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Research Article

Lower Convergence of Minimal Sets in Star-Shaped Vector Optimization Problems

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Let $\{A_n\}$ be a sequence of nonempty star-shaped sets. By using generalized domination property, we study the lower convergence of minimal sets Min A_n. The distinguishing feature of our results lies in disuse of convexity assumptions (only using star-shapedness).

1. Introduction

Stability analysis is one of the most important and interesting subjects and its role has been widely recognized in the theory of optimization. In the literature, two classical approaches can be found to study stability in vector optimization. One is to investigate continuity properties of the optimal multifunctions [1–3]. Another is to study the set-convergence of minimal sets of perturbed sets converging to a given set [4–6]. Bednarczuk [1, 2] obtained some stability results by investigating the Hölder continuity of minimal point functions in vector optimization problems. Bednarczuk [3] established the stability by investigating the lower semicontinuity of minimal points in vector optimization. Luc et al. [4] investigated the stability of vector optimization in terms of the convergence of the efficient sets. Miglierina and Molho [5] obtained some results on stability of convex vector optimization problems by considering the convergence of minimal sets. Convexity is a very common assumption and plays important roles in stability analysis in vector optimization. By using convexity assumptions, Tanino [7] considered the stability of the efficient set in vector optimization. Bednarczuk [8] investigated the stability of Pareto points to finite-dimension parametric convex vector optimization. In [5, 9], the authors used convexity to establish Kuratowski-Painlevé and Attouch-Wets convergence of minimal sets. For more results concerning use of convexity in stability analysis, we refer readers to [10, 11].

However, many practical problems can only be modelled as nonconvex optimization problems. So it is interesting and important to weaken convexity assumption. Star-shapedness is one of the most important generalizations of convexity. Crespi et al. [12, 13] used star-shapedness to study scalar Minty variational inequalities and scalar optimization problems. Fang and Huang [14] used star-shapedness to study the well-posedness of vector optimization problems. Shveidel [15] studied the separability and its application to an optimization problem. In this paper, following the ideas of [5, 9], we investigate the lower convergence of minimal sets in star-shaped vector optimization problems.

2. Preliminaries and Notations

In what follows, unless otherwise specified, we always suppose that X is a normed linear space with dual space X^* and $\mathscr{B}_\rho$ is the closed ball centered at 0 with radius ρ. Let A, B be nonempty subsets of X, let $\{A_n\}$ be a sequence of nonempty subsets of X, and let $K \subset X$ be a pointed, closed, and convex cone with $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} K$ denotes the interior of K. We say that $G \subset X$ is a base of K if and only if G is convex, $0 \notin \operatorname{cl} G$, and $K = \operatorname{cone} G$, where $\operatorname{cl} G$ and $\operatorname{cone} G$ denote the closure and cone hull of G, respectively.

Definition 1. A point $a \in A$ is called a minimal point of A (with respect to K) if and only if $A \cap (a - K) = \{a\}$. Denote by Min A the set of all minimal points of A. A point $a \in A$ is called a weakly minimal point of A if and only if $A \cap (a - \operatorname{int} K) = \emptyset$. Denote by WMin A the set of all weakly minimal points of A. A point $a \in A$ is called strictly minimal point (see

[3]) of A if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $(A - a) \cap (\mathscr{B}_\delta - K) \subset \mathscr{B}_\epsilon$. Denote by StMin A the set of all strictly minimal points of A. Obviously StMin $A \subset$ Min A.

Definition 2. The generalized domination property (GDP) holds for A if and only if $A \subset \mathrm{clMin}\, A + K$.

Remark 3. (i) Clearly the domination property (DP) (see [11]) implies the generalized domination property (GDP). (ii) The containment property (CP) ([16]) implies the generalized domination property (GDP). (iii) The weak containment property (WCP) ([1]) implies the generalized domination property (GDP). For more details on relationship among the domination property, the containment property, and the weak containment property, we refer readers to [8].

