Research Article

On Fuzzy Fixed Points for Fuzzy Maps with Generalized Weak Property

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Let (X, d) be a complex valued metric space and let S, T be mappings from X to a set of all fuzzy subsets of X . We present sufficient conditions for the existence of a common α -fuzzy fixed point of S and T . Our results improve and extend certain recent results in literature. Moreover, we discuss an illustrative example to highlight the realized improvements.

1. Introduction

In 1981, Heilpern [1] used the concept of fuzzy set to introduce a class of fuzzy mappings, which is a generalization of the set-valued mapping, and proved a fixed point theorem for fuzzy contraction mappings in a metric linear space. It is worth noting that the result announced by Heilpern [1] forms a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied existence of fixed points of fuzzy mappings or in fuzzy metric spaces; for example, see the work of Azam et al. [2, 3], Bose et al. [4], Chang et al. [5], Cho and Petrot [6], Hussain et al. [7], Qiu and Shu [8], Rashwan and Ahmed [9], and Zhang [10].

Recently, Azam et al. [11] introduced the concept of complex valued metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type condition involving rational expressions. For more details on complex valued metric space we refer the reader to [12–17].

In [18], Azam obtained some common fuzzy fixed points for fuzzy mappings under a rational contractive condition on a metric space in connection with the Hausdorff metric on the family of fuzzy sets.

The aim of this paper is to obtain a common α -fuzzy fixed point of a pair of fuzzy mappings S and T on a complete complex valued metric space under a generalized rational

contractive condition for α -level sets. Our results generalize the results proved by Azam et al. [11, 18].

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \quad \text{iff} \quad \begin{aligned} \operatorname{Re}(z_1) &\leq \operatorname{Re}(z_2), \\ \operatorname{Im}(z_1) &\leq \operatorname{Im}(z_2). \end{aligned} \quad (1)$$

It follows that

$$z_1 \preceq z_2, \quad (2)$$

if one of the following conditions is satisfied:

- (i) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (ii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (iii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2), \operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (iv) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2), \operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In particular, we will write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 < z_2$ if only (iii) is satisfied. Note that

$$\begin{aligned} 0 \preceq z_1 \preceq z_2 &\implies |z_1| < |z_2|, \\ z_1 \preceq z_2, z_2 < z_3 &\implies z_1 < z_3. \end{aligned} \quad (3)$$

Definition 1. Let X be a nonempty set. Suppose that the mapping

$$d : X \times X \longrightarrow \mathbb{C}, \quad (4)$$

satisfies

- (1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X , and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 < r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A. \quad (5)$$

A point $x \in X$ is called a limit point of A whenever, for every $0 < r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset. \quad (6)$$

A is called open whenever each element of A is an interior point of A . Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B . The family

$$F = \{B(x, r) : x \in X, 0 < r\} \quad (7)$$

is a subbasis for a Hausdorff topology τ on X .

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 < c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x , and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$, or $x_n \rightarrow x$, as $n \rightarrow \infty$. If for every $c \in \mathbb{C}$ with $0 < c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x_{n+m}) < c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d) . If every Cauchy sequence is convergent in (X, d) , then (X, d) is called a complete complex valued metric space. We require the following lemmas.

Lemma 2 (see [11]). Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 3 (see [11]). Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$, where $m \in \mathbb{N}$.

A fuzzy set in X is a function with domain X and values in $[0, 1]$; I^X is the collection of all fuzzy sets in X . If A is a fuzzy

set and $x \in X$, then the function values $A(x)$ are called the grade of membership of x in A . The α -level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\} \quad \text{if } \alpha \in (0, 1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}. \end{aligned} \quad (8)$$

Here $\overline{B}$ denotes the closure of the set B . Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in a metric space X . For $A, B \in \mathcal{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. We denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where $\chi_{\{A\}}$ is the characteristic function of the crisp set A . A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{x \in V} A(x) = 1$. The collection of all approximate quantities in V is denoted by $W(V)$.

