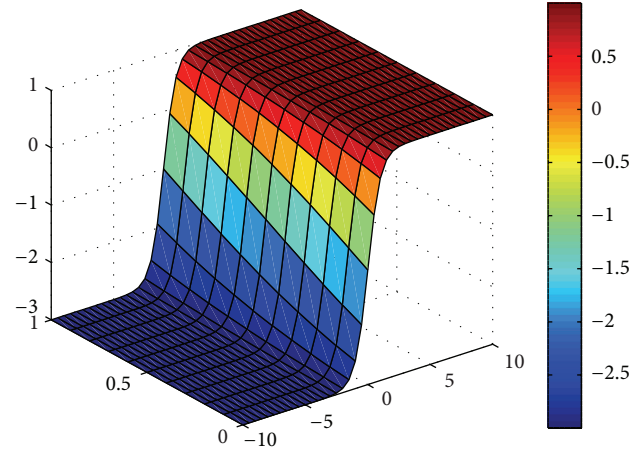

FIGURE 1: The solitary wave and behavior of the solutions $u(x,t) = -1 + 2\tanh(x+t)$ for $0 \le t \le 1$ and $-10 \le x \le 10$.
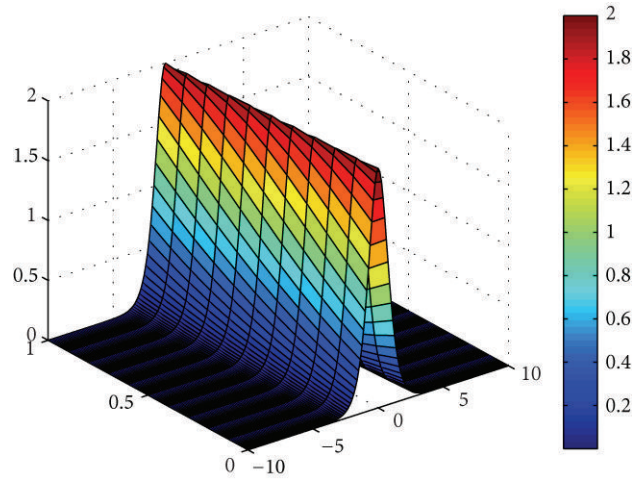

FIGURE 2: The solitary wave and behavior of the solutions $H(x,t) = 2 - 2\tanh^2(x+t)$ for $0 \le t \le 1$ and $-10 \le x \le 10$.

Let the coefficients of all derivatives of $\operatorname{cn}\xi$ be zero and we have

$$a_0 = -\frac{\beta_2}{3\beta_3}, \qquad a_1 = \pm km\sqrt{\frac{2}{\beta_3}},$$

$$k^2 = \frac{\beta_2^2 - 3\beta_1\beta_3}{3\beta_3\left(2m^2 - 1\right)}, \tag{37}$$

$$N = \beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3.$$

Therefore, the solution of (3) is

$$u_3(x,t) = a_0 + a_1 \operatorname{cn} k(x - \lambda t), \tag{38}$$

where a_0, a_1 and k are denoted by (37), and λ is arbitrary constant.

The solutions of (7) are (38) and the following formula:

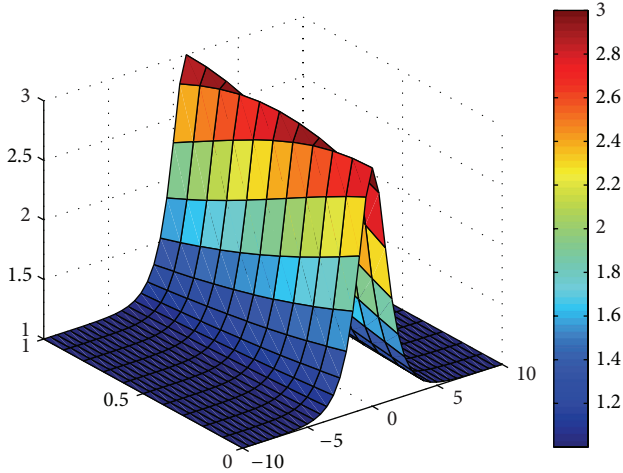$$H_3(x,t) = b_1 + \lambda u_3(x,t) - \frac{1}{2}u_3^2(x,t). \tag{39}$$

FIGURE 3: The solitary wave and behavior of the solutions $u(x, t) = 1 + 2 \operatorname{sech}(x - t)$ for $0 \leqslant t \leqslant 1$ and $-10 \leqslant x \leqslant 10$.