Definition 4. Given a set A and a sequence $\{A_n\}$ of subsets of X, the Kuratowski-Painlevé lower and upper limits are defined as follows:

$$\begin{aligned} \mathrm{Li}A_n &= \Big\{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \\ &\qquad \forall \text{sufficiently large } n\Big\}, \\ \mathrm{Ls}A_n &= \Big\{x \in X : x = \lim_{s\to\infty} x_s, x_s \in A_{n_s}, \{n_s\} \\ &\qquad \text{is a subsequence of } \{n\}\Big\}. \end{aligned} \tag{1}$$

We say that $\{A_n\}$ converges to A in the sense of Kuratowski-Painlevé if and only if $\mathrm{Ls}A_n \subset A \subset \mathrm{Li}A_n$. When we consider the limits in the weak topology on X rather than the original norm topology, we denote the lower and upper limits above by $w - \mathrm{Li}A_n$ and $w - \mathrm{Ls}A_n$, respectively. When $w - \mathrm{Ls}A_n \subset A \subset w - \mathrm{Li}A_n$, we say that $\{A_n\}$ converges to A (denoted by $A_n \to A$) in the sense of Kuratowski-Painlevé with respect to weak topology. We say that $\{A_n\}$ converges to A in the sense of Mosco if and only if $w - \mathrm{Ls}A_n \subset A \subset \mathrm{Li}A_n$.

Definition 5. Give two nonempty subsets A and B of X, and define

$$\begin{gathered} e(A,B) = \sup_{a\in A} d(a,B), \qquad e_\rho(A,B) = e\left(A \cap \mathscr{B}_\rho, B\right), \\ H_\rho(A,B) = \max\left\{e_\rho(A,B), e_\rho(B,A)\right\}, \end{gathered} \tag{2}$$

where $d(x,A) = \inf_{a\in A}\|x - a\|$. One says that a sequence $\{A_n\}$ of subsets of X converges to A in the sense of Attouch-Wets if and only if $\lim_{n\to\infty} H_\rho(A_n, A) \to 0$ for all $\rho > 0$. One says that A is upper (or lower) limit of $\{A_n\}$ in the sense of Attouch-Wets if and only if

$$\lim_{n\to\infty} e_\rho(A_n, A) \longrightarrow 0 \left(\text{or} \lim_{n\to\infty} e_\rho(A, A_n) \longrightarrow 0\right) \tag{3}$$

for all $\rho > 0$.

Remark 6. When X is finite-dimensional, the notions of set-convergence in Definitions 4 and 5 coincide whenever we consider a sequence $\{A_n\}$ of closed sets. For more relationship between the various concepts of set-convergence, we refer readers to [6].

Definition 7. Given a set A, the kernel $\ker A$ of A is defined by

$$\ker A = \{a \in A : a + \lambda(x - a) \in A, \forall x \in A, \ \forall \lambda \in [0,1]\}. \tag{4}$$

A set A is called star-shaped if and only if $\ker A \neq \emptyset$ or $A = \emptyset$. Obviously every convex set is star-shaped and the converse is not true in general.

3. Main Results

In this section, we investigate the lower convergence of minimal sets in star-shaped vector optimization.

The following proposition shows that the limit set of a Kuratowski-Painlevé converging sequence of star-shaped sets is star-shaped.

Proposition 8. *Let X be a normed linear space and let $\{A_n\}$ be a sequence of nonempty star-shaped subsets of X. Then $Li(\ker A_n) \subset \ker(LiA_n)$.*

Proof. By the definition of $\mathrm{Li}(\ker A_n)$, we get $\mathrm{Li}(\ker A_n) \subset \mathrm{Li}A_n$. Suppose to the contrary that there exists $b \in \mathrm{Li}(\ker A_n)$ such that $b \notin \ker(\mathrm{Li}A_n)$. Then there exist $a \in \mathrm{Li}A_n$ and $\lambda \in [0,1)$ such that $b + \lambda(a - b) \notin \mathrm{Li}A_n$. Since $a \in \mathrm{Li}A_n$ and $b \in \mathrm{Li}(\ker A_n)$, there exist sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\begin{gathered} a_n \longrightarrow a, \quad b_n \longrightarrow b, \quad a_n \in A_n, \quad b_n \in \ker A_n \\ \text{for all sufficiently large } n. \end{gathered} \tag{5}$$

It follows that

$$\begin{gathered} b_n + \lambda(a_n - b_n) \longrightarrow b + \lambda(a - b), \\ b_n + \lambda(a_n - b_n) \in A_n \\ \forall \text{sufficiently large } n. \end{gathered} \tag{6}$$

This contradicts $b + \lambda(a - b) \notin \mathrm{Li}A_n$. □

Remark 9. Let $\{A_n\}$ be a sequence of nonempty star-shaped subsets of X and $A_n \to A$ and $\ker A_n \to B$. By Proposition 8, $B \subset \ker A \subset A$. It is known that the limit set of a Kuratowski-Painlevé converging sequence of convex sets is convex (see Proposition 3.1 of [17]). In this sense, Proposition 8 generalizes Proposition 3.1 of [17] to star-shaped case.

