Research Article

The Center Conditions and Bifurcation of Limit Cycles at the Degenerate Singularity of a Three-Dimensional System

Shugang Song,¹ Jingjing Feng,¹ and Qinlong Wang²

¹ School of Information and Mathematics, Yangtze University, Jingzhou 434023, China

² School of Science, Hezhou University, Hezhou 542899, China

Correspondence should be addressed to Qinlong Wang; wqinlong@163.com

Received 25 April 2014; Revised 23 June 2014; Accepted 26 June 2014; Published 9 July 2014

Academic Editor: Junjie Wei

Copyright © 2014 Shugang Song et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate multiple limit cycles bifurcation and center-focus problem of the degenerate equilibrium for a three-dimensional system. By applying the method of symbolic computation, we obtain the first four quasi-Lyapunov constants. It is proved that the system can generate 3 small limit cycles from nilpotent critical point on center manifold. Furthermore, the center conditions are found and as weak foci the highest order is proved to be the fourth; thus we obtain at most 3 small limit cycles from the origin via local bifurcation. To our knowledge, it is the first example of multiple limit cycles bifurcating from a nilpotent singularity for the flow of a high-dimensional system restricted to the center manifold.

1. Introduction

About dynamical behavior of the trajectories of three-dimensional system, bifurcation of limit cycles is one of major concerns; particularly, for Hopf bifurcation of a nondegenerate equilibrium with a pair of pure imaginary roots and a negative one, many investigations have been carried out in the past decades, for example, [1–4] for the three-dimensional chaotic systems, [5–8] for the three-dimensional Lotka-Volterra systems, and [9] for general three-dimensional systems.

However, up till now, study on bifurcation of limit cycles from the degenerate singularity for high-dimensional nonlinear dynamical systems is hardly seen in published references. In this paper, we investigate the following three-dimensional systems:

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{k+2j+l=3}^{\infty} A_{kjl} x^k y^j u^l = X(x, y, u), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{k+2j+l=4}^{\infty} B_{kjl} x^k y^j u^l = Y(x, y, u), \quad (1) \\ \frac{du}{dt} &= -du + \sum_{k+j+l=2}^{\infty} d_{kjl} x^k y^j u^l = U(x, y, u),\end{aligned}$$

where $x, y, u, t, d, A_{kjl}, B_{kjl}, d_{kjl} \in \mathbb{R}$ ($d > 0, k, j, l \in \mathbb{N}$). Obviously system (1) has the Jacobian matrix at the origin as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & -d \end{pmatrix}, \quad \text{where } A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (2)$$

One notices that the origin has two zero eigenvalues and one negative eigenvalue and the block matrix A_1 is nilpotent. From the center manifold theorem, for the system (1), there exists the center manifold: $u = u(x, y)$ with $u(0, 0) = 0$, $Du(0, 0) = 0$, and more the flow on the center manifold is governed by a/a

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} X(x, y, u(x, y)) - y \\ Y(x, y, u(x, y)) \end{pmatrix}, \quad (3)$$

namely,

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{k+2j=3}^{\infty} a_{kj} x^k y^j = \tilde{X}(x, y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{k+2j=4}^{\infty} b_{kj} x^k y^j = \tilde{Y}(x, y),\end{aligned} \quad (4)$$

where $x, y, t, a_{kj}, b_{kj} \in \mathbb{R}$ ($k, j \in \mathbb{N}$). It is usually called the reduced system and system (1) is topological equivalent to system (4) in the vicinity of the origin [10]. Thus by investigating system (4), we can solve Hopf bifurcation of the origin on center manifold of the three-dimensional system (1). At the same time, we also discuss the center-focus problem for the flow restricted to the center manifold, which closely relates to the maximum number of limit cycles bifurcating from the origin.

As far as investigation about limit cycles bifurcation of a nilpotent critical point in planar system is concerned, it is similar to nondegenerate case that detecting nilpotent center and calculating the focal value are needed [11]. There exist some available and classical ways, for instance, Poincare return map [12], Lyapunov function [13], and the normal form theory [14]. At the same time, some good results on the cyclicity were obtained [11, 12, 15, 16]. Recently in [17, 18], the authors gave an inverse integral factor method of calculating the quasi-Lyapunov constants of the three-order nilpotent critical point; it is convenient to compute the higher order focal values and solve the center-focus problem of the equilibrium. Here we extend this method's application to the three-dimensional system (1) and consider its specific example as follows:

$$\begin{aligned}\frac{dx}{dt} &= y + a_0 ux - 2xy = X, \\ \frac{dy}{dt} &= y^2 - 2x^3 + b_0 uy = Y, \\ \frac{du}{dt} &= -u + d_1 xy = U,\end{aligned}\quad (5)$$

where $a_0, b_0, d_1 \in \mathbb{R}$.

The rest of this paper is organized as follows. In Section 2, the corresponding quasi-Lyapunov constants are computed and the center conditions on the center manifold are determined. In Section 3, the multiple local bifurcations at the origin for system (5) are investigated, three limit cycles from the origin are obtained, and it is proved at most three small limit cycles from the origin via local bifurcation. In this work, the system and problem are all considered for the first time.

