Research Article

New Results on Robust Model Predictive Control for Time-Delay Systems with Input Constraints

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This paper investigates the problem of model predictive control for a class of nonlinear systems subject to state delays and input constraints. The time-varying delay is considered with both upper and lower bounds. A new model is proposed to approximate the delay. And the uncertainty is polytopic type. For the state-feedback MPC design objective, we formulate an optimization problem. Under model transformation, a new model predictive controller is designed such that the robust asymptotical stability of the closed-loop system can be guaranteed. Finally, the applicability of the presented results are demonstrated by a practical example.

1. Introduction

The ideas of model predictive control and receding horizon control have been developed since 1960s. It has been shown from [1] that model predictive control (MPC) is an effective way to handle multivariable constrained control problems, which appear in the chemical process control, the petrochemical industries, gas pipeline, and so on. In [2, 3], the authors gave us an overview of the origins of model predictive control and the recent results. The original MPC technique is aimed at solving an open-loop optimization problem with constraints at every sampling instant, implementing only the first control step of solutions.

In practical control systems, parameter uncertainties cannot be avoided. In the literatures, two kinds of parameter uncertainties are often included in the uncertain systems. They are norm-bounded parameter uncertainty and polytopic parameter uncertainty. In addition, time-delay often appears in industrial processes, which results in degradation and instability in such systems [4]. References [5, 6] studied the networked control with time-delay; [7] investigated linear switched systems with time-varying delay. The authors in [8] discussed the problem of dissipativity analysis of stochastic neural networks systems of discrete-time form with time-varying and finite-distributed delays. The authors in [9] designed a novel output-feedback controller for the suspension systems with input delay. Moreover, there exist some physical limits, for instance power limitations and value saturation, in many industrial processes, which result in constraints on input and output. Therefore, considerable researchers have been attracted to study the robust control problem of constrained uncertain systems with state delays [10, 11].

Motivated by the above observation, the problem of MPC for time-varying delay systems with parameter uncertainties
and input constraints is studied in this paper. We summarize the main contributions of this paper as follows. (1) The uncertainty is supposed to be polytopic uncertainty type, and the state with unknown delay with both specified upper and lower bounds is handled by an approximated model. (2) In the controller design process, for the state-feedback MPC design objective, we formulate an optimization problem over an infinite time horizon. A new model predictive controller is designed under the model transformation by approximating the state delay, such that the robust asymptotical stability of the closed-loop system is guaranteed. The existence of the controller can be expressed by the convex optimization algorithm. (3) It is shown that the approach proposed in this paper is effective and performs better with the faster response, smaller overshoot, stronger robustness and so on by a practical example.

The rest of this paper is organized as follows. Section 2 formulates the problem to be solved. Section 3 proposes an MPC method for delay systems with uncertainties and constraints. Section 4 illustrates the effectiveness of the method proposed in this paper with a practical example. This paper is concluded in Section 5.

**Notation.** $\mathbb{R}^n$ stands for the $n$-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $I_n$ denotes the $n \times n$ identity matrix, and diag{⋯} denotes a block-diagonal matrix. $X > 0$ ($X \geq 0$) denotes that the matrix $X$ is a positive definite (resp. a positive semi-definite) matrix. $X^T$ denotes transposition of $X$, and $X^{-1}$ denotes the inverse matrix of $X$. $H^T = H$, when $H$ is symmetric matrix. $\| \cdot \|$ denotes 2-norm, and $\|x\|_P^2 = x^T P x$, where $P > 0$. The symbol $*$ induces a symmetric structure in a matrix.

## 2. Problem Formulation

Consider the following system:

$$ x(k + 1) = A(k) x(k) + \overline{A}(k) x(k - d(k)) + B(k) u(k), $$

$$ x(k) = \phi(k), \quad k \in [-d_M, 0], $$

(1)

where $x(k) \in \mathbb{R}^n$ denotes the state variable with the initial condition $\phi(k) \in \mathbb{R}^n$, $d(k)$ means the unknown value of delay units, being supposed $0 \leq d_m \leq d(k) \leq d_M$ with known integers $d_m$ and $d_M$, and $u(k) \in \mathbb{R}^r$ stands for the control input variable and satisfies

$$ -\overline{u} \leq u(k) \leq \overline{u}, \quad \forall k \in [0, \infty). $$

(2)

$[A(k) \overline{A}(k) B(k)]$ is unknown but belongs to a polytope $\Omega$ at each time $k$, that is

$$ \begin{bmatrix} A(k) & \overline{A}(k) & B(k) \end{bmatrix} \in \Omega $$

$$ \triangleq \text{Co}\left\{ [A_1 \overline{A}_1 B_1], [A_2 \overline{A}_2 B_2], \ldots, [A_q \overline{A}_q B_q] \right\}, $$

