Richardson Cascadic Multigrid Method for 2D Poisson Equation
Based on a Fourth Order Compact Scheme

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1. Introduction

Poisson equation is a partial differential equation (PDE) with broad applications in theoretical physics, mechanical engineering and other fields, such as groundwater flow [1, 2], fluid pressure prediction [3], electromagnetics [4], semiconductor modeling [5], and electrical power network modeling [6].

We consider the following two-dimensional (2D) Poisson equation:

\[-\frac{\partial^2 u(x, y)}{\partial x^2} - \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y), \quad \text{in } \Omega, \]

\[u(x, y) = 0, \quad \text{on } \partial \Omega, \]

where \(\Omega \subseteq R^2\) is a rectangular domain or union of rectangular domains with Dirichlet boundary \(\partial \Omega\). The solution \(u(x, y)\) and the forcing function \(f(x, y)\) are assumed to be sufficiently smooth.

Multigrid (MG) method is one of the most effective algorithms to solve the large scale problem. In 1996, cascadic multigrid (CMG) method proposed by Bornemann and Deuflhard [7] and then analyzed by Shi et al. (see [8–11]) and Shaidurov (see [12]). In the recent years, there have been several theoretical analyses and the applications of these methods for the plate bending problems (see [13]), the parabolic problems (see [10]), the nonlinear problems (see [14, 15]), and the Stokes problems (see [16]). In order to improve the efficiency of the CMG, some new extrapolation formulas and extrapolation cascadic multigrid (EXCMG) methods are proposed by Chen et al. (see [17–20]). These new methods can provide a better initial value for smoothing operator on the refined grid level to accelerate their convergence rate.

Based on the Richardson extrapolation technique, Wang and Zhang [21] presented a multiscale multigrid algorithm. Numerical experiments show that the new method is of higher accuracy solution and higher efficiency.

In this paper, in order to develop a more efficient CMG method, we use the Richardson extrapolation technique presented in [21] and a new extrapolation formula; a new Richardson extrapolation cascadic multigrid (RCMG) method for 2D Poisson equation is proposed.

The sections are arranged as follows: the fourth order compact difference scheme and Richardson extrapolation technique are given in Section 2. Chen’s new extrapolation
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formula and EXCMG method are introduced in Section 3. In Section 4, we present the RCMG method. In Section 5, the numerical experiments show the effectiveness of the new method.

2. Fourth Order Compact Difference Scheme and Richardson Extrapolation Technique

For convenience, we consider the rectangular domain \( \Omega = [0, L_x] \times [0, L_y] \). We discretize \( \Omega \) with uniform mesh sizes \( h_x = L_x/N_x \) and \( h_y = L_y/N_y \) in the \( x \) and \( y \) coordinate directions. The mesh points are \((x_i, y_j)\) with \( x_i = i h_x \) and \( y_j = j h_y \), and \( 0 \leq i \leq N_x, 0 \leq j \leq N_y \). Let’s denote the mesh aspect ratio \( \gamma = h_x/h_y \), and \( u_{i,j} \) be the solution at the grid point \((x_i, y_j)\), we can rewrite the fourth order compact difference scheme of (1) into the following form [22]:

\[
a u_{i,j} + b \left(u_{i+1,j} + u_{i-1,j} \right) + d \left(u_{i+1,j-1} + u_{i-1,j-1} + u_{i,j-1} + u_{i,j-1} \right)
+ \frac{h_x^2}{2} \left(8 f_{i,j} + f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} \right).
\]

(2)

The coefficients in (2) are

\[
a = -10 \left(1 + \gamma^2 \right), \quad b = 5 - \gamma^2,
\]

\[
c = 5 \gamma^2 - 1, \quad d = \frac{1 + \gamma^2}{2}.
\]

(3)

If the domain \( \Omega \) is subdivided into a sequence of grids \( Z_{h_l} \) (or \( Z_{2h_l} \)), \( l = 0, 1, 2, \ldots, L \) with step length \( h_l = h_l/2^l = h_{l,x} = h_{l,y} \) (namely, \( \gamma = 1 \)), by using the fourth order compact difference scheme (see (2)), a series of linear equations of the model problem (1) are given as follows

\[A^l \mathbf{u}^l = \mathbf{f}^l, \quad l = 0, 1, 2, \ldots, L.\]

(4)

Assume the fourth order accurate solutions \( u_{i,j}^{2h_l} \) and \( u_{i,j}^{h_l} \) on the \( Z_{2h_l} \) grid and the \( Z_{h_l} \) grid are given, respectively (Figure 1). In 2009, Wang and Zhang [21] applied the Richardson extrapolation (where \( p = 4 \))

\[
\overline{u}_{i,j}^{2h_l} = \frac{\left(2^p u_{i,j}^{h_l} - u_{i,j}^{2h_l}\right)}{2^p - 1} = \frac{\left(16 u_{i,j}^{2h_l} - u_{i,j}^{2h_l}\right)}{15}
\]

(5)

to get a sixth order accurate solution \( \overline{u}_{i,j}^{2h_l} \) on \( Z_{2h_l} \).