Theorem 10. *Let X be a normed linear space, let K be a pointed, closed, and convex cone with a sequentially weakly compact base G, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) *$A = w - LsA_n$, $B = Li(\ker A_n)$.*

Then $B \cap \mathrm{Min}\, A \subset Li(\mathrm{Min}\, A_n)$.

Proof. If $B \cap \operatorname{Min} A = \emptyset$, then the conclusion holds trivially. Let $B \cap \operatorname{Min} A \neq \emptyset$. Suppose to the contrary that there exists $a \in B \cap \operatorname{Min} A$ such that $a \notin \operatorname{Li}(\operatorname{Min} A_n)$. Without loss of generality, we can assume that $a = 0$. Since $B = \operatorname{Li}(\ker A_n)$, there exists a sequence $\{a_n\}$ of X such that $a_n \to a = 0$ and $a_n \in \ker A_n$, for all sufficiently large n. Let $J = \{j \in N : a_j \in A_j \setminus \operatorname{Min} A_j\}$, where N is the set of all natural numbers. J can be regarded as a subsequence of N since $0 \notin \operatorname{Li}(\operatorname{Min} A_n)$. By the generalized domination property (GDP) for A_n, for every $j \in J$, there exist $b_j \in \operatorname{clMin} A_j$ and $k_j \in K$ such that $a_j = b_j + k_j$. Consider the following two cases.

(I) $\{b_j\}_{j\in J}$ converges to $a = 0$. Since $b_j \in \operatorname{clMin} A_j$, there exists a sequence $\{b_j^k\} \subset \operatorname{Min} A_j$ such that $b_j^k \to b_j$ as $k \to \infty$. Then there exists a strictly increasing function $\phi : J \to J$ such that $b_j^{\phi(j)} \to a = 0$. Let

$$\tilde{a}_n = \begin{cases} a_n, & n \in N \setminus J, \\ b_n^{\phi(n)}, & n \in J. \end{cases} \tag{7}$$

It is easy to see that $\tilde{a}_n \to a = 0$ and $\tilde{a}_n \in \operatorname{Min} A_n$ for all sufficiently large n. Thus, $a \in \operatorname{Li}(\operatorname{Min} A_n)$, a contradiction.

(II) $\{b_j\}_{j\in J}$ does not converge to $a = 0$. By the closedness of A_j, we have $b_j \in A_j$. Since A_j is star-shaped,

$$\begin{aligned} \left[a_j, b_j\right] &= \left\{x \in X : x = a_j + (1-\alpha)\left(b_j - a_j\right), \alpha \in [0,1]\right\} \\ &\subset A_j \cap \left(a_j - K\right) \end{aligned} \tag{8}$$

for all sufficiently large $j \in J$. Since G is a base of K, for every $j \in J$, there exist $g_j \in G$ and $\lambda_j > 0$ such that $b_j = a_j - \lambda_j g_j$. If $\lambda_j \to 0$, then $b_j \to 0$, a contradiction. Then there exists $\epsilon > 0$ such that, up to a subsequence, $\lambda_j > \epsilon$ for all $j \in J$. Take $\tilde{\lambda}_j = \lambda_j/\epsilon$ and $\tilde{g}_j = \epsilon g_j \in \epsilon G$. It follows that

$$\begin{aligned} c_j &:= a_j + \left(1 - \left(1 - \frac{1}{\tilde{\lambda}_j}\right)\right)\left(b_j - a_j\right) \\ &= a_j - \tilde{g}_j \in A_j \cap \left(a_j - K\right) \end{aligned} \tag{9}$$

for all sufficiently large $j \in J$. By the sequentially weak compactness of G, up to a subsequence, $\tilde{g}_j$ converges weakly to $g \neq 0$. In another word, $\{c_j\}_{j\in J}$ admits a subsequence converging weakly to $-g \neq 0$. We have $-g \in A \cap (-K)$, since $c_j \in A_j \cap (a_j - K)$ and $A = w - \operatorname{Ls} A_n$. It contradicts the minimality of $a = 0$. □

Theorem 11. *Let X be a normed linear space, let K be a pointed, closed, and convex cone with a sequentially weakly compact base G, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) $A = w - \operatorname{Ls} A_n$, $B = w - \operatorname{Li}(\ker A_n)$.