2. Quasi-Lyapunov Constants and Center Conditions

In this part, we firstly investigate the singular point quantities of the origin. For the center manifold of system (5), one can determine the formal expression $u = \tilde{u}(x, y)$ and obtain the same form as system (4):

$$\begin{aligned}\frac{dx}{dt} &= y + a_0 \tilde{u}x - 2xy = \tilde{X}, \\ \frac{dy}{dt} &= y^2 - 2x^3 + b_0 \tilde{u}y = \tilde{Y}.\end{aligned}\quad (6)$$

Furthermore, we give the definition of quasi-Lyapunov constants for system (6) and the way of computing them; more details can be found in [17–19].

Lemma 1 (Theorems 8.7.1 and 8.7.2 in [17]). *For system (6), any positive integer s and a given number sequence $\{c_{0\beta}\}$, $\beta \geq 3$, we can derive successively and uniquely the terms of the following formal series with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$,*

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^\alpha y^\beta \quad (7)$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=6}^{\infty} \omega_m x^m. \quad (8)$$

And, if $\alpha \neq 0$, $c_{\alpha\beta}$ is determined by the following recursive formula:

$$\begin{aligned}c_{\alpha\beta} &= \frac{1}{(s+1)\alpha} \sum_{k+j=2}^{\alpha+\beta-1} \{[k - (s+1)(\alpha - k)] a_{kj} c_{\alpha-k, \beta-j+1} \\ &\quad + [j - (s+1)(\beta - j + 2)] \\ &\quad \times b_{kj} c_{\alpha-k-1, \beta-j+2}\}\end{aligned}\quad (9)$$

and, for any positive integer $m \geq 6$, ω_m is determined by the following recursive formula:

$$\begin{aligned}\omega_m &= \sum_{k+j=2}^{m-1} [(k - (s+1)(\alpha - k + 1)) a_{kj} c_{m-k+1, -j} \\ &\quad + (j - (s+1)(1 - j)) b_{kj} c_{m-k, 1-j}]\end{aligned}\quad (10)$$

and, when $\alpha < 0$ or $\beta < 0$, we have let $c_{\alpha\beta} = 0$.

Particularly, by choosing appropriate s and number sequence $\{c_{0\beta}\}$ ($\beta \geq 3$), we can make $\omega_{2i+1}(s) = 0$ and then let

$$\omega_{2i+4}(s) = (2i - 4s - 1) \lambda_i. \quad (11)$$

Definition 2. The λ_i in (11) is called the i th quasi-Lyapunov constant of the origin of system (6), $i = 1, 2, \dots$, and more if all the quasi-Lyapunov constants vanish, that is, $\lambda_m = 0$, $m = 1, 2, \dots$, then the origin of system (5) is a center on the local center manifold at the origin.

Lemma 3. *For system (6), the m th focal value $v_{2m}(-2\pi)$ at the origin of system (6) is algebraic equivalent to $\sigma_m \lambda_m$; that is, for any positive integer $m = 2, 3, \dots$, if $v_2 = v_4 = \dots = v_{2m-2} = 0$ and $\lambda_1 = \lambda_2 = \dots = \lambda_{m-1} = 0$ hold, then $v_{2m} = \sigma_m \lambda_m$, where σ_m is a defined constant (which is given in [17, 18]).*

Remark 4. According to Lemma 1, one sees that each ω_m in (10) is related to only the coefficients of the first $m - 1$ degree terms of system (6). Here we determine the above $\tilde{u}(x, y)$ only up to a tenth degree polynomial with respect to x, y as follows:

$$\begin{aligned}\tilde{u}(x, y) &= d_1 y(x - y) + d_1 y^2(x + y) \\ &\quad + u_4 + u_5 + \dots + u_{10},\end{aligned}\quad (12)$$

where $u_i, i = 4, \dots, 10$ is a homogeneous polynomial in x, y of degree i , respectively, (which can be seen in the Appendix); thus $\bar{X}$ and $\bar{Y}$ in system (6) are two polynomials with degree 11. And more all a_{kj}, b_{kj} of $c_{\alpha\beta}$ in (9) are given definitely by the coefficients a_0, b_0, d_1 (the specific $c_{\alpha\beta}$ is available in Email address of the corresponding author); the ω_m in (10) is given by the following specific form:

$$\begin{aligned} \omega_m = & -32d_1(686 - 94a_0d_1 - 31b_0d_1) \\ & \times (21a_0 + b_0 - a_0m + 10a_0s - a_0ms) \\ & \times c[m - 10, 0] \\ & - 2208d_1(19a_0 + b_0 - a_0m + 9a_0s - a_0ms) \\ & \times c[m - 9, 0] \\ & + 4d_1(144 - 11a_0d_1 - 3b_0d_1) \\ & \times (17a_0 + b_0 - a_0m + 8a_0s - a_0ms) \\ & \times c[m - 8, 0] \\ & + 40d_1(15a_0 + b_0 - a_0m + 7a_0s - a_0ms) \\ & \times c[m - 7, 0] \\ & - 24d_1(13a_0 + b_0 - a_0m + 6a_0s - a_0ms) \\ & \times c[m - 6, 0] \\ & + 2d_1(9a_0 + b_0 - a_0m + 4a_0s - a_0ms) \\ & \times c[m - 4, 0] + 2(1 + s)c[m - 3, 0]. \end{aligned} \quad (13)$$

Applying the powerful symbolic computation function of Mathematica system and the recursive formulas in Lemma 1, and from Remark 4, we obtain the first 7 quantities as follows:

$$\begin{aligned} \omega_6 &= 0, \\ \omega_7 &= 3(1 + s)c_{03}, \\ \omega_8 &= \frac{4}{5}d_1(2a_0 - b_0)(3 - 4s), \\ \omega_9 &= \frac{2}{3}a_0d_1(1 - s), \\ \omega_{10} &= \frac{2}{7}a_0d_1(1 - 6a_0d_1), \\ \omega_{11} &= \frac{1}{112}(73a_0d_1 - 704a_0^2d_1^2 + 840c_{05}), \\ \omega_{12} &= -\frac{397}{105}a_0d_1. \end{aligned} \quad (14)$$

In the above expression of each $\omega_k, k = 7, \dots, 12$, we have already let $\omega_6 = \dots = \omega_{k-1} = 0$.