(3)

in which $\text{Co}$ indicates the convex hull and $[A_i \overline{A}_i B_i]$ are vertices of the convex hull. The nonnegative coefficients $\lambda_i(k)$ ($i = 1, 2, \ldots, q$) for each time $k$ satisfies the following:

$$ \begin{bmatrix} A(k) & \overline{A}(k) & B(k) \end{bmatrix} = \sum_{i=1}^q \lambda_i(k) [A_i \overline{A}_i B_i], $$

(4)

$$ \sum_{i=1}^q \lambda_i(k) = 1. $$

In this paper we aims at designing the following controller for system in (1):

$$ u(k) = K(k) x(k), $$

(5)

with the performance index as follows at every time $k$:

$$ \min_{u(k+1), s \geq 0, [A(k+s) \overline{A}(k+s) B(k+s)] \in \Omega} \max_{s \geq 0} J(k), $$

(6)

where

$$ J(k) = \sum_{s=0}^{\infty} \left\{ \|x(k+s|k)\|_Q^2 + \|u(k+s|k)\|_R^2 \right\}, $$

(7)

$$ x(k+s+1|k) = A(k+s) x(k+s|k) + \overline{A}(k+s) x(k+s-d(k)|k) + B(k+s) u(k+s|k), $$

(8)

$$ -\overline{u} \leq u(k+s|k) \leq K(k) x(k+s|k) \leq \overline{u}, $$

(9)

where $Q$ and $R$ are known positive definite symmetric weighting matrices, $x(k+s|k)$ denotes the predicted state at time $k+s$ and $u(k+s+1|k)$ denotes the control signal at time $k+s$, when $x(k-s|k) = x(k-s)$ ($s \geq 0$). Based on the concept of MPC, before the next sampling time comes, we just implement the first compute input signal $u(k|k)$. Then we repeat the aforementioned optimization problem after updating it with the actual state.

**Remark 1.** The input-output technique is one of the most effective ways to handle the time delay, which was presented in the robust control theory [13, 14]. Before using the approach to dispose time-varying delay, a proper approximation with small error for $x(k-d(k))$ should be found. In [15, 16], the authors used different variants of the state variable as the approximation of $x(k-d(k))$. In this paper, we utilize $(x(k-d_m)+x(k-d_M))/2$ as the approximation of $x(k-d(k))$.

Next, the lower and upper bounds $d_m$ and $d_M$ are used to estimate $x(k-d(k))$. The two-term approximation $(x(k-d_m)+x(k-d_M))/2$ leads to the following error:

$$ \sigma_{d_m}(k) = \frac{2}{d_0} \left\{ x(k-d(k)) - \frac{1}{2} [x(k-d_m)+x(k-d_M)] \right\}, $$

$$ = \frac{1}{d_0} \left\{ \sum_{i=k-d_m}^{k-d_M-1} \beta(i) \zeta_{d_m}(i) \right\}, $$

(10)
where \( d_0 = d_M - d_m \), and
\[
\beta(i) = \begin{cases} 
1, & i \leq k - d(k) - 1; \\
-1, & i > k - d(k) - 1. 
\end{cases}
\]

Then, system (1) is replaced by the following one:
\[
x(k+1) = A(k)x(k) + \frac{1}{2} A(k) \left[ x(k-d_m) + x(k-d_M) + d \sigma_{d_0}(k) \right] + B(k) u(k) + \frac{1}{2} x(k-d_m) + \frac{d_0}{2} d \sigma_{d_0}(k) + B(k) u(k).
\]

(12)

Now let \( x(k+s+1 | k) = x(k+s | k) + y(k+s | k) \). Then we gain the following:
\[
x(k+s-d_m | k) = x(k+s | k) - \sum_{i=1}^{d_m} y(k+s-i | k),
\]
\[
x(k+s-d_M | k) = x(k+s | k) - \sum_{i=1}^{d_M} y(k+s-i | k).
\]

(13)

As in [17], (8) can be converted into the following equivalent descriptor form:
\[
\begin{bmatrix} A(k+s) + B(k+s) K(k) + \overline{A}(k+s) - I \end{bmatrix} x(k+s | k)
- \frac{1}{2} \overline{A}(k+s) \sum_{i=1}^{d_m} y(k+s-i | k)
- \frac{1}{2} \overline{A}(k+s) \sum_{i=1}^{d_M} y(k+s-i | k)
+ \frac{d_0}{2} d \sigma_{d_0}(k) \overline{A}(k+s) - y(k+s | k) = 0.
\]

(14)

We introduce the following lemma for our main result.