Then $B \cap \operatorname{Min} A \subset w - \operatorname{Li}(\operatorname{Min} A_n)$.

Proof. The conclusion follows from almost the same arguments as in Theorem 10. □

Remark 12. Theorems 10 and 11 generalize Theorems 3.1 and 3.2 of [5], respectively.

Remark 13. Note that if $B \cap \operatorname{Min} A = \emptyset$, the results of Theorems 10 and 11 are trivial. In the sequel we present some conditions under which the intersection is nonempty. We first recall some concepts and results.

Definition 14. Let $D \subset X$ and $x \in X$. The set $D \cap (x - K)$ is called a section of D at x and denoted by D_x.

Definition 15. A nonempty convex set D is said to be rotound when its boundary does not contain line segments.

Proposition 16. *If D is nonempty, closed, and star-shaped, then* $\ker D$ *is closed and convex.*

Proof. Let $x, y \in \ker D$ and $x(t) = ty + (1-t)x$, for all $t \in [0,1]$. For any $z \in D$ and any $\lambda \in [0,1]$, it follows that

$$\begin{aligned} &x(t) + \lambda(z - x(t)) \\ &\quad = x + [1 - (1-t)(1-\lambda)] \\ &\qquad \times \left\{\left[y + \frac{\lambda}{1-(1-t)(1-\lambda)}(z-y)\right] - x\right\}. \end{aligned} \tag{10}$$

Let

$$\begin{gathered} \alpha = 1 - (1-t)(1-\lambda), \qquad \beta = \frac{\lambda}{1-(1-t)(1-\lambda)}, \\ y(\beta) = y + \beta(z-y). \end{gathered} \tag{11}$$

Clearly $\alpha, \beta \in [0,1]$. It follows that

$$x(t) + \lambda(z - x(t)) = x + \alpha\left(y(\beta) - x\right). \tag{12}$$

Since $\{x, y\} \subset \ker D$, we have $y(\beta) \in D$ and

$$x(t) + \lambda(z - x(t)) \in D, \quad \forall \lambda \in [0,1]. \tag{13}$$

Therefore, $x(t) \in \ker D$ for all $t \in [0,1]$ and so $\ker D$ is convex.

Let $\{x_n\} \subset \ker D$ such that $x_n \to x^*$. Obviously $x^* \in D$ since D is closed. We will prove $x^* \in \ker D$. For any $z \in D$ and any $t \in [0,1]$, we have $x_n + t(z - x_n) \in D$. Letting $n \to \infty$, we have $x^* + t(z - x^*) \in D$ since D is closed. Thus, $\ker D$ is closed. □

Remark 17. Let $\{A_n\}$ be a sequence of nonempty closed and star-shaped subsets of X and $A_n \to A$ and $\ker A_n \to B$. By Proposition 3.1 of [17] and Propositions 8 and 16, B is a closed convex subset of $\ker A$.

The following proposition presents some conditions under which the intersection $B \cap \operatorname{Min} A$ is nonempty.

Proposition 18. *Let X be a normed linear space, let K be a pointed, closed, and convex cone with* $\operatorname{int} K \neq \emptyset$, *and let $\{A_n\}$ be a sequence of nonempty closed and star-shaped subsets of X. Let A and B be nonempty subsets of X. Assume that*

(i) $A_n \to A$ *and* $\ker A_n \to B$;

(ii) $Ls(\ker A_n \cap \operatorname{Min} A_n) \neq \emptyset$;

(iii) *B is rotound and $B = A_x$ for some $x \in X$, where A_x is the section of A at x (see Definition 14).*

Then $B \cap \operatorname{Min} A \neq \emptyset$.