Taking $m \to 1$, $\operatorname{cn}\xi \to \operatorname{sech}\xi$, (38) can be rewritten as follows:

$$u_3(x, t) = a_0 + a_1 \operatorname{sech}k(x - \lambda t). \tag{40}$$

Assuming $\lambda = 1$, $\beta_1 = 0.5$, $\beta_2 = -1.5$, and $\beta_3 = 0.5$, it can be obtained that $a_0 = 1$, $a_1 = 2$, and $k = 1$. The solution of (3) is $u(x, t) = 1 + 2 \operatorname{sech}(x - t)$. The solitary wave and behavior of the solutions are shown in Figure 3, respectively, for $0 \leq t \leq 1$ and $-10 \leq x \leq 10$.

4.3. Third Kind of Jacobi Elliptic Function Method. Let

$$u = a_0 + a_1 \operatorname{dn}\xi. \tag{41}$$

Thus

$$u' = -m^2 a_1 \operatorname{sn}\xi, \qquad u'' = \left(2 - m^2\right) a_1 \operatorname{dn}\xi - 2a_1 \operatorname{dn}^3\xi,$$

$$u^3 = a_0^3 + 3a_0^2 a_1 \operatorname{dn}\xi + 3a_0 a_1^2 \operatorname{dn}^2\xi + a_1^3 \operatorname{dn}^3\xi. \tag{42}$$

Substituting them into (3), we can obtain

$$\beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3$$

$$- N + a_1 \left[k^2 \left(2 - m^2\right) + \beta_1 + 2\beta_2 a_0 + 3\beta_3 a_0^2\right] \operatorname{dn}\xi$$

$$+ a_1^2 \left(\beta_2 + 3\beta_3 a_0\right) \operatorname{dn}^2\xi + a_1 \left(-2k^2 + a_1^2 \beta_3\right) \operatorname{dn}^3\xi = 0. \tag{43}$$

Letting the coefficients of all derivatives of $\operatorname{dn}\xi$ be zero, we have

$$a_0 = \frac{-\beta_2}{3\beta_3}, \qquad a_1 = \pm k \sqrt{\frac{2}{\beta_3}},$$

$$k^2 = \frac{1}{m^2 - 2} \left(\frac{3\beta_1 \beta_3 - \beta_2^2}{3\beta_3}\right), \tag{44}$$

$$N = \beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3.$$

Therefore the solution of (3) is

$$u_4(x, t) = a_0 + a_1 \operatorname{dn}k(x - \lambda t), \tag{45}$$

where a_0, a_1, and k are denoted by (44), and λ is arbitrary constant.

Thus the solutions of (7) are (45) and the following formula:

$$H_4(x, t) = b_1 + \lambda u_4(x, t) - \frac{1}{2}u_4^2(x, t). \tag{46}$$

Letting $m \to 1$, then (45) equals (38).

4.4. Jacobi Elliptic Function $\operatorname{cs}\xi$ Method. Let

$$u = a_0 + a_1 \operatorname{cs}\xi, \qquad \operatorname{cs}\xi \equiv \frac{\operatorname{cn}\xi}{\operatorname{sn}\xi}. \tag{47}$$

Then

$$u' = -a_1 \left(1 + \operatorname{cs}^2\xi\right) \operatorname{dn}\xi,$$

$$u'' = \left(2 - m^2\right) a_1 \operatorname{cs}\xi + 2a_1 \operatorname{cs}^3\xi. \tag{48}$$