**Lemma 2.** According to [18], suppose that \( \alpha \in \mathbb{R}^n \), \( \beta \in \mathbb{R}^n \), and \( N \in \mathbb{R}^{n \times n} \); then for any matrices \( X \in \mathbb{R}^{n \times n} \), \( Y \in \mathbb{R}^{n \times n} \), and \( Z \in \mathbb{R}^{n \times n} \), the following inequality holds:
\[
-2 \alpha^T N \beta \leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},
\]
\[
\text{where } \begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \succeq 0.
\]

(15)

### 3. Main Results

The following function is introduced in order to obtain the main results
\[
V(x(k+s | k)) = x^T(k+s | k) p_1 x(k+s | k)
+ \sum_{i=1}^{d_m} \sum_{r=1}^{\theta} y^T(k+s-r | k) g_1 y(k+s-r | k)
+ \sum_{i=1}^{d_M} \sum_{r=1}^{\theta} y^T(k+s-r | k) g_2 y(k+s-r | k)
\]
\[
+ \sum_{i=1}^{d_m} x^T(k+s-r | k) h_1 x(k+s-r | k)
+ \sum_{i=1}^{d_M} x^T(k+s-r | k) h_2 x(k+s-r | k).
\]

(16)

For every \( \{ A(k+s), \overline{A}(k+s), B(k+s) \} \in \Omega, s \geq 0 \) and satisfies the following:
\[
V(x(k+s+1 | k)) - V(x(k+s | k)) \leq -\left\| x(k+s | k) \right\|_Q^2 + \left\| u(k+s | k) \right\|_R^2;
\]
\[
\text{we obtain the upper bound of } J(k). \text{ Since } J(k) \text{ should be limited, one can get } x(\infty | k) = 0. \text{ Then, one can get } y(\infty | k) = 0. \text{ Therefore, } V(x(\infty | k)) = 0. \text{ It follows that } -V(x(k | k)) \leq -J(k), \text{ by calculating the sum of inequality (17) from } s = 0 \text{ to } s = \infty. \text{ Hence}
\]
\[
\max_{\{ A(k+s), \overline{A}(k+s), B(k+s) \} \in \Omega, s \geq 0} J(k) \leq V(x(k | k)),
\]
\[
\text{where}
\]
\[
V(x(k | k)) = x^T(k | k) p_1 x(k | k)
+ \sum_{i=1}^{d_m} \sum_{r=1}^{\theta} y^T(k-r | k) g_1 y(k-r | k)
+ \sum_{i=1}^{d_M} \sum_{r=1}^{\theta} y^T(k-r | k) g_2 y(k-r | k)
\]
\[
+ \sum_{i=1}^{d_m} x^T(k-r | k) h_1 x(k-r | k)
+ \sum_{i=1}^{d_M} x^T(k-r | k) h_2 x(k-r | k).
\]

(17)

(18)

(19)

Therefore, it can be seen from (18) that we transform the organical min-max optimization problem in (6)–(9) into the following optimization problem which can minimize the upper bound of \( V(x(k | k)) \) as follows:
\[
\min_{K(k), p_1, g_1, g_2, h_1, h_2} V(x(k | k)).
\]

(20)

Equations (9), (14), and (17) are the constraint conditions.
Theorem 3. Considering the uncertain system in (1) with time-varying delay and input constraints (2), if there exist matrices $X > 0, Y, Z, K, U_1 > 0, U_2 > 0, U_3 > 0, U_4 > 0, W_1, W_2, W_3,$ and $E$ with appropriate dimensions and the scalar $\gamma > 0$, the following problem is solvable:

\[
\min_{\gamma, X, Y, Z, K, U_1, U_2, U_3, U_4, W_1, W_2, W_3, E} \gamma,
\]

subject to

\[
\begin{bmatrix}
\Theta_1 & \Theta_2 & 0 & 0 & Z^T & d_mZ^T & d_MZ^T & X & X^TQ^{1/2} & \bar{K}^T & R^{1/2}
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
\Omega_1 & \Omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
W_1 & W_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
W_1 & W_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
E & K & X
\end{bmatrix} \geq 0, \quad E_{ii} \leq \bar{u}_i^2, \quad i = 1, \ldots, n,
\]

where
\[
\Gamma_3 = \begin{bmatrix}
    d_m^1 U_1 + d_M^2 U_2 \\
    (d_m - 1)^{-1} U_1 + (d_M - 1)^{-1} U_2 \\
    \vdots \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    0 \\
    \vdots \\
    0
\end{bmatrix},
\]

\[
\Theta_1 = Z + Z^T + d_m \overline{W}_1 + d_M \overline{W}_2,
\]

\[
\Theta_2 = X \left( A_o^T + 2 \sigma A_o^T - I \right) + Y + K^T B_o^T - Z^T + d_m \overline{W}_2 + d_M \overline{W}_3,
\]

\[
\Theta_3 = -Y - Y^T + d_m \overline{W}_2 + d_M \overline{W}_3,
\]

\[
\Theta_4 = (1 - \varepsilon) \overline{A}_3 U_3,
\]

\[
\sigma = 1, \ldots, q.
\]

(27)

and \( \overline{a}_i \) indicates the \( i \)-th element of \( \overline{a} \) and \( E_{i,i} \) represents the \( i \)-th diagonal element of \( E \); then the upper bound \( V(x(k | k)) \) of the desired performance index is minimized by the MPC law \( u(k+1 | k) = K X^{-1} x(k+1 | k) \), \( s \geq 0 \).