The above extrapolation operator is rewritten as the following iterative operator RET.

**Algorithm 1.** Consider \( \overline{u}_{i,j}^{h_{k+1}} \leftarrow \text{RET}(\overline{u}_{i,j}^{h_k}, \overline{u}_{i,j}^{2h_k}, \varepsilon, k_{\max}) \).

**Step 1.** Set \( \overline{u}_{i,j}^{h_0} := \overline{u}_{i,j}^{h}, k := 0 \).

**Step 2.** Update every (even, even) grid point on \( Z_{h_k} \) by Richardson extrapolation formula (see (5)); then use direct interpolation to get \( \overline{u}_{i,j}^{h_{k+1}} \in Z_{h_{k+1}} \). Consider

\[
\overline{u}_{i,j}^{h_{k+1}} := \frac{\left(16 u_{i,j}^{2h_{k+1}} - u_{i,j}^{2h_k}\right)}{15}.
\]

(6)

**Step 3.** Update every (odd, odd) grid point on \( Z_{h_k} \). From (2), for each (odd, odd) point \((i, j)\), the updated solution is

\[
\overline{u}_{i,j}^{h_{k+1}} := \frac{1}{a} \left[F_{i,j} - b \left(\overline{u}_{i+1,j}^{h_{k+1}} + \overline{u}_{i-1,j}^{h_{k+1}}\right)
- c \left(\overline{u}_{i,j+1}^{h_{k+1}} + \overline{u}_{i,j-1}^{h_{k+1}}\right)
- d \left(\overline{u}_{i+1,j-1}^{h_{k+1}} + \overline{u}_{i-1,j-1}^{h_{k+1}}\right)\right].
\]

(7)

Here, \( F_{i,j} \) represents the right-hand side part of (2).

**Step 4.** Update every (odd, even) grid point on \( Z_{h_k} \). From (2), for each (odd, even) grid point, the updated value is

\[
\overline{u}_{i,j}^{h_{k+1}} := \frac{1}{a} \left[F_{i,j} - b \left(\overline{u}_{i+1,j}^{h_{k+1}} + \overline{u}_{i-1,j}^{h_{k+1}}\right)
- c \left(\overline{u}_{i,j+1}^{h_{k+1}} + \overline{u}_{i,j-1}^{h_{k+1}}\right)
- d \left(\overline{u}_{i+1,j-1}^{h_{k+1}} + \overline{u}_{i-1,j-1}^{h_{k+1}}\right)\right].
\]

(8)

**Step 5.** Update every (even, odd) grid point on \( Z_{h_k} \). From (2), the idea is similar to the (odd, even) grid point. Let \( k := k + 1 \).

**Step 6.** If \( ||\overline{u}_{i,j}^{h_{k+1}} - \overline{u}_{i,j}^{h_k}|| \leq \varepsilon \) or \( k = k_{\max} \), stop. Else, let \( \overline{u}_{i,j}^{h_{k+1}} := \overline{u}_{i,j}^{h_{k+1}} \) and return to Step 3.

3. New Extrapolation Formula and EXCMG Method

Based on an asymptotic expansion of finite element method, a new extrapolation formula and an extrapolation cascade
multigrid (EXCMG) method are proposed by Chen et al. (see [17–20]). The numerical experiments show that the EXCMG method is of high accuracy and efficiency. Now we rewrite the new extrapolation formula as follows.

\[
\begin{align*}
\text{Ex}^h_{2i,2j,2} &= \left( 5u^{h}_{2i,2j,2} - u^{2h}_{i,j} \right) / 6, \\
\text{Ex}^h_{2i+1,2j,2} &= u^{h}_{2i+1,2j,2} + \left[ \left( u^{h}_{2i,2j,2} - u^{2h}_{i,j} \right) + \left( u^{h}_{2i+2,2j,2} - u^{2h}_{i+1,j} \right) \right] / 8, \\
\text{Ex}^h_{2i,2j+1,2} &= u^{h}_{2i,2j+1,2} + \left[ \left( u^{h}_{2i,2j,2} - u^{2h}_{i,j} \right) + \left( u^{h}_{2i,2j+2,2} - u^{2h}_{i+1,j+1} \right) \right] / 8, \\
\text{Ex}^h_{2i+1,2j+1,2} &= u^{h}_{2i+1,2j+1,2} + \left[ \left( u^{h}_{2i,2j,2} - u^{2h}_{i,j} \right) + \left( u^{h}_{2i+2,2j,2} - u^{2h}_{i+1,j+1} \right) \right] \times 16^{-1}. 
\end{align*}
\]

Let us denote the above new extrapolation formula by operator

\[
\text{Ex}^h : F\left( u^{2h}, u^h \right). 
\]

Now let \( \tilde{u}^l \), on \( Z_l \), \( l = 0, 1 \) denote the exact solutions, the EXCMG method is as following:

**Algorithm 2 (EXCMG).** For \( l = 2, \ldots, L \), consider the following

**Step 1.** Extrapolate by using the new extrapolation formula (see (10))

\[
\text{Ex}^{l-1} = F\left( \tilde{u}^{l-2}, \tilde{u}^{l-1} \right). 
\]

**Step 2.** Compute the initial value

\[
u^0_l := I_2 \text{Ex}^{l-1} \]

on the grid level \( Z_{l+1} \).