Substituting (47) and (48) into (3), we can obtain

$$D = \beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3$$

$$- N + a_1 \left[k^2 \left(2 - m^2\right) + \beta_1 + 2\beta_2 a_0 + 3\beta_3 a_0^2\right] \operatorname{cs}\xi$$

$$+ a_1^2 \left(\beta_2 + 3\beta_3 a_0\right) \operatorname{cs}^2\xi + a_1 \left(2k^2 + a_1^2 \beta_3\right) \operatorname{cs}^3\xi. \tag{49}$$

Let the coefficients in above formula be zero, and we have

$$a_0 = \frac{-\beta_2}{3\beta_3}, \qquad a_1 = \pm k^2 \sqrt{\frac{2}{-\beta_3}},$$

$$k^2 = \frac{1}{m^2 - 2} \left(\frac{3\beta_1 \beta_3 - \beta_2^2}{3\beta_3}\right), \tag{50}$$

$$N = \beta_1 a_0 + \beta_2 a_0^2 + \beta_3 a_0^3.$$

According to (47), the solution of (3) is

$$u_5(x, t) = a_0 + a_1 \operatorname{cs}k(x - \lambda t), \tag{51}$$

where a_0, a_1, and k are denoted by (50), and λ is arbitrary constant.

Therefore, the solutions of (7) are (51) and the following formula:

$$H_5 = b_1 + \lambda u_5(x, t) - \frac{1}{2}u_5^2(x, t). \tag{52}$$

When $m \to 1$,

$$u_5(x, t) = a_0 + a_1 \operatorname{csch}k(x - \lambda t). \tag{53}$$

Assuming $\beta_1 = (1 - n\omega^2)/n$, $\beta_2 = -(3/2)\omega$, $\beta_3 = -(1/2)$, $N = -(N_1/n)$, and $k = 1$, we can obtain (3). Choosing
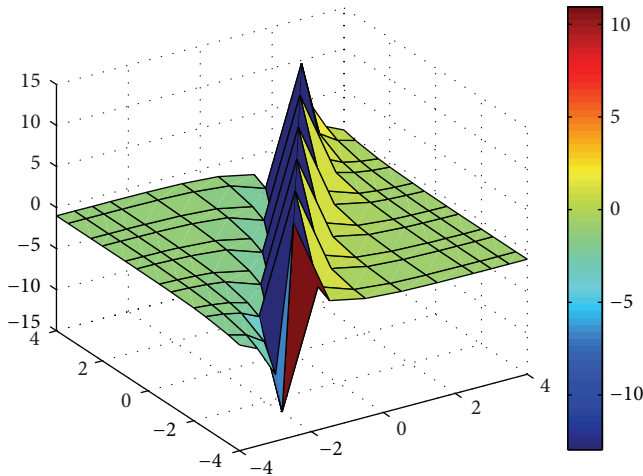
FIGURE 4: The solitary wave and behavior of the solutions $u(x, y, t) = -1 + 2 \operatorname{csch}(x - 2/3y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.
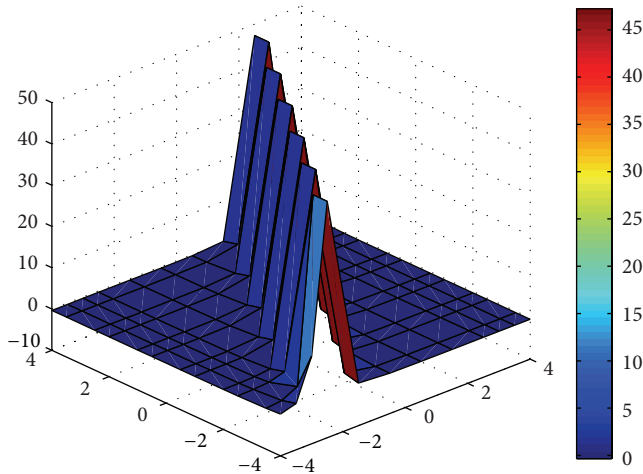

FIGURE 5: The solitary wave and behavior of the solutions $h(x, y, t) = -1/3 + 4/3 \operatorname{csch}^2(x - 2/3y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.