Proof. Since $B = A_x$, it follows from Propositions 2.6 and 2.8 of Luc [11] that

$$B \cap \operatorname{Min} A \subset \operatorname{Min} B \subset \operatorname{Min} A. \tag{14}$$

This yields

$$B \cap \operatorname{Min} A = \operatorname{Min} B. \tag{15}$$

Taking into account the assumptions from Theorem 4.4 of Miglierina and Molho [5], we get

$$\operatorname{Ls} \operatorname{Min}(\ker A_n) \subset \operatorname{Min} B = B \cap \operatorname{Min} A. \tag{16}$$

By Proposition 2.6 of Luc [11],

$$\ker A_n \cap \operatorname{Min} A_n \subset \operatorname{Min}(\ker A_n). \tag{17}$$

It follows that

$$\begin{aligned} \emptyset &\neq \operatorname{Ls}(\ker A_n \cap \operatorname{Min} A_n) \\ &\subset \operatorname{Ls} \operatorname{Min}(\ker A_n) \subset B \cap \operatorname{Min} A. \end{aligned} \tag{18}$$

□

The following example further illustrates the results of Theorems 10 and 11.

Example 19. Let $X = R^2, K = R^2_+$, and

$$\begin{aligned} A_n = &\left\{(x,y) : x + y \geq -\frac{1}{n}, -\frac{1}{n} \leq x \leq 0, -\frac{1}{n} \leq y \leq 0\right\} \\ &\cup \left\{(x,0) : -\frac{2}{n} \leq x \leq -\frac{1}{n}\right\}, \quad \forall n \in N. \end{aligned} \tag{19}$$

Then K has a compact base, A_n is closed and star-shaped, and the generalized domination property (GDP) holds for A_n. By Theorem 10, we have

$$B \cap \operatorname{Min} A \subset \operatorname{Li}(\operatorname{Min} A_n). \tag{20}$$

Indeed, it is easy to see that

$$\begin{gathered} \ker A_n = \left\{(x,0) : -\frac{1}{n} \leq x \leq 0\right\}, \\ \operatorname{Min} A_n = \left\{\left(-\frac{2}{n}, 0\right)\right\} \cup \left\{(x,y) : x + y = -\frac{1}{n}, -\frac{1}{n} < x \leq 0\right\}, \\ B = \operatorname{Li}(\ker A_n) = \{(0,0)\}, \qquad A = \operatorname{Ls} A_n = \{(0,0)\}, \\ \operatorname{Min} A = \{(0,0)\}. \end{gathered} \tag{21}$$

Therefore,

$$B \cap \operatorname{Min} A \subset \operatorname{Li}(\operatorname{Min} A_n) = \{(0,0)\}. \tag{22}$$

The following example shows that the sequentially weak compactness of G is essential in Theorems 10 and 11.

Example 20. Let $X = l^2$ be endowed with the usual norm; let K be the nonnegative orthant. Let $\{e_n\}_n \in N$ be the canonical orthonormal base of K and

$$A_n = [-ne_n, 0] \cup [-ne_{n+1}, 0]. \tag{23}$$

It is easy to see that A_n is not convex but star-shaped and $\ker A_n = \{0\}$. Further we have

$$\begin{gathered} A = w - \operatorname{Ls}(A_n) = \{0\}, \qquad B = \operatorname{Li}(\ker A_n) = \{0\}, \\ \operatorname{Min} A_n = \{-ne_n, -ne_{n+1}\}, \\ A = \operatorname{Min} A = \{0\}, \qquad w - \operatorname{Li}(\operatorname{Min} A_n) = \emptyset. \end{gathered} \tag{24}$$

Theorem 21. *Let X be a normed linear space, let $A \subset X$, $K \subset X$ be a pointed, closed, and convex cone, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) *$B = Li(\ker A_n)$ and $e_\rho(A_n, A) \to 0$ for all $\rho > 0$.*

Then $B \cap \operatorname{StMin} A \subset Li(\operatorname{Min} A_n)$.