**Step 3.** Smooth \( m_l \) times by using the classical iterative operator \( S_l \),

\[
u^l := S_l^{m_l} u^0_l 
\]

on the level \( Z_{l+1} \). Set \( l := l + 1 \);

**Step 4.** Return to Step 2, if \( l < L \).

The difference between RCMG and EXCMG methods is that

\[
\begin{align*}
\text{RCMG} &= \text{RET} + \text{cubic interpolation} \\
&+ \text{classical iterative operator} + \text{CMG}, \\
\text{EXCMG} &= \text{new extrapolation} + \text{quadratic interpolation} \\
&+ \text{classical iterative operator} + \text{CMG}. 
\end{align*}
\]

**5. Numerical Experiment and Comparison**

Numerical experiments are conducted to solve a 2D Poisson equation (1) on the unit square domain \([0, 1] \times [0, 1]\).
Example 4. The exact solution \( u = \sin(y)(1 - e^x)(1 - x^2)(1 - y^2) \); the forcing function

\[
\begin{align*}
  f &= 2 \sin(y) (e^x - 1) (x^2 - 1) + 2 \sin(y) (e^x - 1) (y^2 - 1) \\
  &\quad - \sin(y) (e^x - 1) (x^2 - 1) (y^2 - 1) \\
  &\quad + 4y \cos(y) (e^x - 1) (x^2 - 1) \\
  &\quad + 4xe^x \sin(y) (y^2 - 1) + e^x \sin(y) (x^2 - 1) (y^2 - 1) .
\end{align*}
\]

(18)

Example 5. The exact solution \( u = \ln(1 + \sin(\pi x^2)) (\cos(\sin(x)) - 1) \sin(\pi y) \); the forcing function

\[
\begin{align*}
  f &= \pi^2 \sin(\pi y) \log(\sin(\pi x^2) + 1) (\cos(\sin(x)) - 1) \\
  &\quad - \sin(\sin(x)) \sin(\pi y) \log(\sin(\pi x^2) + 1) \sin(x) \\
  &\quad + \cos(\sin(x)) \sin(\pi y) \log(\sin(\pi x^2) + 1) \cos^2(x) \\
  &\quad - 2\pi \sin(\pi y) \cos(\pi x^2) (\cos(\sin(x)) - 1) \\
  &\quad + \left(4\pi^2 x^2 \sin(\pi y) \cos^2(\pi x^2) (\cos(\sin(x)) - 1) \right) \\
  &\quad \div (\sin(\pi x^2) + 1)^2 \\
  &\quad + \left(4\pi^2 x^2 \sin(\pi y) \sin(\pi x^2) (\cos(\sin(x)) - 1) \right) \\
  &\quad \div \sin(\pi x^2) + 1 \\
  &\quad + \left(4\pi x \sin(\sin(x)) \sin(\pi y) \cos(x) \cos(\pi x^2) \right) \\
  &\quad \div \sin(\pi x^2) + 1 .
\end{align*}
\]

(19)

We use the conjugate gradient (CG) method as a smoothing iterative operator \( S \) in EXCMG method and RCMG method. In EXCMG method, the number of iterations \( \hat{m}_l \) on each grid level has to increase from finer to coarser grids; in this paper let \( \hat{m}_l = 8 \times 2^{L-l+1} \). And in RCMG, we set the number of iteration \( k_{max}^l \) (Step 2) and \( w_l \) (Step 4) be \( 8 \times 2^{L-l} \).

We set \( \varepsilon = 10^{-8} \) of RET in the RCMG method (on Step 2).

5.1. Comparison of the Initial Errors. Assume that the exact solutions of the difference equation on grids \( 16 \times 16 \) and \( 32 \times 32 \) are given. We compare EXCMG method with RCMG method for the initial error \( \| \mathbf{E}_{16}^0 \| = \| u_{16}^0 - u_{16} \| \) on grid \( 64 \times 64 \).

From Figure 2, the accuracy of the initial error on the next grid of RCMG method is higher than EXCMG method.
Namely, a better initial value on the fine grid can be got by using RCMG method. Based on the results of the literature [17–20], the RCMG method can obtain good convergence.

5.2. Comparison between EXCMG Method and RCMG Method. Let \( \| \text{Error} \|_\infty = \| \hat{u} - u \|_\infty \) denote the maximum absolute error between the computed solution \( \hat{u} \) and the exact solution \( u \) on the finest grid points. The “cpu” denotes the computing time (unit: second) of EXCMG method and RCMG method.

From Figures 3 and 4 and Tables 1 and 2, we see that, under the same conditions, the RCMG method can obtain higher computational precision and spend less computing time than EXCMG method.

6. Conclusion

In this paper, based on a fourth order compact scheme, we present a Richardson cascade multigrid method for 2D Poisson problem by using Richardson technique presented by [21]. The numerical results show that RCMG method has higher computational accuracy and higher efficiency.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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