Proof. Set \( X = \gamma P^{-1} \), \( U_1 = \gamma S_1^{-1} \), \( U_2 = \gamma S_2^{-1} \), \( U_3 = \gamma H_1^{-1} \) and \( U_4 = \gamma H_2^{-1} \). Then, the problem of minimizing \( V(x(k | k)) \) can be regarded as follows:

\[
\min_{\gamma, P, G_1, G_2, H_1, H_2} \gamma
\]

subject to

\[
V(x(k | k)) = x^T(k | k) P_1 x(k | k)
\]

+ \[
\sum_{\varepsilon=1}^{d_m} \sum_{r=1}^\theta y^T(k-r | k) G_1 y(k-r | k)
\]

+ \[
\sum_{\varepsilon=1}^{d_M} \sum_{r=1}^\theta y^T(k-r | k) G_2 y(k-r | k)
\]

+ \[
\sum_{i=1}^{d_m} x^T(k-r | k) H_1 x(k-r | k)
\]

+ \[
\sum_{i=1}^{d_M} x^T(k-r | k) H_2 x(k-r | k) \leq \gamma.
\]

(28)

We can easily deduce (21) and (22) by using the Schur complement. According to the definition of \( V(x(k+s | k)) \) in (16), one can obtain the following:

\[
V(x(k+s+1 | k)) - V(x(k+s | k)) = 2 x^T(k+s | k) P_1 y(k+s | k)
\]

+ \[
\sum_{i=1}^{d_m} x^T(k-r | k) (H_1 + H_2) x(k-s | k)
\]

+ \[
\sum_{i=1}^{d_M} x^T(k-r | k) H_2 x(k-s | k) \leq \gamma.
\]

(29)

By Lemma 2 and the descriptor system (14), we have

\[
2 x^T(k+s | k) P_1 y(k+s | k)
\]

= \[
2 \eta^T(k+s | k) P^T
\]

× \[
\left[ y(k+s | k) - \sum_{i=1}^{d_m} \frac{1}{2} A(k+s) y(k+s-i | k) \right]
\]

\[
\left[ y(k+s | k) - \sum_{i=1}^{d_M} \frac{1}{2} A(k+s) y(k+s-i | k) \right]
\]

\[
\leq 2 \eta^T(k+s | k) P^T
\]

× \[
\left[ A(k+s) + B(k+s) K(k) - I \right] \eta(k+s | k)
\]

+ \[
2 \eta^T(k+s | k) P^T \left[ \frac{1}{2} A(k+s) \right] x(k+s | k)
\]

+ \[
2 \eta^T(k+s | k) P^T \left[ \frac{d_0}{2} d_t A(k+s) \right]
\]
where
\[ \Theta_5 = \begin{bmatrix} A (k+s) + B (k+s) K (k) + \hat{A} (k+s) - I \\ x(k+s | k) + \frac{d_0}{2} \sigma_{d_0} \hat{A} (k+s) - y(k+s | k) \end{bmatrix} \]
\[ \eta (k+s | k) = \begin{bmatrix} x(k+s | k) \\ y(k+s | k) \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \]
and matrices $W$ and $M$ satisfy the following conditions:
\[ \begin{bmatrix} W & M \\ * & G_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} W & M \\ * & G_2 \end{bmatrix} \geq 0. \]

After substituting (31) into (30), we obtain the following inequality:
\[ V (x(k+s+1 | k)) - V (x(k+s | k)) \]
\[ \leq \eta^T (k+s | k) \Phi \eta (k+s | k) + 2 \eta^T (k+s | k) \]
\[ \times \left( P^T \left[ \frac{1}{2} \hat{A} (k+s) \right] - M \right) \times (k+s-d_M | k) \]
\[ + 2 \eta^T (k+s | k) \left( P^T \left[ \frac{1}{2} \hat{A} (k+s) \right] - M \right) \times x(k+s - d_M | k) \]
\[ - x^T (k+s-d_M | k) H_2 x (k+s - d_M | k) \]
\[ + 2 \eta^T (k+s | k) P^T \left[ \frac{d_0}{2} \sigma_{d_0} \hat{A} (k+s) \right] \]
\[ = \varepsilon^T (k+s | k) \]
\[ \times \begin{bmatrix} \Phi & P^T \left[ \frac{1}{2} \hat{A} (k+s) \right] - M & P^T \left[ \frac{1}{2} \hat{A} (k+s) \right] - M \\ * & -H_1 & * \\ * & * & -H_2 \end{bmatrix} \times \varepsilon (k+s | k) \]
\[ + 2 \eta^T (k+s | k) P^T \left[ \frac{d_0}{2} \sigma_{d_0} \hat{A} (k+s) \right] \],