$n = -(2/3)$, $\omega = 1$, then $\beta_1 = -(5/2)$, $\beta_2 = -(3/2)$. Substituting β_1, β_2, β_3 into (50), it can be obtained that $a_0 = -1$, $a_1 = 2$, and $k = 1$. The solutions of (8) are $u(x, y, t) = -1 + 2\operatorname{csch}(x - (2/3)y + t)$ and $h(x, y, t) = -(1/3) + (4/3)\operatorname{csch}^2(x - (2/3)y + t)$. The solitary wave and behavior of the solutions $u(x, t) = -1 + 2\operatorname{csch}(x - (2/3)y + t)$ and $h(x, y, t) = -(1/3) + (4/3)\operatorname{csch}^2(x - (2/3)y + t)$ are shown in Figures 4 and 5, respectively, for $t = 0$, $-4 \leq x \leq 4$, $-4 \leq y \leq 4$, and $x - (2/3)y \neq 0$.

5. Fixed Point Method

In general, fixed point theory can be divided into two types. The first type is only used to discuss the existence of solution. The second type is used not only to discuss the existence and uniqueness of solution but also to search the fixed point. We are more interested in the second type.

Lemma 1 (see [18–20]). *Suppose E is a real Banach space which has normal cone. Consider $u_0, v_0 \in E$, $u_0 < v_0$. The operator $A : [u_0, v_0] \times [u_0, v_0] \to E$ is mixed monotone and satisfies*

(1)

$$u_0 \leq A(u_0, v_0), \qquad A(v_0, u_0) \leq v_0 \qquad (54)$$

(2) *for all u, v ($u_0 \leq u \leq v \leq v_0$), $\exists$ constant $\alpha \in (0, 1)$, satisfies*

$$\|A(v, u) - A(u, v)\| \leq \alpha \|v - u\|. \qquad (55)$$

Then there exists a unique $\overline{u}$ ($\overline{u} \in [u_0, v_0]$) which satisfies

$$A(\overline{u}, \overline{u}) = \overline{u}, \qquad (56)$$

where $\overline{u}$ is called the fixed point of A.

In $[u_0, v_0]$ for all w_0, letting $w_n = A(w_{n-1}, w_{n-1})$, $n = 1, 2, \ldots$, we have

$$\overline{u} = \lim_{n \to \infty} w_n. \qquad (57)$$

In this paper, E is taken as a continuous function space $C[0, 1]$ which is defined in the closed interval $[0, 1]$. Introducing equivalent norm, it is easy to know that $C[0, 1]$ is real Banach space which has normal cone. Let

$$u_0 = 0, \qquad v_0 = 1. \qquad (58)$$

Integrating (3) with respect to ξ, we can obtain

$$\frac{du}{d\xi} = -k^{-2} \int_0^\xi \left[\beta_1 u(s) + \beta_2 u^2(s) + \beta_3 u^3(s) \right] d\xi + N\xi + C_1, \qquad (59)$$

where C_1 is integral constant.

Integrating the above formula with respect to ξ and using the formula in [12], we have

$$u = -k^{-2} \int_0^\xi (\xi - s) \left[\beta_1 u(s) + \beta_2 u^2(s) + \beta_3 u^3(s) \right] ds + \frac{1}{2} N\xi^2 + C_1\xi + C_2, \qquad (60)$$

where C_2 is integral constant.

If $\beta_2 > 0$, $\beta_3 > 0$ (i.e., $\lambda^2 > c_1$), and $\beta_1 < 0$ (i.e., $\lambda^2 < c_0^2$), suppose

$$A(u, v) = -k^{-2} \int_0^\xi (\xi - s) \left[\beta_1 u(s) + \beta_2 v^2(s) + \beta_3 v^3(s) \right] ds + \frac{1}{2} N\xi^2 + C_1\xi + C_2. \qquad (61)$$