Proof. If $B \cap \operatorname{StMin} A = \emptyset$, then the conclusion holds trivially. Let $B \cap \operatorname{StMin} A \neq \emptyset$. Suppose to the contrary that there exists $a \in B \cap \operatorname{StMin} A$ such that $a \notin \operatorname{LiMin} A_n$. Without loss of generality, we can suppose that $a = 0$. Then there exists a sequence $\{a_n\}$ of X such that $a_n \to a = 0$ and $a_n \in \ker A_n$, for all sufficiently large n. Let $J = \{j \in N : a_j \notin \operatorname{Min} A_j\}$. J can be regarded as a subsequence of N since $a \notin \operatorname{LiMin} A_n$. Since the generalized domination property (GDP) holds for A_j, there exists $b_j \in \operatorname{clMin} A_j$ such that $b_j \in a_j - K$, for all $j \in J$. The closedness of A_j implies $b_j \in A_j$. It follows from the star-shapedness of A_j that

$$[a_j, b_j] \subset A_j \cap (a_j - K) \tag{25}$$

for all sufficiently large $j \in J$. By assumption (iii), for any $\rho > 0$ and for any $\epsilon > 0$, we have

$$A_n \cap \mathscr{B}_\rho \subset A + \mathscr{B}_\epsilon \quad \forall \text{sufficiently large } n. \tag{26}$$

Since $a_n \to 0$, it follows from (25) and (26) that, for any $\epsilon > 0$,

$$\begin{gathered} [a_j, b_j] \cap \mathscr{B}_\rho \subset (A + \mathscr{B}_\epsilon) \cap (\mathscr{B}_\epsilon - K) \\ \forall \text{sufficiently large } j \in J. \end{gathered} \tag{27}$$

Now we prove that the following property holds: for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$(A + \mathscr{B}_\eta) \cap (\mathscr{B}_\eta - K) \subset \mathscr{B}_\epsilon. \tag{28}$$

If it is not the case, then $\exists \epsilon_0 > 0$, for all $\eta > 0$, $\exists x \in A, b_\eta, b'_\eta \in \mathscr{B}_\eta, k \in K$, such that

$$x + b_\eta = b'_\eta - k, \qquad \left\|x + b_\eta\right\| > \epsilon_0. \tag{29}$$

Since $0 \in \mathrm{StMin}\, A$, there exists $\eta_0 > 0$ such that

$$A \cap \left(\mathscr{B}_{\eta_0} - K\right) \subset \mathscr{B}_{\epsilon_0/2}. \tag{30}$$

We can choose η in (29) such that $\eta < \min\{\eta_0/2, \epsilon_0/2\}$. It follows that

$$x = b'_\eta - b_\eta - k \in \left(\mathscr{B}_{2\eta} - K\right) \cap A \subset \left(\mathscr{B}_{\eta_0} - K\right) \cap A. \tag{31}$$

This together with (30) implies that $\|x\| \le \epsilon_0/2$. But from (29), one has

$$\|x\| \ge \epsilon_0 - \left\|b_\eta\right\| > \epsilon_0 - \frac{\epsilon_0}{2} = \frac{\epsilon_0}{2}, \tag{32}$$

a contradiction. Thus, (28) holds. It follows from (27) and (28) that, for any $\epsilon > 0$,

$$\left[a_j, b_j\right] \cap \mathscr{B}_\rho \subset \mathscr{B}_\epsilon \quad \forall \text{sufficiently large } j \in J. \tag{33}$$

This arrives at a contradiction since $0 \notin \mathrm{Li}(\mathrm{Min}\, A_n)$ and $\{b_j\}_{j\in J}$ does not converge to 0. □

Remark 22. Theorem 21 generalizes Theorem 3.5 of [5] to the star-shaped case.

Example 23. Let $X = R^2, K = R^2_+$,

$$\begin{aligned} A_n &= \{(x, y) : x + y = 1, 0 \le x \le 1\} \\ &\quad \cup \left\{(x, 0) : 1 \le x \le 2 + \frac{1}{n}\right\}, \\ &\qquad\qquad \forall n \in N, \\ A &= \{(x, y) : x + y = 1, 0 \le x \le 1\} \\ &\quad \cup \{(x, 0) : 1 \le x \le 2\}. \end{aligned} \tag{34}$$

It is easy to see that all assumptions of Theorem 21 hold. By Theorem 21, we have

$$B \cap \mathrm{StMin}\, A \subset \mathrm{Li}\left(\mathrm{Min}\, A_n\right). \tag{35}$$