Definition 4. Let X be a nonempty set and let (Y, d) be a complex valued metric space. A mapping T is called fuzzy mapping if T is a mapping from X into (Y) . A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 5. Let (X, d) be a complex valued metric space and let S, T be fuzzy mappings from X into (X) . A point $z \in X$ is called a fuzzy fixed point of T if $z \in [Tz]_\alpha$, for some $\alpha \in [0, 1]$. The point $z \in X$ is called a common fuzzy fixed point of S and T if $z \in [Sz]_\alpha \cap [Tz]_\alpha$ for some $\alpha \in [0, 1]$. When $\alpha = 1$, z is called a common fixed point of fuzzy mappings.

3. Main Result

Let (X, d) be a complex valued metric space. We denote the family of all nonempty, closed and bounded subsets of a complex valued metric space X by $\mathfrak{CB}(X)$.

From now on, we denote $s(z_1) = \{z_2 \in \mathbb{C} : z_1 \preceq z_2\}$ for $z_1 \in \mathbb{C}$ and $s(a, B) = \cup_{b \in B} s(d(a, b)) = \cup_{b \in B} \{z \in \mathbb{C} : d(a, b) \preceq z\}$ for $a \in X$ and $B \in \mathfrak{CB}(X)$.

For $A, B \in \mathfrak{CB}(X)$, we denote

$$s(A, B) = \left(\bigcap_{a \in A} s(a, B) \right) \cap \left(\bigcap_{b \in B} s(b, A) \right). \quad (9)$$

Lemma 6. Let (X, d) be a complex valued metric space.

- (i) Let $p, q \in \mathbb{C}$. If $p \preceq q$, then $s(q) \subset s(p)$.
- (ii) Let $x \in X$ and $A \in \mathfrak{CB}(X)$. If $\theta \in s(x, A)$, then $x \in A$.
- (iii) Let $q \in \mathbb{C}$ and let $A, B \in \mathfrak{CB}(X)$ and $a \in A$. If $q \in s(A, B)$, then $q \in s(a, B)$ for all $a \in A$ or $q \in s(A, b)$ for all $b \in B$.

Remark 7. If (X, d) is a metric space, for $A, B \in \mathfrak{CB}(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by the metric d .

Let (X, d) be a complex valued metric space and $\mathfrak{C}(X)$ be a collection of nonempty closed subsets of X . Let

$T : X \rightarrow \mathfrak{CB}(X)$ be a multivalued map. For $x \in X$ and $A \in \mathfrak{CB}(X)$, define

$$W_x(A) = \{d(x, a) : a \in A\}. \quad (10)$$

Thus for $x, y \in X$

$$W_x(Ty) = \{d(x, u) : u \in Ty\}. \quad (11)$$

Definition 8. Let (X, d) be a complex valued metric space. A subset A of X is called bounded from below if there exists some $z \in X$, such that $z \leq a$ for all $a \in A$.

Definition 9. Let (X, d) be a complex valued metric space. A multivalued mapping $F : X \rightarrow 2^{\mathbb{C}}$ is called bounded from below if for each $x \in X$ there exists $z_x \in \mathbb{C}$ such that

$$z_x \leq u, \quad (12)$$

for all $u \in Fx$.

Definition 10. Let (X, d) be a complex valued metric space. The fuzzy mapping $T : X \rightarrow \mathcal{F}(X)$ is said to have lower bound property (l.b property) on (X, d) , if, for any $x \in X$ associated with some $\alpha \in (0, 1]$, the multivalued mapping $F_x : X \rightarrow 2^{\mathbb{C}}$ defined by

$$F_x(y) = W_x([Ty]_{\alpha}) \quad (13)$$

is bounded from below. That is, for $x, y \in X$ there exists an element $l_x([Ty]_{\alpha}) \in \mathbb{C}$ such that

$$l_x([Ty]_{\alpha}) \leq u, \quad (14)$$

for all $u \in W_x([Ty]_{\alpha})$, where $l_x([Ty]_{\alpha})$ is called lower bound of T associated with (x, y) .