Theorem 2. *In (61), supposing $\beta_2 > 0$, $\beta_3 > 0$, $\beta_1 < 0$, $C_1 \geq 0$, $C_2 \geq 0$, and $k^{-2} \leq N/(\beta_2 + \beta_3) \leq 2(1 + C_1 + C_2)/(\beta_2 + \beta_3 - \beta_1)$, there exists unique fixed point $\overline{u} \in [u_0, v_0]$ for the operator A which is defined by (61). Then (57) is true.*

Proof. Obviously, $A : [u_0, v_0] \times [u_0, v_0] \to C[0, 1]$ is mixed monotone. Consider

$$A(u_0, v_0) = A(0, 1)$$

$$= \frac{1}{2}\left[N - k^{-2}(\beta_2 + \beta_3)\right]\xi^2 + C_1\xi + C_2 \geq 0 = u_0,$$

$$A(v_0, u_0) = \frac{1}{2}\left(N - \beta_1 k^{-2}\right)\xi^2 + C_1\xi + C_2$$

$$\leq \frac{1}{2}N\left(1 - \frac{\beta_1}{\beta_2 + \beta_3}\right) + C_1 + C_2 \leq 1 = v_0. \tag{62}$$

Thus A satisfies the term (1) of Lemma 1.

For all $u, v \in [u_0, v_0], u \leq v$,

$$A(v, u) - A(u, v)$$

$$= -k^{-2}\int_0^\xi (\xi - s)$$

$$\times \left[\beta_1 v(s) + \beta_2 u^2(s) + \beta_2 u^3(s)\right.$$

$$\left. -\beta_1 u(s) - \beta_2 v^2(s) - \beta_3 v^3(s)\right] ds$$

$$= k^{-2}\int_0^\xi (\xi - s) e^{Ms} e^{-Ms} [v(s) - u(s)]$$

$$\times \left\{-\beta_1 + \beta_2 [v(s) + u(s)]\right.$$

$$\left. +\beta_3 \left[v^2(s) + u(s)v(s) + u^2(s)\right]\right\} ds$$

$$\leq (-\beta_1 + 2\beta_2 + 3\beta_3) k^{-2} \|v - u\| \int_0^\xi e^{MS} ds$$

$$\leq \frac{-\beta_1 + 2\beta_2 + 3\beta_3}{k^2 M} \|v - u\| e^{M\xi}. \tag{63}$$

Then, we have

$$\|A(v, u) - A(u, v)\| \leq \alpha, \tag{64}$$

where $\alpha = (-\beta_1 + 2\beta_2 + 3\beta_3)/(k^2 M)$, and when M is large enough, $\alpha \in (0, 1)$.

Therefore, operator A which is defined by (61) satisfies the term (2) of Lemma 1. The proof is completed. □

Obviously, the fixed point of the operator defined by (61) is equivalent to the solution of (3). We denote

$$u_6(x, t) = \bar{u}(\xi). \tag{65}$$

By using the classical iteration algorithms, such as Mann iteration method, Ishikawa iteration method, and Noor iteration method, the solutions can be obtained. Then, the traveling wave solutions of (5) are (65) and the following formula:

$$H_6(x, t) = b_1 + \lambda u_6(x, t) - \frac{1}{2}u_6^2(x, t). \tag{66}$$

6. Modified Mapping Method

The modified mapping method was introduced and the nonlinear evolution equation was solved in [21, 22]. In [22], several hundreds of Jacobi elliptic function expansion solutions were obtained. We also use this method in this paper.

Let

$$u = A_0 + A_1 f + B_1 f^{-1}, \tag{67}$$

where $f'' = pf + qf^3 \Leftrightarrow f'^2 = r + pf^2 + (1/2)qf^4$. r, p, and q are constants.