Indeed, it is easily seen that

$$\begin{aligned} \ker A_n &= \{(1, 0)\}, \\ \mathrm{Min}\, A_n &= \{(x, y) : x + y = 1, 0 \le x \le 1\}, \\ B &= \mathrm{Li}\left(\ker A_n\right) = \{(1, 0)\}, \\ \mathrm{Min}\, A = \mathrm{StMin}\, A &= \{(x, y) : x + y = 1, 0 \le x \le 1\}. \end{aligned} \tag{36}$$

Thus,

$$\begin{aligned} \{(1, 0)\} &= B \cap \mathrm{StMin}\, A \subset \mathrm{Li}\left(\mathrm{Min}\, A_n\right) \\ &= \{(x, y) : x + y = 1, 0 \le x \le 1\}. \end{aligned} \tag{37}$$

Theorem 24. *Let X be a normed linear space, let $K \subset X$ be a pointed, closed, and convex cone, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) *$B = Li(\ker A_n)$ and $e_\rho(A_n, A) \to 0$ for all $\rho > 0$;*

(iv) *$\mathrm{StMin}\, A \cap \mathscr{B}_\rho \cap B$ is relatively compact for every $\rho > 0$.*

Then, for each $\rho > 0$, $\lim_{n\to\infty} e_\rho(B \cap \mathrm{StMin}\, A, \mathrm{Min}\, A_n) = 0$.

Proof. If $B \cap \mathrm{StMin}\, A = \emptyset$, then the conclusion holds trivially. Let $B \cap \mathrm{StMin}\, A \ne \emptyset$. Suppose on the contrary that the conclusion of the theorem does not hold. Then there exist $\rho > 0, \epsilon > 0$, and a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that

$$e_\rho\left(B \cap \mathrm{StMin}\, A, \mathrm{Min}\, A_{n_k}\right) > 2\epsilon, \quad \forall k. \tag{38}$$

This yields that, for every k, there exists $m_k \in \mathscr{B}_\rho \cap B \cap \mathrm{StMin}\, A$ such that

$$d\left(m_k, \mathrm{Min}\, A_{n_k}\right) > 2\epsilon. \tag{39}$$

Since $\mathscr{B}_\rho \cap B \cap \mathrm{StMin}\, A$ is relatively compact, up to a subsequence, $m_k \to a \in \mathrm{cl}(\mathscr{B}_\rho \cap B \cap \mathrm{StMin}\, A)$. By Theorem 21, for each k,

$$m_k \in B \cap \mathrm{StMin}\, A \subset \mathrm{Li}\left(\mathrm{Min}\, A_n\right). \tag{40}$$

Then there exists a sequence $\{a^s_k\}_{s\in N}$ such that $a^s_k \to m_k$ as $s \to \infty$ and $a^s_k \in \mathrm{Min}\, A_s$, for all sufficiently large s. We can choose a strictly increasing function $\phi : N \to N$ such that $a^{\phi(k)}_k \to a$ as $k \to \infty$. Thus, $d(a, \mathrm{Min}\, A_{n_k}) \to 0$. It follows that

$$\begin{aligned} d\left(a, \mathrm{Min}\, A_{n_k}\right) &\ge d\left(m_k, \mathrm{Min}\, A_{n_k}\right) - d\left(m_k, a\right) > \epsilon \\ &\qquad \forall \text{sufficiently large } k, \end{aligned} \tag{41}$$

a contradiction. □

Remark 25. Theorem 24 generalizes Theorem 3.7 of [5] to the star-shaped case.

Proposition 26. *Let X be a normed linear space, let $K \subset X$ be a pointed, closed, and convex cone, and let A be a nonempty, closed, and star-shaped subset of X. Assume that, for every $x_0 \in \ker A \cap \mathrm{Min}\, A$, there exists a nondecreasing function $\delta_{x_0} : [0, +\infty) \to [0, +\infty)$ satisfying $\delta_{x_0}(0) = 0, \delta_{x_0}(t) > 0$, for all $t > 0$ and*

$$\frac{1}{2}\left(x_0 + x\right) + \mathscr{B}_{\delta_{x_0}(\|x_0 - x\|)} \subset A, \quad \forall x \in A. \tag{42}$$