Definition 11. Let (X, d) be a complex valued metric space. The fuzzy mapping $T : X \rightarrow \mathcal{F}(X)$ is said to have greatest lower bound property (g.l.b property) on (X, d) , if for any $x \in X$ and any $\alpha \in (0, 1]$, greatest lower bound of $W_x([Ty]_{\alpha})$ exists in $\mathbb{C}$ for all $y \in X$. One denotes $d(x, [Ty]_{\alpha})$ by the g.l.b of $W_x([Ty]_{\alpha})$. That is,

$$d(x, [Ty]_{\alpha}) = \inf \{d(x, u) : u \in [Ty]_{\alpha}\}. \quad (15)$$

3.1. Banach Type Fuzzy Fixed Point Result

Theorem 12. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_{\alpha}$ and $[Tx]_{\alpha}$ are nonempty closed bounded subsets of X ; greatest lower bound of $W_x([Ty]_{\alpha})$, $W_x([Sy]_{\alpha})$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned} & \zeta d(x, y) \\ & + \frac{\kappa d(x, [Sx]_{\alpha}) d(y, [Ty]_{\alpha}) + \varsigma d(y, [Sx]_{\alpha}) d(x, [Ty]_{\alpha})}{1 + d(x, y)} \\ & \in s([Sx]_{\alpha}, [Ty]_{\alpha}), \end{aligned} \quad (16)$$

for all $x, y \in X$, where ζ, κ, ς are nonnegative real numbers with $\zeta + \kappa + \varsigma < 1$. Then there exists some $u \in [Su]_{\alpha} \cap [Tu]_{\alpha}$.

Proof. Let x_0 be an arbitrary point in X . By assumption, we can find $x_1 \in [Sx_0]_{\alpha}$. So, we have

$$\begin{aligned} & \zeta d(x_0, x_1) + \left((\kappa d(x_0, [Sx_0]_{\alpha}) d(x_1, [Tx_1]_{\alpha}) \right. \\ & \quad \left. + \varsigma d(x_1, [Sx_0]_{\alpha}) d(x_0, [Tx_1]_{\alpha}) \right) \\ & \quad \times (1 + d(x_0, x_1))^{-1} \in s([Sx_0]_{\alpha}, [Tx_1]_{\alpha}). \end{aligned} \quad (17)$$

By Lemma 6(iii), we have

$$\begin{aligned} & \zeta d(x_0, x_1) + \left((\kappa d(x_0, [Sx_0]_{\alpha}) d(x_1, [Tx_1]_{\alpha}) \right. \\ & \quad \left. + \varsigma d(x_1, [Sx_0]_{\alpha}) d(x_0, [Tx_1]_{\alpha}) \right) \\ & \quad \times (1 + d(x_0, x_1))^{-1} \in s(x_1, [Tx_1]_{\alpha}). \end{aligned} \quad (18)$$

By definition there exists some $x_2 \in [Tx_1]_{\alpha}$, such that

$$\begin{aligned} & \zeta d(x_0, x_1) + \left((\kappa d(x_0, [Sx_0]_{\alpha}) d(x_1, [Tx_1]_{\alpha}) \right. \\ & \quad \left. + \varsigma d(x_1, [Sx_0]_{\alpha}) d(x_0, [Tx_1]_{\alpha}) \right) \\ & \quad \times (1 + d(x_0, x_1))^{-1} \in s(d(x_1, x_2)). \end{aligned} \quad (19)$$