Thus

$$u' = A_1 f' - B_1 f^{-2} f',$$

$$u'' = A_1 pf + A_1 qf^3 + B_1 pf^{-1} + 2B_1 rf^{-3},$$

$$u^2 = A_0^2 + 2A_1 B_1 + 2A_0 A_1 f + 2A_0 B_1 f^{-1} + A_1^2 f^2 + B_1^2 f^{-2},$$

$$u^3 = A_0^3 + 6A_0 A_1 B_1 + 3A_1\left(A_0^2 + A_1 B_1\right)f$$

$$+ 3A_0 A_1^2 f^2 + A_1^3 f^3 + 3B_1\left(A_0^2 + A_1 B_1\right)f^{-1}$$

$$+ 3A_0 B_1^2 f^{-2} + B_1^3 f^{-3}. \tag{68}$$

Substituting (68) into (3), we have

$$\beta_1 A_0 + \beta_2\left(A_0^2 + 2A_1 B_1\right) + \beta_3\left(A_0^3 + 6A_0 A_1 B_1\right) - N$$

$$+ A_1\left[k^2 p + \beta_1 + \beta_2 A_0 + 3\beta_3\left(A_0^2 + A_1 B_1\right)\right]f$$

$$+ A_1^2\left(\beta_2 + 3\beta_3 A_0\right)f^2 + A_1\left(k^2 q + \beta_3 A_1^2\right)f^3$$

$$+ B_1\left[k^2 p + \beta_1 + 2\beta_2 A_0 + 3\beta_3\left(A_0^2 + A_1 B_1\right)\right]f^{-1}$$

$$+ B_1^2\left(\beta_2 + 3\beta_3 A_0\right)f^{-2} + B_1\left(2k^2 r + \beta_3 B_1^2\right)f^{-3} = 0. \tag{69}$$

Notice that we get the same algebraic equations when the coefficients of f, f^{-1} and f^2, f^{-2} in (69) are zero. Therefore there are only five algebra equations. Solve them and we have

$$A_0 = \frac{-\beta_2}{3\beta_3}, \qquad A_1 = \pm k\sqrt{\frac{-q}{\beta_3}}, \qquad B_1 = \pm k\sqrt{\frac{-2r}{\beta_3}},$$

$$k = \pm\sqrt{\frac{\beta_2^2 - 3\beta_1\beta_3 - 3k^2\beta_3\sqrt{2qr}}{3\beta_3 p}}. \tag{70}$$

In order to make A_1, B_1 be real number, q and r should be different signs with β_3.

(i) Choosing $r = 1$, $p = -2$, and $q = 2$, we have $f = \tanh\xi$. Then the solitary wave solutions of (3) are

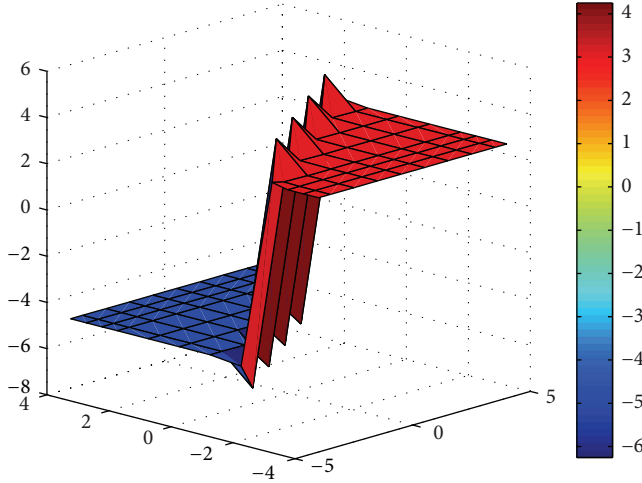$$u_7(x, t) = A_0 + A_1 \tanh\xi + B_1\tanh^{-1}\xi. \tag{71}$$

FIGURE 6: The solitary wave and behavior of the solutions $u(x, y, t) = -1 + 2\tanh(x - 2y + t) + 2\tanh^{-1}(x - 2y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.