Particularly, in order to make $\omega_{2i+1} = 0, i = 1, \dots, 5$, we let $s = 1$ and choose

$$c_{03} = 0, \quad c_{05} = \frac{1}{840}(704a_0^2d_1^2 - 73a_0d_1). \quad (15)$$

Thus from the expression (11) in Lemma 1, we have the following.

Theorem 5. For the flow on center manifold of system (5), we get the first 4 quasi-Lyapunov constants of the origin as follows:

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \frac{4}{5}d_1(2a_0 - b_0), \\ \lambda_3 &= \frac{2}{7}a_0d_1(1 - 6a_0d_1), \\ \lambda_4 &= -\frac{397}{315}a_0d_1. \end{aligned} \quad (16)$$

Theorem 6. For system (5), the origin is a center on the local center manifold if and only if the following condition is satisfied:

$$d_1 = 0 \quad \text{or} \quad a_0 = b_0 = 0. \quad (17)$$

Proof. The proof of the necessity is easy; then we omit it. Now we prove the sufficient condition; this technique derives from the method of Darboux (also see [20–23]). Obviously, if $d_1 = 0$ in the conditions (17) holds, then system (5) has the corresponding form as follows:

$$\frac{dx}{dt} = y + a_0ux - 2xy, \quad (18)$$

$$\frac{dy}{dt} = y^2 - 2x^3 + b_0uy, \quad \frac{du}{dt} = -u.$$

And more we can figure out easily one algebraic invariant surface for system (18): $F(x, y, u) = u$; in fact, there exists a polynomial $K(x, y, u) = -1$, the cofactor of $F(x, y, u)$, such that $dF/dt|_{(17)} = KF$. One can observe that $F(x, y, u) = 0$ is tangent to the center eigenspace, the (x, y) -plane, at the origin. Thus it forms a local center manifold in a neighborhood of the origin. From $F(x, y, u) = u = 0$, we substitute it into the first and second equations of the system defined by system (18) and we have the differential equations

$$\frac{dx}{dt} = y - 2xy, \quad \frac{dy}{dt} = y^2 - 2x^3 \quad (19)$$

which is a Hamiltonian system with Hamiltonian function: $H(x, y) = x^4 - 2xy^2 + y^2$. Therefore the origin is a center for systems (18) or (19) as shown in Figure 1(a).

And if $a_0 = b_0 = 0$ in the conditions (17) holds, then system (5) has the corresponding form as follows:

$$\frac{dx}{dt} = y - 2xy, \quad \frac{dy}{dt} = y^2 - 2x^3, \quad \frac{du}{dt} = -u + d_1xy. \quad (20)$$

Clearly, the above $H = x^4 - 2xy^2 + y^2$ is also a first integral of system (20); thus the origin is a center for the flow of system (5) restricted to a center manifold. $\square$

According to Lemma 3 and Theorem 6, then we have the following.

FIGURE 1: (a) The phase portrait of system (19) on the (x, y) -plane which shows that it is integrable at the origin; (b) the stable limit cycle bifurcated from the origin of system (24) with $\delta = 0.001$, $d_1 = 1$, $a_0 = 1/6$, $b_0 = 1/3 + 0.001$.

Theorem 7. For the origin of system (5) as a weak focus on center manifold, the highest is the fourth order and its first 4 focal values are as follows:

$$\begin{aligned} v_2(-2\pi) &= \sigma_1 \lambda_1, & v_4(-2\pi) &= \sigma_2 \lambda_2, \\ v_6(-2\pi) &= \sigma_3 \lambda_3, & v_8(-2\pi) &= \sigma_4 \lambda_4, \end{aligned} \quad (21)$$

where, for the expression of v_4 , we have let $v_2 = 0$, and $v_2 = v_4 = 0$ for v_6 , $v_2 = v_4 = v_6 = 0$ for v_8 .

3. Multiple Limit Cycle Bifurcation of the System

In this section, we apply the focal values obtained in last section to discuss multiple bifurcation of the equilibrium and demonstrate there exist 3 and at most 3 limit cycles.