(35)

where $\varepsilon^T (k+s | k) = [x^T (k+s | k) \ y^T (k+s | k) \ x^T (k+s-d_M | k) \ x^T (k+s-d_M | k)]$.

Replacing (35) with (36), it can be found that
\[ V (x(k+s+1 | k)) - V (x(k+s | k)) \]
\[ \leq \varepsilon^T (k+s | k) \]
Replacing (17) with (37), we have

\begin{align*}
\Phi P^{-1} A (k+s) - M P^{-1} A (k+s) \\
* & - H_1 \\
* & - H_2 \\
\end{align*}

\begin{align*}
= \epsilon^T (k+s | k) \\
\times [ \Phi P^T \left[ \begin{array}{c} 0 \\
\frac{1}{2} \overline{A} (k+s) \end{array} \right] - M P^T \left[ \begin{array}{c} 0 \\
\frac{1}{2} \overline{A} (k+s) \end{array} \right] - M ] \\
\times \epsilon (k+s | k).
\end{align*}

For obtaining LMI, we give the definition as follows:

\begin{align*}
M = \epsilon P^T \left[ \begin{array}{c} 0 \\
\frac{1}{2} \overline{A} (k+s) \end{array} \right], \\
K = K X,
\end{align*}

\begin{align*}
W_1 &= X Z Y, \\
W_2 &= Y Z X, \\
W &= \gamma (P^{-1})^T W (P^{-1}) = \left[ \begin{array}{cc} W_1 & W_2 \\
W_2 & W_3 \end{array} \right].
\end{align*}

Pre- and postmultiplying (38), (33), and (34) by diag\{\gamma^{1/2}(P^{-1}), X Z, Y Z\} and diag\{\gamma^{1/2}(P^{-1}), X Z, Y Z\}, respectively, then (38) is equivalent to the following:

\begin{align*}
\begin{bmatrix}
\Theta_7 & \Theta_8 & 0 & 0 & Z^T d_m Z^T & d_m Y^T & d_M Z^T & X & X^T Q^{1/2} & \overline{A}^T R^{1/2}
* & \Theta_9 & \Theta_{10} & \Theta_{31} & Y^T d_m Y^T & d_M Y^T & 0 & 0 & 0 & 0
* & * & * & -U_3 & 0 & 0 & 0 & 0 & 0 & 0
* & * & * & * & -U_4 & 0 & 0 & 0 & 0 & 0
* & * & * & * & * & -X & 0 & 0 & 0 & 0
* & * & * & * & * & * & -d_m U_1 & 0 & 0 & 0
* & * & * & * & * & * & * & -d_M U_2 & 0 & 0
* & * & * & * & * & * & * & * & -U_3 - U_4 & 0 & 0
* & * & * & * & * & * & * & * & * & -y I & 0
* & * & * & * & * & * & * & * & * & * & -y I
\end{bmatrix} \\
\leq 0,
\end{align*}

Equation (33) is equivalent to the following:

\begin{align*}
\begin{bmatrix}
W_1 & W_2 & 0 \\
* & W_3 & \epsilon \overline{A} U_1 \\
* & * & U_1 \end{bmatrix} \geq 0,
\end{align*}

Equation (34) is equivalent to the following:

\begin{align*}
\begin{bmatrix}
W_1 & W_2 & 0 \\
* & W_3 & \epsilon \overline{A} U_2 \\
* & * & U_2 \end{bmatrix} \geq 0.
\end{align*}

If and only if inequalities in (23)–(25) hold, respectively, inequalities in (40)–(43) are fulfilled for any
\[ A(k) \overline{A}(k) B(k) \in \Omega, \] for that (40)–(43) are affine on the basis of matrices \[ A(k) \overline{A}(k) B(k) \].