That is,

$$\begin{aligned} d(x_1, x_2) & \leq \zeta d(x_0, x_1) + \left((\kappa d(x_0, [Sx_0]_{\alpha}) d(x_1, [Tx_1]_{\alpha}) \right. \\ & \quad \left. + \varsigma d(x_1, [Sx_0]_{\alpha}) d(x_0, [Tx_1]_{\alpha}) \right) \\ & \quad \times (1 + d(x_0, x_1))^{-1}. \end{aligned} \quad (20)$$

By the meaning of $W_x([Ty]_{\alpha})$ and $W_x([Sy]_{\alpha})$ for $x, y \in X$, we get

$$\begin{aligned} d(x_1, x_2) & \leq \zeta d(x_0, x_1) \\ & + \frac{\kappa d(x_0, x_1) d(x_1, x_2) + \varsigma d(x_1, x_1) d(x_0, x_2)}{1 + d(x_0, x_1)} \\ & = \zeta d(x_0, x_1) + \frac{\kappa d(x_0, x_1) d(x_1, x_2)}{1 + d(x_0, x_1)}, \end{aligned} \quad (21)$$

which implies that

$$\begin{aligned} |d(x_1, x_2)| & \leq \zeta |d(x_0, x_1)| + \frac{\kappa |d(x_0, x_1)| |d(x_1, x_2)|}{|1 + d(x_0, x_1)|} \\ & = \zeta |d(x_0, x_1)| + \kappa |d(x_1, x_2)| \left| \frac{d(x_0, x_1)}{1 + d(x_0, x_1)} \right|, \\ |d(x_1, x_2)| & \leq \zeta |d(x_0, x_1)| + \kappa |d(x_1, x_2)|, \\ (1 - \kappa) |d(x_1, x_2)| & \leq \zeta |d(x_0, x_1)|, \end{aligned} \quad (22)$$

where

$$h = \frac{\zeta}{1 - \kappa} < 1. \quad (23)$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that, for $n = 0, 1, 2, \dots$,

$$|d(x_n, x_{n+1})| \leq h^n |d(x_0, x_1)|, \quad (24)$$

with $h = \zeta/(1 - \kappa) < 1$, for $x_{2n+1} \in [Sx_{2n}]_\alpha$ and $x_{2n+2} \in [Tx_{2n+1}]_\alpha$.

Now for $m > n$, we get

$$\begin{aligned} |d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| \\ &\quad + |d(x_{n+1}, x_{n+2})| + \dots + |d(x_{m-1}, x_m)| \\ &\leq [h^n + h^{n+1} + \dots + h^{m-1}] |d(x_0, x_1)| \quad (25) \\ &\leq \left[\frac{h^n}{1 - h} \right] |d(x_0, x_1)|, \end{aligned}$$

and so

$$|d(x_n, x_m)| \leq \frac{h^n}{1 - h} |d(x_0, x_1)| \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty. \quad (26)$$

This implies that $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, so there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. We now show that $u \in [Tu]_\alpha$ and $u \in [Su]_\alpha$. From (16), we have

$$\begin{aligned} \zeta d(x_{2k}, u) + & \left((\kappa d(x_{2k}, [Sx_{2k}]_\alpha) d(u, [Tu]_\alpha) \right. \\ & + \zeta d(u, [Sx_{2k}]_\alpha) d(x_{2k}, [Tu]_\alpha) \\ & \left. \times (1 + d(x_{2k}, u))^{-1} \right) \in s([Sx_{2k}]_\alpha, [Tu]_\alpha). \end{aligned} \quad (27)$$

By Lemma 6(iii), we have

$$\begin{aligned} \zeta d(x_{2k}, u) + & \left((\kappa d(x_{2k}, [Sx_{2k}]_\alpha) d(u, [Tu]_\alpha) \right. \\ & + \zeta d(u, [Sx_{2k}]_\alpha) d(x_{2k}, [Tu]_\alpha) \\ & \left. \times (1 + d(x_{2k}, u))^{-1} \right) \in s(x_{2k+1}, [Tu]_\alpha). \end{aligned} \quad (28)$$