From Theorems 5 and 7, we have

Theorem 8. For the flow on center manifold of (5), the origin is a 4th-order fine focus; that is, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ if and only if

$$\begin{aligned} b_0 &= 2a_0, & 6a_0d_1 &= 1 \\ \text{or } b_0 &= \frac{1}{3d_1}, & a_0 &= \frac{1}{6d_1} \end{aligned} \quad (22)$$

$(d_1 \neq 0).$

Proof. Firstly, we prove the necessity, from the expressions (16) in Theorem 5; we let $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and consider $\lambda_4 \neq 0$ and then only $b_0 = 2a_0$ and $6a_0d_1 = 1$ are obtained at the same time. Then for the proof of the sufficiency, by substituting the above conditions into $\lambda_i, i = 1, 2, 3, 4$, one can get the conclusion easily. Furthermore we can transform

the above conditions into the following form: $b_0 = 1/3d_1$, $a_0 = (1/6d_1)$ ($d_1 \neq 0$). Thus the proof is completed. $\square$

Remark 9. For the coefficients of system (5) or (6), when $d_1 \neq 0$, there always exists a group of critical values: $a_0 = 1/6d_1$, $b_0 = 1/3d_1$ such that the conditions (22) hold.

From Theorem 5, one calculates easily the Jacobian determinant with respect to the functions λ_2, λ_3 and variables a_0, b_0 ,

$$J = \frac{\partial(\lambda_2, \lambda_3)}{\partial(a_0, b_0)} = \frac{8}{35}d_1^2(1 - 12a_0d_1). \quad (23)$$

Considering the conditions (22) and when $d_1 \neq 0$ holds, we obtain $J = -(8/35)d_1^2 \neq 0$. At the same time, one can get that $\lambda_4 = -(397/1890) \neq 0$ holds.

Thus we take some appropriate perturbations for the coefficients of system (6) and get the perturbed system with the following form:

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y + a_0 \tilde{u}x - 2xy, \\ \frac{dy}{dt} &= 2\delta y + y^2 - 2x^3 + b_0 \tilde{u}y, \end{aligned} \quad (24)$$

where $0 < \delta \ll 1$. According to Theorem 2.2 in [18], system (6) has 3 limit cycles in the neighborhood of the origin; that is, system (5) has 3 limit cycles on the center manifold necessarily. One can refer to [9, 17, 18] for more details about constructing of limit cycles. And using numerical method, we get an example of one stable limit cycle for system (24) as shown in Figure 1(b).

Therefore, from the above discussion and Theorem 7, we have the following.

Theorem 10. For system (5), 3 and at most 3 small amplitude limit cycles can be bifurcated from the origin on the center manifold.

4. Conclusions

In summary, based on precise symbolic computation, we have investigated deeply multiple limit cycles bifurcation in the vicinity of the degenerate equilibrium for a three-dimensional system. Firstly by computing its quasi-Lyapunov constants, we solved its center-focus problem on center manifold; thus the center conditions are found and as weak foci the highest order is proved to be the fourth. Then we obtained that the system can generate 3 small limit cycles from nilpotent critical point and at most 3 small limit cycles from the origin via local bifurcation. For the flow of a high-dimensional system restricted to the center manifold, only one interesting example of multiple limit cycles bifurcating from its nilpotent singularity was given here, but we believe that more outcomes on this problem will be shown in the near future.

Appendix

Consider

$$\begin{aligned}
 u_4 &= -d_1 \left(-2x^4 + 12x^3y - 36x^2y^2 + a_0d_1x^2y^2 \right. \\
 &\quad \left. + b_0d_1x^2y^2 + 72xy^3 - 3a_0d_1xy^3 \right. \\
 &\quad \left. - 5b_0d_1xy^3 - 69y^4 + 3a_0d_1y^4 + 7b_0d_1y^4 \right), \\
 u_5 &= d_1y \left(20x^4 - 134x^3y + 474x^2y^2 - 4a_0d_1x^2y^2 \right. \\
 &\quad \left. - 5b_0d_1x^2y^2 - 876xy^3 + 5a_0d_1xy^3 \right. \\
 &\quad \left. + 5b_0d_1xy^3 + 600y^4 + 7a_0d_1y^4 + 28b_0d_1y^4 \right), \\
 u_6 &= -d_1 \left(24x^6 - 288x^5y + 14a_0d_1x^5y + 6b_0d_1x^5y \right. \\
 &\quad \left. + 1732x^4y^2 - 144a_0d_1x^4y^2 - 88b_0d_1x^4y^2 \right. \\
 &\quad \left. - 6944x^3y^3 + 744a_0d_1x^3y^3 + 552b_0d_1x^3y^3 \right. \\
 &\quad \left. - 3a_0^2d_1^2x^3y^3 - 6a_0b_0d_1^2x^3y^3 - 3b_0^2d_1^2x^3y^3 \right. \\
 &\quad \left. + 20358x^2y^4 - 2443a_0d_1x^2y^4 - 2081b_0d_1x^2y^4 \right. \\
 &\quad \left. + 17a_0^2d_1^2x^2y^4 + 46a_0b_0d_1^2x^2y^4 + 33b_0^2d_1^2x^2y^4 \right. \\
 &\quad \left. - 42468xy^5 + 5038a_0d_1xy^5 + 4882b_0d_1xy^5 \right. \\
 &\quad \left. - 40a_0^2d_1^2xy^5 - 134a_0b_0d_1^2xy^5 - 126b_0^2d_1^2xy^5 \right. \\
 &\quad \left. + 45468y^6 - 5003a_0d_1y^6 - 5153b_0d_1y^6 \right. \\
 &\quad \left. + 40a_0^2d_1^2y^6 + 152a_0b_0d_1^2y^6 + 168b_0^2d_1^2y^6 \right),
 \end{aligned}$$