Now we consider the input constraints (9) and discuss how to transform it into LMI. First we introduce the following invariant ellipsoid:
\[ \omega = \{ z \in \mathbb{R}^{n_2} \mid z^T \Psi^{-1} z \leq 1 \}, \tag{44} \]
where
\[
\Psi = \text{diag} \{ X_1, U_3 + U_4, \ldots, U_3 + U_4 \}
\]
\[
\begin{align*}
U_4 & \cdots U_4 d_m^{-1} U_1 + d_m^{-1} U_2 \\
& \cdots (d_m - 1)^{-1} U_1 + (d_m - 1)^{-1} U_2, \\
& \cdots U_1 + (d_m - d_m + 1)^{-1} U_2 (d_m - d_m), U_2 \\
& \cdots (d_m - d_m - 1) U_2, U_2, \\
\end{align*}
\]
\[ z = \begin{bmatrix} x(k + s | k) & x(k + s - 1 | k) & \cdots & x(k + s - d_m | k) & x(k + s - d_m - 1 | k) & \cdots & x(k + s - d_m | k) \end{bmatrix}^T \begin{bmatrix} y(k + s - 1 | k) & y(k + s - d_m | k) & \cdots & y(k + s - 2 | k) & y(k + s - d_m | k) & \cdots & y(k + s - d_m | k) \end{bmatrix} \] \tag{45}

As discussed in [12], it is shown that
\[
\max_{s \geq 0} \left[ u_t(k + s | k) \right]^2 \\
= \max_{s \geq 0} \left[ (\overline{K}X^{-1} x(k + s | k)) \right]^2 \\
\leq \max_{s \geq 0} \left[ (\overline{K} \Psi^{-1} z) \right]^2 \\
= \max_{s \geq 0} \left[ (\overline{K} \Psi^{-1/2} \Psi^{-1/2} z) \right]^2 \\
\leq \left\| (\overline{K} \Psi^{-1/2}) \right\|_2^2 \\
\text{(using the Cauchy-Schwarz inequality)} \\
= \left\| (\overline{K} \Psi^{-1/2}) \right\|_2^2,
\]
where \[ \overline{K} = \begin{bmatrix} \overline{K} & 0 & \cdots & 0 \end{bmatrix} \].

Therefore, there exists a symmetric matrix \( E \) and the following inequality is satisfied:
\[
\begin{bmatrix} E & \overline{K} \\
\overline{K}^T & \Psi \end{bmatrix} \succeq 0, \quad E_{ii} \leq \overline{u}_i^2, \quad i = 1, \ldots, n_\omega. \tag{47} \]

It is easily shown that (47) is equivalent to (26) with the definitions of \( \overline{K} \) and \( \Psi \). This completes the proof. \( \square \)

**Theorem 4.** If the optimization problem in (21)–(26) is solved with a feasible solution, the MPC law designed in Theorem 3 can make the resulting closed-loop system robustly asymptotically stable.

**Proof.** At first, let one verify the feasibility of optimization problem (21)–(26). A feasible sequence \( u(k + s | k), s \geq 0 \) is supposed to exist in (21)–(26) at time \( k \). At time \( k + 1 \), we choose the following feasible control sequence obtained at time \( k \) to guarantee the existence of feasible solutions:
\[
u(k + s + 1 | k + 1) = u^*(k + s + 1 | k), \quad s \geq 0, \tag{48} \]

where \( u^*(k + s + 1 | k) \) is a solution obtained at time \( k \). The input constraint (9) at time \( k + 1 \) is satisfied. It indicates the feasibility of the optimization problem at all future instants. Then the stability of the closed-loop system is given. Let \( P^*_1(k), G^*_1(k), G^*_2(k), H^*_1(k), H^*_2(k), \) and \( P^*_1(k + 1), G^*_1(k + 1), G^*_2(k + 1), H^*_1(k + 1), \) and \( H^*_2(k + 1) \) indicate the optimal values at time \( k \) and \( k + 1 \), respectively. Now we take the following quadratic function into account:
\[
V^*(x(k | k)) = x^T(k | k) P^*_1(k) x(k | k) \\
+ \sum_{d=1}^{d_m} \sum_{r=1}^{d_m} y^T(k - r | k) G^*_1(k) y(k - r | k) \\
+ \sum_{d=1}^{d_m} \sum_{r=1}^{d_m} y^T(k - r | k) G^*_2(k) y(k - r | k) \
+ \sum_{r=1}^{d_m} x^T(k - r | k) H^*_1(k) x(k - r | k) \\
+ \sum_{r=1}^{d_m} x^T(k - r | k) H^*_2(k) x(k - r | k).
\] \tag{49}