By definition there exists some $u_k \in [Tu]_\alpha$ such that

$$\begin{aligned} \zeta d(x_{2k}, u) + & \left((\kappa d(x_{2k}, [Sx_{2k}]_\alpha) d(u, [Tu]_\alpha) \right. \\ & + \zeta d(u, [Sx_{2k}]_\alpha) d(x_{2k}, [Tu]_\alpha) \\ & \left. \times (1 + d(x_{2k}, u))^{-1} \right) \in s(d(x_{2k+1}, u_k)). \end{aligned} \quad (29)$$

That is,

$$\begin{aligned} d(x_{2k+1}, u_k) &\leq \zeta d(x_{2k}, u) + \left((\kappa d(x_{2k}, [Sx_{2k}]_\alpha) d(u, [Tu]_\alpha) \right. \\ &\quad + \zeta d(u, [Sx_{2k}]_\alpha) d(x_{2k}, [Tu]_\alpha) \\ &\quad \left. \times (1 + d(x_{2k}, u))^{-1} \right). \end{aligned} \quad (30)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_{2k+1}, u_k) &\leq \zeta d(x_{2k}, u) \\ &\quad + \frac{\kappa d(x_{2k}, x_{2k+1}) d(u, u_k) + \zeta d(u, x_{2k+1}) d(x_{2k}, u_k)}{1 + d(x_{2k}, u)}. \end{aligned} \quad (31)$$

Since by triangle inequality, we get

$$d(u, u_k) \leq d(u, x_{2k+1}) + d(x_{2k+1}, u_k). \quad (32)$$

So using (31) in (32), we get

$$\begin{aligned} d(u, u_k) &\leq d(u, x_{2k+1}) + \zeta d(u, x_{2k+1}) \\ &\quad + \frac{\kappa d(x_{2k}, x_{2k+1}) d(u, u_k) + \zeta d(u, x_{2k+1}) d(x_{2k}, u_k)}{1 + d(x_{2k}, u)}, \\ |d(u, u_k)| &\leq |d(u, x_{2k+1})| + \zeta |d(u, x_{2k+1})| \\ &\quad + \frac{\kappa |d(x_{2k}, x_{2k+1})| |d(u, u_k)| + \zeta |d(u, x_{2k+1})| |d(x_{2k}, u_k)|}{|1 + d(x_{2k}, u)|}. \end{aligned} \quad (33)$$

Taking the limit as $k \rightarrow \infty$, we get $|d(u, u_k)| \rightarrow 0$ as $k \rightarrow \infty$. By Lemma 2 [11], we have $u_k \rightarrow u$ as $k \rightarrow \infty$. Since $[Tu]_\alpha$ is closed, so $u \in [Tu]_\alpha$. Similarly, it follows that $u \in [Su]_\alpha$. Thus S and T have a common fuzzy fixed point. $\square$

By setting $\varsigma = 0$ in Theorem 12, we get the following corollary.

Corollary 13. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_\alpha$ and $[Tx]_\alpha$ are nonempty closed bounded subsets of X ; greatest lower bound of $W_x([Ty]_\alpha)$, $W_x([Sy]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\zeta d(x, y) + \frac{\kappa d(x, [Sx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \in s([Sx]_\alpha, [Ty]_\alpha), \quad (34)$$

for all $x, y \in X$, where ζ and κ are nonnegative real numbers with $\zeta + \kappa < 1$. Then there exists some $u \in [Su]_\alpha \cap [Tu]_\alpha$.

By setting $S = T$ in Theorem 12, we get the following corollary.

Corollary 14. Let (X, d) be a complete complex valued metric space and let T be fuzzy mapping from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Tx]_\alpha$ is nonempty closed bounded subset of X ;

greatest lower bound of $W_x([Ty]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned} & \zeta d(x, y) \\ & + \frac{\kappa d(x, [Tx]_\alpha) d(y, [Ty]_\alpha) + \varsigma d(y, [Tx]_\alpha) d(x, [Ty]_\alpha)}{1 + d(x, y)} \\ & \in s([Tx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (35)$$

for all $x, y \in X$, where ζ, κ , and ς are nonnegative real numbers with $\zeta + \kappa + \varsigma < 1$. Then there exists some $u \in [Tu]_\alpha$.