$$\begin{aligned}
 u_7 &= d_1 \left(40x^7 - 1104x^6y + 12060x^5y^2 \right. \\
 &\quad \left. - 260a_0d_1x^5y^2 - 128b_0d_1x^5y^2 - 77700x^4y^3 \right. \\
 &\quad \left. + 2860a_0d_1x^4y^3 + 1700b_0d_1x^4y^3 + 337632x^3y^4 \right. \\
 &\quad \left. - 15498a_0d_1x^3y^4 - 10704b_0d_1x^3y^4 + 24a_0^2d_1^2x^3y^4 \right. \\
 &\quad \left. + 56a_0b_0d_1^2x^3y^4 + 33b_0^2d_1^2x^3y^4 - 1012896x^2y^5 \right. \\
 &\quad \left. + 49266a_0d_1x^2y^5 + 39042b_0d_1x^2y^5 - 94a_0^2d_1^2x^2y^5 \right. \\
 &\quad \left. - 251a_0b_0d_1^2x^2y^5 - 169b_0^2d_1^2x^2y^5 + 1898388xy^6 \right. \\
 &\quad \left. - 84891a_0d_1xy^6 - 72276b_0d_1xy^6 + 66a_0^2d_1^2xy^6 \right. \\
 &\quad \left. + 67a_0b_0d_1^2xy^6 - 166b_0^2d_1^2xy^6 - 1625580y^7 \right. \\
 &\quad \left. + 54873a_0d_1y^7 + 45075b_0d_1y^7 + 174a_0^2d_1^2y^7 \right. \\
 &\quad \left. + 915a_0b_0d_1^2y^7 + 1419b_0^2d_1^2y^7 \right), \\
 u_8 &= -d_1 \left(-576x^8 + 44a_0d_1x^8 + 12b_0d_1x^8 \right. \\
 &\quad \left. + 10976x^7y - 1264a_0d_1x^7y - 496b_0d_1x^7y \right. \\
 &\quad \left. - 106352x^6y^2 + 16048a_0d_1x^6y^2 \right. \\
 &\quad \left. + 7696b_0d_1x^6y^2 \right. \\
 &\quad \left. - 114a_0^2d_1^2x^6y^2 - 116a_0b_0d_1^2x^6y^2 \right. \\
 &\quad \left. - 34b_0^2d_1^2x^6y^2 + 704496x^5y^3 \right. \\
 &\quad \left. - 126632a_0d_1x^5y^3 - 69528b_0d_1x^5y^3 \right. \\
 &\quad \left. + 1700a_0^2d_1^2x^5y^3 + 2172a_0b_0d_1^2x^5y^3 \right. \\
 &\quad \left. + 816b_0^2d_1^2x^5y^3 - 3558660x^4y^4 \right. \\
 &\quad \left. + 710088a_0d_1x^4y^4 + 428808b_0d_1x^4y^4 \right. \\
 &\quad \left. - 12764a_0^2d_1^2x^4y^4 - 19064a_0b_0d_1^2x^4y^4 \right. \\
 &\quad \left. - 8340b_0^2d_1^2x^4y^4 + 14a_0^3d_1^3x^4y^4 \right. \\
 &\quad \left. + 42a_0^2b_0d_1^3x^4y^4 + 42a_0b_0^2d_1^3x^4y^4 \right. \\
 &\quad \left. + 14b_0^3d_1^3x^4y^4 + 14104992x^3y^5 \right. \\
 &\quad \left. - 2960538a_0d_1x^3y^5 - 1907570b_0d_1x^3y^5 \right. \\
 &\quad \left. + 61685a_0^2d_1^2x^3y^5 + 103855a_0b_0d_1^2x^3y^5 \right. \\
 &\quad \left. + 50130b_0^2d_1^2x^3y^5 - 125a_0^3d_1^3x^3y^5 \right. \\
 &\quad \left. - 463a_0^2b_0d_1^3x^3y^5 - 591a_0b_0^2d_1^3x^3y^5 \right. \\
 &\quad \left. - 261b_0^3d_1^3x^3y^5 - 43327872x^2y^6 \right. \\
 &\quad \left. + 9065880a_0d_1x^2y^6 + 6166752b_0d_1x^2y^6 \right)
 \end{aligned}$$

$$\begin{aligned}
& -200895a_0^2d_1^2x^2y^6 - 374342a_0b_0d_1^2x^2y^6 \\
& -196423b_0^2d_1^2x^2y^6 + 504a_0^3d_1^3x^2y^6 \\
& + 2186a_0^2b_0d_1^3x^2y^6 + 3360a_0b_0^2d_1^3x^2y^6 \\
& + 1854b_0^3d_1^3x^2y^6 + 94249296xy^7 \\
& - 18564504a_0d_1xy^7 - 13274868b_0d_1xy^7 \\
& + 412530a_0^2d_1^2xy^7 + 833201a_0b_0d_1^2xy^7 \\
& + 468601b_0^2d_1^2xy^7 - 1097a_0^3d_1^3xy^7 \\
& - 5317a_0^2b_0d_1^3xy^7 - 9303a_0b_0^2d_1^3xy^7 \\
& - 6011b_0^3d_1^3xy^7 - 105628356y^8 \\
& + 18948615a_0d_1y^8 + 13977981b_0d_1y^8 \\
& - 411312a_0^2d_1^2y^8 - 868420a_0b_0d_1^2y^8 \\
& - 503532b_0^2d_1^2y^8 + 1097a_0^3d_1^3y^8 \\
& + 5673a_0^2b_0d_1^3y^8 + 10687a_0b_0^2d_1^3y^8 \\
& + 7551b_0^3d_1^3y^8),
\end{aligned}$$