Since \( P^*_1(k + 1), G^*_1(k + 1), G^*_2(k + 1), H^*_1(k + 1) \), and \( H^*_2(k + 1) \) are optimal, while \( P^*_1(k), G^*_1(k), G^*_2(k), H^*_1(k), \) and \( H^*_2(k) \) are also feasible at time \( k + 1 \), we have
\[
V^*(x(k + 1 | k + 1)) \\
\begin{aligned}
= & \ x^T(k + 1 | k + 1) P^*_1(k + 1) x(k + 1 | k + 1) \\
+ \sum_{d=1}^{d_m} \sum_{r=1}^{d_m} y^T(k + 1 - r | k) G^*_1(k + 1) \
& \quad \times y(k + 1 - r | k + 1) \\
+ \sum_{d=1}^{d_m} \sum_{r=1}^{d_m} y^T(k + 1 - r | k) G^*_2(k + 1) \\
& \quad \times y(k + 1 - r | k + 1) \\
\end{aligned}
\]
\[
+ \sum_{r=1}^{d_m} x^T(k+1-r | k+1) H_r^*(k+1) \\
\times x(k+1-r | k+1) \\
+ \sum_{r=1}^{d_M} x^T(k+1-r | k+1) H_r^*(k+1) \\
\times x(k+1-r | k+1) \\
\leq x^T(k+1 | k+1) P^*_1(k) x(k+1 | k+1) \\
+ \sum_{r=1}^{d_m} \sum_{\theta=1}^{\theta} y^T(k+1-r | k) G^*_1(k) \\
\times y(k+1-r | k) \\
+ \sum_{r=1}^{d_M} \sum_{\theta=1}^{\theta} y^T(k+1-r | k) G^*_2(k) \\
\times y(k+1-r | k) \\
+ \sum_{r=1}^{d_m} x^T(k+1-r | k) H_r^*(k+1) \\
\times x(k+1-r | k+1) \\
+ \sum_{r=1}^{d_M} x^T(k+1-r | k) H_r^*(k) \\
\times x(k+1-r | k+1).
\]

(50)

Furthermore, from (17), it follows that \(V(x(k+1)) - V(x(k | k)) \leq 0\). Thus, one can obtain

\[
x^T(k+1 | k) P^*_1(k) x(k+1 | k) \\
+ \sum_{r=1}^{d_m} \sum_{\theta=1}^{\theta} y^T(k+1-r | k) G^*_1(k) \\
\times y(k+1-r | k) \\
+ \sum_{r=1}^{d_M} \sum_{\theta=1}^{\theta} y^T(k+1-r | k) G^*_2(k) \\
\times y(k+1-r | k) \\
+ \sum_{r=1}^{d_m} x^T(k+1-r | k) H_r^*(k+1) \\
\times x(k+1-r | k+1) \\
+ \sum_{r=1}^{d_M} x^T(k+1-r | k) H_r^*(k) \\
\times x(k+1-r | k) \\
\leq x^T(k | k) P^*_1(k) x(k | k) \\
+ \sum_{r=1}^{d_m} \sum_{\theta=1}^{\theta} y^T(k-r | k) G^*_1(k) y(k-r | k) \\
+ \sum_{r=1}^{d_M} \sum_{\theta=1}^{\theta} y^T(k-r | k) G^*_2(k) y(k-r | k) \\
+ \sum_{r=1}^{d_m} x^T(k-r | k) H_r^*(k) x(k-r | k) \\
+ \sum_{r=1}^{d_M} x^T(k-r | k) H_r^*(k) x(k-r | k)
\]

\[
= V^*(x(k | k)).
\]

(51)

From the definition of \(y(k+1 | k)\) above, it follows that \(y(k+1-s | k) = x(k+2-s | k+1) - x(k+1-s | k) + x(k+1-s | k)\), \(s = 1, \ldots, d_0\), for \([A(k) \quad \overline{A}(k) \quad B(k)] \in \Omega\) at each time \(k\). \(x(k+1-s | k+1)\) is defined in (50). As \(x(k+1 | k+1) = x(k+1 | k)\) and \(x(k+1-s | k+1) = x(k+1-s | k)\), \(s = 1, \ldots, d_0\), respectively.

Thus, according to (51), one can get

\[
V^*(x(k+1 | k+1)) \leq V^*(x(k | k)).
\]

(52)

It is shown that the Lyapunov function \(V^*(x(k | k))\) is bounded and monotonically nonincreasing. Therefore, \(x(k)\) approaches 0 when \(k\) approaches \(\infty\). And the proof is completed.

\[\square\]

Remark 5. According to the condition in Theorem 3, \(x(k-r | k) = x(k-r)\) and \(y(k-r | k) = x(k-r+1)-x(k-r)\), \(r = 1, \ldots, d_0\), are decided by the previous time and remained fixed. Thus, for given \(\epsilon\), (21)–(26) can be regarded as a LMI optimization problem. And we can obtain the numerical solution efficiently in terms of LMIs.

Remark 6. Different from the delay-independent MPC methods in [4,12], Theorem 3 indicates that the proposed MPC technique is decided by the value of delay. The delay considered in this paper is unknown, while the delay is known in the reference [10].