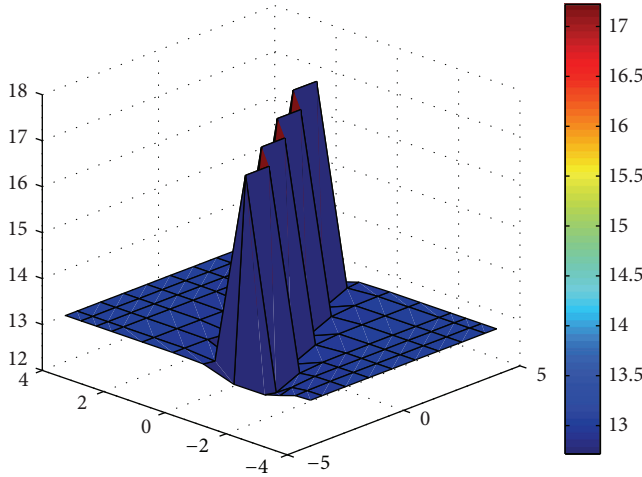

FIGURE 7: The solitary wave and behavior of the solutions $h(x, y, t) = 7 + 4\tanh^2(x - 2y + t) + 4\tanh^{-2}(x - 2y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.

Assuming $n = -2$, $\omega = 1$, $k = 1$, and $\beta_3 = -(1/2)$, it can be obtained that $\beta_1 = -(3/2)$, $\beta_2 = -(3/2)$, $A_0 = -1$, $A_1 = 2$, and $B_1 = 2$. The solutions of (8) are $u(x, y, t) = -1 + 2\tanh(x - 2y + t) + 2\tanh^{-1}(x - 2y + t)$ and $h(x, y, t) = 7 + 4\tanh^2(x - 2y + t) + 4\tanh^{-2}(x - 2y + t)$. The solitary wave and behavior of them are shown in Figures 6 and 7, respectively, for $t = 0$, $-4 \leq x \leq 4$, $-4 \leq y \leq 4$, and $x - 2y \neq 0$.

(ii) Choosing $r = 1$, $p = -(1 + m^2)$, and $q = 2m^2$, thus

$$f = \mathrm{sn}\xi, \qquad (72)$$

where m $(0 < m < 1)$ is the module of Jacobi elliptic function.

Therefore the expansion solutions of (3) are

$$u_8(x, t) = A_0 + A_1\mathrm{sn}\xi + B_1\mathrm{sn}^{-1}\xi. \qquad (73)$$
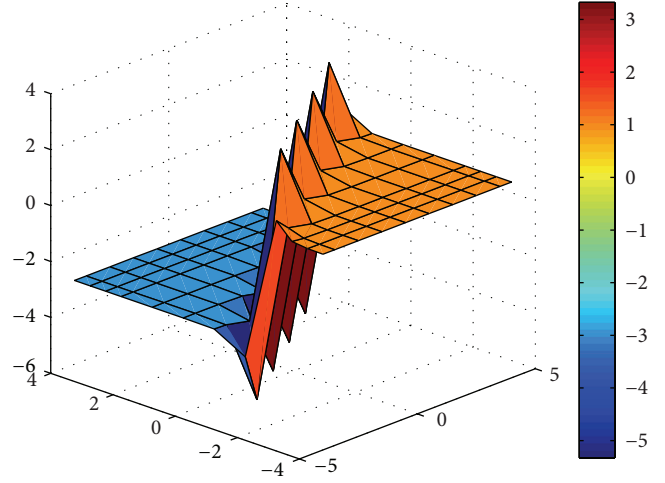
Letting $m \to 1$, then (73) equals (71).

FIGURE 8: The solitary wave and behavior of the solutions $u(x, y, t) = -1 + \tanh(x - 2y + t)/(1 + \mathrm{sech}(x - 2y + t)) + (1 + \mathrm{sech}(x - 2y + t))/\tanh(x - 2y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.

(iii) Choosing $r = 1/4$, $p = (m^2 - 2)/2$, and $q = m^2/2$, we have

$$f = \frac{\mathrm{sn}\xi}{(1 + \mathrm{dn}\xi)}. \qquad (74)$$

Therefore the solutions of (3) are

$$u_9(x, t) = A_0 + \frac{A_1\mathrm{sn}\xi}{1 + \mathrm{dn}\xi} + \frac{B_1(1 + \mathrm{dn}\xi)}{\mathrm{sn}\xi}, \qquad (75)$$

where A_0, A_1, B_1 of u_i $(i = 7, 8, 9)$ are denoted by (70).