$$\begin{aligned}
u_9 = d_1 & (-2208x^9 + 77328x^8y - 2384a_0d_1x^8y \\
& - 784b_0d_1x^8y - 1227512x^7y^2 \\
& + 64396a_0d_1x^7y^2 + 26272b_0d_1x^7y^2 \\
& + 12357160x^6y^3 - 835460a_0d_1x^6y^3 \\
& - 396608b_0d_1x^6y^3 + 3144a_0^2d_1^2x^6y^3 \\
& + 3498a_0b_0d_1^2x^6y^3 + 1096b_0^2d_1^2x^6y^3 \\
& - 89203392x^5y^4 + 6916384a_0d_1x^5y^4 \\
& + 3676396b_0d_1x^5y^4 - 49214a_0^2d_1^2x^5y^4 \\
& - 63780a_0b_0d_1^2x^5y^4 - 22834b_0^2d_1^2x^5y^4 \\
& + 483032256x^4y^5 - 40282356a_0d_1x^4y^5 \\
& - 23266264b_0d_1x^4y^5 + 378254a_0^2d_1^2x^4y^5 \\
& + 553724a_0b_0d_1^2x^4y^5 + 217218b_0^2d_1^2x^4y^5 \\
& - 180a_0^3d_1^3x^4y^5 - 607a_0^2b_0d_1^3x^4y^5 \\
& - 687a_0b_0^2d_1^3x^4y^5 - 261b_0^3d_1^3x^4y^5 \\
& - 1968992136x^3y^6 + 169196340a_0d_1x^3y^6 \\
& + 104279988b_0d_1x^3y^6 - 1780906a_0^2d_1^2x^3y^6 \\
& - 2890212a_0b_0d_1^2x^3y^6 - 1233114b_0^2d_1^2x^3y^6
\end{aligned}$$

$$\begin{aligned}
& + 1292a_0^3d_1^3x^3y^6 + 4961a_0^2b_0d_1^3x^3y^6 \\
& + 6477a_0b_0^2d_1^3x^3y^6 + 2877b_0^3d_1^3x^3y^6 \\
& + 5820320664x^2y^7 - 495558804a_0d_1x^2y^7 \\
& - 321861864b_0d_1x^2y^7 + 5224500a_0^2d_1^2x^2y^7 \\
& + 9158682a_0b_0d_1^2x^2y^7 + 4161294b_0^2d_1^2x^2y^7 \\
& - 3274a_0^3d_1^3x^2y^7 - 12840a_0^2b_0d_1^3x^2y^7 \\
& - 16145a_0b_0^2d_1^3x^2y^7 - 5819b_0^3d_1^3x^2y^7 \\
& - 11169394848xy^8 + 901925700a_0d_1xy^8 \\
& + 606394284b_0d_1xy^8 - 8530348a_0^2d_1^2xy^8 \\
& - 15428426a_0b_0d_1^2xy^8 - 6980247b_0^2d_1^2xy^8 \\
& + 949a_0^3d_1^3xy^8 - 2716a_0^2b_0d_1^3xy^8 \\
& - 23065a_0b_0^2d_1^3xy^8 - 31601b_0^3d_1^3xy^8 \\
& + 10324368000y^9 - 750336780a_0d_1y^9 \\
& - 509164044b_0d_1y^9 + 5239852a_0^2d_1^2y^9 \\
& + 8941749a_0b_0d_1^2y^9 + 3331701b_0^2d_1^2y^9 \\
& + 7827a_0^3d_1^3y^9 + 50215a_0^2b_0d_1^3y^9 \\
& + 119361a_0b_0^2d_1^3y^9 + 108056b_0^3d_1^3y^9),
\end{aligned}$$

$$\begin{aligned}
u_{10} = -d_1 & (21952x^{10} - 3008a_0d_1x^{10} - 992b_0d_1x^{10} \\
& - 605184x^9y + 107232a_0d_1x^9y + 42144b_0d_1x^9y \\
& - 1104a_0^2d_1^2x^9y - 752a_0b_0d_1^2x^9y - 160b_0^2d_1^2x^9y \\
& + 8513712x^8y^2 - 1841888a_0d_1x^8y^2 \\
& - 822896b_0d_1x^8y^2 + 34440a_0^2d_1^2x^8y^2 \\
& + 29616a_0b_0d_1^2x^8y^2 + 7896b_0^2d_1^2x^8y^2 \\
& - 81848832x^7y^3 + 20765520a_0d_1x^7y^3 \\
& + 10179424b_0d_1x^7y^3 - 527376a_0^2d_1^2x^7y^3 \\
& - 530448a_0b_0d_1^2x^7y^3 - 162816b_0^2d_1^2x^7y^3 \\
& + 1050a_0^3d_1^3x^7y^3 + 1774a_0^2b_0d_1^3x^7y^3 \\
& + 1102a_0b_0^2d_1^3x^7y^3 + 250b_0^3d_1^3x^7y^3 \\
& + 602777304x^6y^4 - 173386212a_0d_1x^6y^4 \\
& - 90723084b_0d_1x^6y^4 + 5270270a_0^2d_1^2x^6y^4 \\
& + 5948592a_0b_0d_1^2x^6y^4 + 2010036b_0^2d_1^2x^6y^4
\end{aligned}$$