4. Simulation Results

In this section, the truck-trailer system used in [4] is provided to illustrate the effectiveness of the method proposed in this paper:

\[
\begin{align*}
\dot{x}_1(t) &= -\delta \frac{v\ell}{\ell t_0} x_1(t) - (1-\delta) \frac{v\ell}{\ell t_0} x_1(t-\tau) + \frac{v\ell}{\ell t_0} u(t), \\
\dot{x}_2(t) &= \delta \frac{v\ell}{\ell t_0} x_1(t) + (1-\delta) \frac{v\ell}{\ell t_0} x_1(t-\tau),
\end{align*}
\]
\[
\dot{x}_3(t) = \frac{v T}{t_0} \sin \left[ x_2(t) + \delta \frac{v T}{2L} x_1(t) \right] \\
+ (1 - \delta) \frac{v T}{2L} x_1(t - \tau),
\]
\[ (53) \]
where the variables \(x_1(t), x_2(t), x_3(t),\) and \(u(t)\), respectively, denote the angle difference between the trailer and truck, the angle of the trailer, the \(y\)-coordinate of the rear end of the trailer, and the steering angle. Delays exist in \(x_1\); \(|u(t)| \leq \pi\). The system parameters are given as \(r = 2.8 \text{ m}, L = 5.5 \text{ m}, v = -1.0 \text{ m/s}, \tau = 2.0 \text{ s},\) and \(t_0 = 0.5 \text{ s}.\) The constant \(\delta\) denotes the retarded coefficient varying \(\delta \in [0, 1]\). In our numerical example, we would like to suppose \(\delta = 0.7\). The time-delay system with nonlinearity is transformed into discrete-time polytopic uncertain system as follows with the sampling time \(T = 0.1 \text{ s}:\)

\[
x(k + 1) = A(k) x(k) + \bar{A}(k) x(k - d(k)) + B(k) u(k),
\]
\[ (54) \]
where \([A(k) \quad \bar{A}(k) \quad B(k)]\) satisfies (3) with \(q = 2,\) and

\[
A_1 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1.0000 & 0 \\ 0.0509 & -0.4000 & 1.0000 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1.0000 & 0 \\ 0.0509 & -0.6366 & 1.0000 \end{bmatrix},
\]

\[
\bar{A}_1 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0218 & 0 & 0 \end{bmatrix},
\]

\[
\bar{A}_2 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0347 & 0 & 0 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix}.
\]

We choose the initial state \(x(0) = [0.5 \quad 0.75\pi \quad -5]^T\), the delay lower bound \(d_m = 1\), the delay upper bound \(d_{\bar{m}} = 10\), the input and state weighting matrices \(R = 1\), and \(Q = \text{diag}(10, 10, 10)\), respectively. In addition, we choose \(\varepsilon = 1.5\). Figure 1 shows the state response of the open-loop system which is unstable. In order to stabilize the system, we implement the MPC strategy at each step to design the controller \(u(k+s | k) = Kx(k+s | k), s \geq 0\). It can be found from Theorem 3 that the MPC state-feedback law is obtained as follows:

\[
K = \begin{bmatrix} 2.3627 & -1.0863 & 0.1195 \end{bmatrix}.
\]

\[ (56) \]

In order to show the advantages of this paper over the existing results [4, 19], we give the comparison simulation results as follows. Figures 2(a) and 2(b) plot the state response of the closed loop system acquired through the MPC methods presented in this paper and in [4], respectively. It is obvious that the system employing our MPC method performs better with the faster response, smaller overshoot, stronger robustness, and so on than the existence results in [4]. Figures 3(a) and 3(b) show the control inputs obtained by the MPC algorithms in this paper and in [4], respectively. It is shown that both of them are not out of the input constraint. However, in this paper, the value of the input is smaller and the input trajectory is smoother. Figure 4 show the cost function trajectories gained by the two MPC methods. In addition, it is found that the method presented in [19] is not feasible in general numerical example.

5. Conclusion

The problem of MPC for a class of uncertain systems subject to time-varying delays and input constraints has been studied in this paper. \(x(k - d(k))\) is estimated by its lower and upper bounds. Then the system is transformed into an equivalent descriptor system. Using some advanced method, the MPC law has been designed to guarantee the resulting closed-loop system to be asymptotically stable. The merits of the MPC approach proposed in this work have been demonstrated by a practical example. In future work, the proposed MPC technique will be applied to some kinds of systems, such as Markovian jumping systems [20] and fuzzy systems [21–24].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Figure 2: Comparison of the closed-loop state responses ((a) using our method and (b) using the method proposed in [4]).

Figure 3: Comparison of the cost function ((a) using our method and (b) using the method proposed in [4]).

Figure 4: Comparison of the cost function ((a) using our method and (b) using the method proposed in [4]).
References