Taking $m \to 1$, (75) can be rewritten as follows:

$$u_9(x, t) = A_0 + \frac{A_1\tanh\xi}{1 + \mathrm{sech}\xi} + \frac{B_1(1 + \mathrm{sech}\xi)}{\tanh\xi}. \qquad (76)$$

Assuming $n = -2$, $\omega = 1$, $k = 1$, and $\beta_3 = -(1/2)$, it can be obtained that $\beta_1 = -(3/2)$, $\beta_2 = -(3/2)$, $A_0 = -1$, $A_1 = 1$, and $B_1 = 1$. The solutions of (8) are $u(x, y, t) = -1 + \tanh(x - 2y + t)/(1 + \mathrm{sech}(x - 2y + t)) + (1 + \mathrm{sech}(x - 2y + t))/\tanh(x - 2y + t)$ and $h(x, y, t) = 1 + [\tanh(x - 2y + t)/(1 + \mathrm{sech}(x - 2y + t))]^2 + [(1 + \mathrm{sech}(x - 2y + t))/\tanh(x - 2y + t)]^2$. The solitary wave and behavior of them are shown in Figures 8 and 9, respectively, for $t = 0$, $-4 \leq x \leq 4$, $-4 \leq y \leq 4$, and $x - 2y \neq 0$.

7. Summary and Conclusions

In summary, by using different methods, nonlinear wave equation of a finite deformation elastic circular rod, Boussinesq equations, and dispersive long wave equations are solved in this paper, and several analytical solutions are obtained. Kink soliton solutions are obtained when we use the homogenous balance method. The Jacobi elliptic function method is utilized and periodic solutions are obtained. When $m \to 1$, the solutions reduce to solitary solution. Their wave forms are the bell type, kink type, exotic type, and
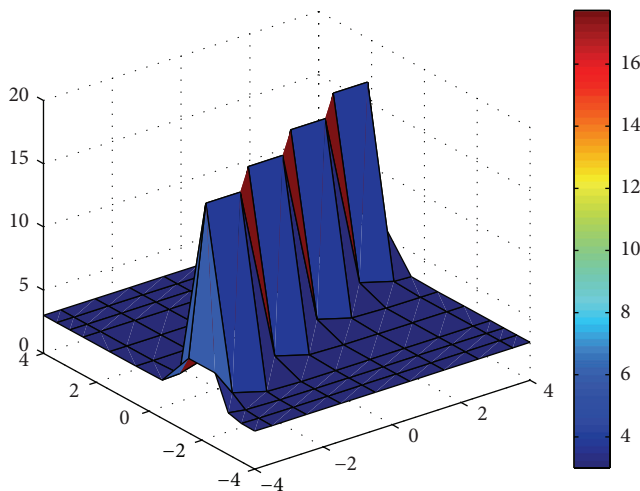
Figure 9: The solitary wave and behavior of the solutions $h(x, y, t) = 1 + \tanh^2(x - 2y + t)/(1 + \text{sech}(x - 2y + t))^2 + (1 + \text{sech}(x - 2y + t))^2/\tanh^2(x - 2y + t)$ for $t = 0$, $-4 \leqslant x \leqslant 4$, and $-4 \leqslant y \leqslant 4$.

peakon type. By using the modified mapping method, peakon solutions of (3), (5), (8) are obtained. On the other hand, after proving the existence and uniqueness of fixed point of operator A, we can use the classical iteration algorithms to get the solutions. The method has wide application and can be used to solve other nonlinear wave equations. Fixed point method may be significant and important for the explanation of some special physical problems whose analytical solution cannot be obtained. The results indicate the following. First, the natural phenomena and the physical properties of the equations are abundant, which should be further discovered. Second, there are many kinds of methods for solving these equations. Third, the arbitrary constant can be determined by the initial and boundary conditions and we can obtain the exact solutions for certain engineering or scientific problems. It is shown that the methods, proposed in this paper for three types of nonlinear wave equations, are feasible for determining exact solutions of other nonlinear equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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