$$\begin{aligned}
& -21020a_0^3d_1^3x^6y^4 - 41960a_0^2b_0d_1^3x^6y^4 \\
& -31132a_0b_0^2d_1^3x^6y^4 - 8640b_0^3d_1^3x^6y^4 \\
& -3601377936x^5y^5 + 1131810888a_0d_1x^5y^5 \\
& + 620482512b_0d_1x^5y^5 - 38357066a_0^2d_1^2x^5y^5 \\
& - 47295826a_0b_0d_1^2x^5y^5 - 17155148b_0^2d_1^2x^5y^5 \\
& + 212538a_0^3d_1^3x^5y^5 + 481702a_0^2b_0d_1^3x^5y^5 \\
& + 405182a_0b_0^2d_1^3x^5y^5 + 128818b_0^3d_1^3x^5y^5 \\
& - 85a_0^4d_1^4x^5y^5 - 340a_0^3b_0d_1^4x^5y^5 \\
& - 510a_0^2b_0^2d_1^4x^5y^5 - 340a_0b_0^3d_1^4x^5y^5 \\
& - 85b_0^4d_1^4x^5y^5 \\
& + 17877283056x^4y^6 - 5874570084a_0d_1x^4y^6 \\
& - 3333151404b_0d_1x^4y^6 + 212121436a_0^2d_1^2x^4y^6 \\
& + 280352816a_0b_0d_1^2x^4y^6 + 107174808b_0^2d_1^2x^4y^6 \\
& - 1401156a_0^3d_1^3x^4y^6 - 3518659a_0^2b_0d_1^3x^4y^6 \\
& - 3250715a_0b_0^2d_1^3x^4y^6 - 1130262b_0^3d_1^3x^4y^6 \\
& + 1085a_0^4d_1^4x^4y^6 + 5094a_0^3b_0d_1^4x^4y^6 \\
& + 9160a_0^2b_0^2d_1^4x^4y^6 + 7498a_0b_0^3d_1^4x^4y^6 \\
& + 2363b_0^4d_1^4x^4y^6 - 73199185920x^3y^7 \\
& + 23982302232a_0d_1x^3y^7 \\
& + 13978222776b_0d_1x^3y^7 \\
& - 892808908a_0^2d_1^2x^3y^7 - 1249355132a_0b_0d_1^2x^3y^7 \\
& - 497385264b_0^2d_1^2x^3y^7 + 6463496a_0^3d_1^3x^3y^7 \\
& + 17699203a_0^2b_0d_1^3x^3y^7 + 17662897a_0b_0^2d_1^3x^3y^7 \\
& + 6564610b_0^3d_1^3x^3y^7 - 6527a_0^4d_1^4x^3y^7 \\
& - 34734a_0^3b_0d_1^4x^3y^7 - 72008a_0^2b_0^2d_1^4x^3y^7 \\
& - 69410a_0b_0^3d_1^4x^3y^7 - 26441b_0^4d_1^4x^3y^7 \\
& + 237058519752x^2y^8 - 73717828776a_0d_1x^2y^8 \\
& - 44037236592b_0d_1x^2y^8 + 2751173532a_0^2d_1^2x^2y^8 \\
& + 4044825060a_0b_0d_1^2x^2y^8 \\
& + 1668601218b_0^2d_1^2x^2y^8 \\
& - 20693125a_0^3d_1^3x^2y^8 - 60906347a_0^2b_0d_1^3x^2y^8 \\
& - 64886985a_0b_0^2d_1^3x^2y^8 - 25527395b_0^3d_1^3x^2y^8 \\
& + 23205a_0^4d_1^4x^2y^8 + 135806a_0^3b_0d_1^4x^2y^8 \\
& + 313360a_0^2b_0^2d_1^4x^2y^8 + 341634a_0b_0^3d_1^4x^2y^8 \\
& + 150475b_0^4d_1^4x^2y^8 - 541133408592xy^9 \\
& + 153053040600a_0d_1xy^9 + 93565298520b_0d_1xy^9 \\
& - 5592047067a_0^2d_1^2xy^9 - 8557415979a_0b_0d_1^2xy^9 \\
& - 3635890530b_0^2d_1^2xy^9 + 42251824a_0^3d_1^3xy^9 \\
& + 131348672a_0^2b_0d_1^3xy^9 + 147010021a_0b_0^2d_1^3xy^9 \\
& + 60353479b_0^3d_1^3xy^9 - 48844a_0^4d_1^4xy^9 \\
& - 305846a_0^3b_0d_1^4xy^9 - 760242a_0^2b_0^2d_1^4xy^9 \\
& - 901682a_0b_0^3d_1^4xy^9 - 438506b_0^4d_1^4xy^9 \\
& + 634052720592y^{10} - 159806071620a_0d_1y^{10} \\
& - 99249887196b_0d_1y^{10} + 5639205735a_0^2d_1^2y^{10} \\
& + 8832784710a_0b_0d_1^2y^{10} + 3813012729b_0^2d_1^2y^{10} \\
& - 42181381a_0^3d_1^3y^{10} - 135185562a_0^2b_0d_1^3y^{10} \\
& - 155218542a_0b_0^2d_1^3y^{10} - 64874815b_0^3d_1^3y^{10} \\
& + 48844a_0^4d_1^4y^{10} + 318016a_0^3b_0d_1^4y^{10} \\
& + 824332a_0^2b_0^2d_1^4y^{10} + 1024232a_0b_0^3d_1^4y^{10} \\
& + 525776b_0^4d_1^4y^{10}).
\end{aligned}
\tag{A.1}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by Scientific Research Foundation of Guangxi Education Department (no. ZD2014131), Research Foundation of Hezhou University (no. HZUBS201302), and Natural Science Foundation of China grants nos. 11261013 and 11371373.