Then

$$\ker A \cap \mathrm{Min}\, A = \ker A \cap \mathrm{StMin}\, A. \tag{43}$$

Proof. It is sufficient to prove $\ker A \cap \operatorname{Min} A \subset \ker A \cap \operatorname{StMin} A$. Suppose on the contrary that there exists $a \in \ker A \cap \operatorname{Min} A$ such that $a \notin \operatorname{StMin} A$. By the definition of StMin A, there exist $\delta > 0, \{q_n\} \subset X, \{k_n\} \subset K$, such that $q_n \to 0, q_n - k_n + a \in A$, and $\|a_n - k_n\| > \eta$, for all n. Take $x_n = q_n - k_n + a$. Then $d((x_n - a)/2, -K) \to 0$. By the minimality of a, we have $(-K) \cap (A - a) = \{0\}$. It follows that

$$d\left(\frac{x_n - a}{2}, X \setminus (A - a)\right) \longrightarrow 0. \tag{44}$$

By the assumption, there exists a nondecreasing function $\delta_a : [0, +\infty) \to [0, +\infty)$ satisfying $\delta_a(0) = 0$ and $\delta_a(t) > 0$ for all $t > 0$ such that

$$\frac{x_n + a}{2} + \mathscr{B}_{\delta_a(\|x_n - a\|)} \subset A. \tag{45}$$

This implies that

$$d\left(\frac{x_n - a}{2}, X \setminus (A - a)\right) \geq \delta_a\left(\|x_n - a\|\right), \tag{46}$$

contradicting (44). □

Remark 27. Proposition 26 is inspired by Proposition 3.9 of [5].

Corollary 28. *Let X be a normed linear space, let K be a pointed, closed, and convex cone, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) *$B = Li(\ker A_n)$ and $e_\rho(A_n, A) \to 0$ for all $\rho > 0$;*

(iv) *for every $x_0 \in \ker A \cap \operatorname{Min} A$, there exists a nondecreasing function $\delta_{x_0} : [0, +\infty) \to [0, +\infty)$ satisfying $\delta_{x_0}(0) = 0, \delta_{x_0}(t) > 0$ for all $t > 0$ and*

$$\frac{1}{2}(x_0 + x) + \mathscr{B}_{\delta_{x_0}(\|x_0 - x\|)} \subset A, \quad \forall x \in A. \tag{47}$$

Then

$$B \cap \operatorname{Min} A \subset Li\left(\operatorname{Min} A_n\right). \tag{48}$$

Proof. From Proposition 8, $B \subset \ker A$. By Theorem 21, $B \cap \operatorname{StMin} A \subset \operatorname{Li}(\operatorname{Min} A_n)$. By using assumption (iv), from Proposition 26, we have

$$\begin{aligned} B \cap \operatorname{StMin} A &= B \cap \ker A \cap \operatorname{StMin} A \\ &= B \cap \ker A \cap \operatorname{Min} A \\ &= B \cap \operatorname{Min} A \subset \operatorname{Li}\left(\operatorname{Min} A_n\right). \end{aligned} \tag{49}$$

The proof is complete. □

Corollary 29. *Let X be a normed linear space, let K be a pointed, closed, and convex cone, and let $\{A_n\}$ be a sequence of nonempty subsets of X. Assume that*

(i) *A_n is closed and star-shaped for all n;*

(ii) *the generalized domination property (GDP) holds for all A_n;*

(iii) *$B = Li(\ker A_n)$ and $e_\rho(A_n, A) \to 0$ for all $\rho > 0$;*

(iv) *$\operatorname{StMin} A \cap B \cap \mathscr{B}_\rho$ is relatively compact for all $\rho > 0$;*

(v) *for every $x_0 \in \ker A \cap \operatorname{Min} A$, there exists a nondecreasing function $\delta_{x_0} : [0, +\infty) \to [0, +\infty)$ satisfying $\delta_{x_0}(0) = 0, \delta_{x_0}(t) > 0$ for all $t > 0$ and*

$$\frac{1}{2}(x_0 + x) + \mathscr{B}_{\delta_{x_0}(\|x_0 - x\|)} \subset A, \quad \forall x \in A. \tag{50}$$

Then, for each $\rho > 0$,

$$\lim_{n \to \infty} e_\rho\left(B \cap \operatorname{Min} A, \operatorname{Min} A_n\right) = 0. \tag{51}$$

Proof. The conclusion follows from Propositions 8 and 26 and Theorem 24. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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