References

- [1] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer, New York, NY, USA, 1982.
- [2] T. Li, G. Chen, and Y. Tang, "On stability and bifurcation of Chen's system," *Chaos, Solitons and Fractals*, vol. 19, no. 5, pp. 1269–1282, 2004.

- [3] L. F. Mello and S. F. Coelho, "Degenerate Hopf bifurcations in the Lü system," *Physics Letters A*, vol. 373, no. 12-13, pp. 1116–1120, 2009.
- [4] M. Messias, D. D. C. Braga, and L. F. Mello, "Degenerate Hopf bifurcations in Chua's system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 19, no. 2, pp. 497–515, 2009.
- [5] J. Hofbauer and J. W.-H. So, "Multiple limit cycles for three-dimensional Lotka-Volterra equations," *Applied Mathematics Letters*, vol. 7, no. 6, pp. 65–70, 1994.
- [6] Z. Lu and Y. Luo, "Two limit cycles in three-dimensional Lotka-Volterra systems," *Computers & Mathematics with Applications*, vol. 44, no. 1-2, pp. 51–66, 2002.
- [7] M. Gyllenberg and M. Yan, "Four limit cycles for a three-dimensional competitive Lotka-Volterra system with a heteroclinic cycle," *Computers & Mathematics with Applications*, vol. 58, no. 4, pp. 649–669, 2009.
- [8] Q. Wang, W. Huang, and B. Li, "Limit cycles and singular point quantities for a 3D Lotka-Volterra system," *Applied Mathematics and Computation*, vol. 217, no. 21, pp. 8856–8859, 2011.
- [9] Q. Wang, Y. Liu, and H. Chen, "Hopf bifurcation for a class of three-dimensional nonlinear dynamic systems," *Bulletin des Sciences Mathématiques*, vol. 134, no. 7, pp. 786–798, 2010.
- [10] J. Carr, *Applications of Centre Manifold Theory*, vol. 35 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1981.
- [11] M. J. Álvarez and A. Gasull, "Generating limit cycles from a nilpotent critical point via normal forms," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 271–287, 2006.
- [12] M. J. Álvarez and A. Gasull, "Monodromy and stability for nilpotent critical points," *International Journal of Bifurcation and Chaos*, vol. 15, no. 4, pp. 1253–1265, 2005.
- [13] J. Chavarriga, H. Giacomini, J. Giné, and J. Llibre, "Local analytic integrability for nilpotent centers," *Ergodic Theory and Dynamical Systems*, vol. 23, no. 2, pp. 417–428, 2003.
- [14] R. Moussu, "Symétrie et forme normale des centres et foyers dégénérés," *Ergodic Theory and Dynamical Systems*, vol. 2, no. 2, pp. 241–251, 1982.
- [15] A. F. Andreev, A. P. Sadovskii, and V. A. Tsikalyuk, "The center-focus problem for a system with homogeneous nonlinearities in the case of zero eigenvalues of the linear part," *Differential Equations*, vol. 39, no. 2, pp. 155–164, 2003.
- [16] H. Giacomini, J. Giné, and J. Llibre, "The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems," *Journal of Differential Equations*, vol. 227, no. 2, pp. 406–426, 2006.
- [17] Y. Liu and J. Li, "Bifurcations of limit cycles and center problem for a class of cubic nilpotent system," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 20, no. 8, pp. 2579–2584, 2010.
- [18] Y. Liu and J. Li, "On third-order nilpotent critical points: integral factor method," *International Journal of Bifurcation and Chaos*, vol. 21, no. 5, pp. 1293–1309, 2011.
- [19] F. Li, Y. Liu, and Y. Wu, "Center conditions and bifurcation of limit cycles at three-order nilpotent critical point in a seventh degree Lyapunov system," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 6, pp. 2598–2608, 2011.
- [20] A. Buică, I. A. García, and S. Maza, "Existence of inverse Jacobi multipliers around Hopf points in $\mathbb{R}^3$: emphasis on the center problem," *Journal of Differential Equations*, vol. 252, no. 12, pp. 6324–6336, 2012.
- [21] V. F. Edneral, A. Mahdi, V. G. Romanovski, and D. S. Shafer, "The center problem on a center manifold in $\mathbb{R}^3$," *Nonlinear Analysis*, vol. 75, no. 4, pp. 2614–2622, 2012.
- [22] F. S. Dias and L. F. Mello, "Analysis of a quadratic system obtained from a scalar third order differential equation," *Electronic Journal of Differential Equations*, vol. 2010, 6 pages, 2010.
- [23] J. Llibre, "On the integrability of the differential systems in dimension two and of the polynomial differential systems in arbitrary dimension," *The Journal of Applied Analysis and Computation*, vol. 1, no. 1, pp. 33–52, 2011.