By Definition 11, one can have the following corollaries easily from Theorem 12.

Corollary 15. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ with g.l.b property such that, for each $x, y \in X$ and $\alpha \in (0, 1]$ and $[Sx]_\alpha, [Ty]_\alpha$ are nonempty closed bounded subsets of X and

$$\begin{aligned} & \zeta d(x, y) \\ & + \frac{\kappa d(x, [Sx]_\alpha) d(y, [Ty]_\alpha) + \varsigma d(y, [Sx]_\alpha) d(x, [Ty]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (36)$$

for all $x, y \in X$, and ζ, κ , and ς are nonnegative real numbers with $\zeta + \kappa + \varsigma < 1$. Then there exists some $u \in [Su]_\alpha \cap [Tu]_\alpha$.

Corollary 16. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ with g.l.b property such that, for each $x, y \in X$ and $\alpha \in (0, 1]$ and $[Sx]_\alpha, [Ty]_\alpha$ are nonempty closed bounded subsets of X and

$$\zeta d(x, y) + \frac{\kappa d(x, [Sx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \in s([Sx]_\alpha, [Ty]_\alpha), \quad (37)$$

for all $x, y \in X$, and ζ and κ are nonnegative real numbers with $\zeta + \kappa < 1$. Then there exists some $u \in [Su]_\alpha \cap [Tu]_\alpha$.

Corollary 17. Let (X, d) be a complete complex valued metric space and let T be fuzzy mapping from X into $\mathcal{F}(X)$ with g.l.b property such that, for each $x, y \in X$ and $\alpha \in (0, 1]$, $[Ty]_\alpha$ is nonempty closed bounded subset of X and

$$\begin{aligned} & \zeta d(x, y) \\ & + \frac{\kappa d(x, [Tx]_\alpha) d(y, [Ty]_\alpha) + \varsigma d(y, [Tx]_\alpha) d(x, [Ty]_\alpha)}{1 + d(x, y)} \\ & \in s([Tx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (38)$$

for all $x, y \in X$, and ζ, κ , and ς are nonnegative real numbers with $\zeta + \kappa + \varsigma < 1$. Then there exists some $u \in [Tu]_\alpha$.

Corollary 18 (see [19]). Let (X, d) be a complete complex valued metric space and let $F, G : X \rightarrow CB(X)$ be multivalued mappings with g.l.b property such that

$$\begin{aligned} & \zeta d(x, y) \\ & + \frac{\kappa d(x, Fx) d(y, Gy) + \varsigma d(y, Fx) d(x, Gy)}{1 + d(x, y)} \\ & \in s(Fx, Gy), \end{aligned} \quad (39)$$

for all $x, y \in X$, where ζ, κ , and ς are nonnegative real numbers with $\zeta + \kappa + \varsigma < 1$. Then there exists some $u \in Fu \cap Tu$.

Proof. Consider a pair of fuzzy mappings $S, T : X \rightarrow \mathcal{F}(X)$ defined by

$$\begin{aligned} S(x)(t) &= \begin{cases} \alpha, & t \in Fx \\ 0, & t \notin Fx, \end{cases} \\ T(x)(t) &= \begin{cases} \alpha, & t \in Gx \\ 0, & t \notin Gx, \end{cases} \end{aligned} \quad (40)$$

where $\alpha \in (0, 1]$. Then

$$[Sx]_\alpha = \{t : S(x)(t) \geq \alpha\} = Fx, \quad [Tx]_\alpha = Gx. \quad (41)$$

Thus, Theorem 12 can be applied to obtain $u \in X$ such that

$$u \in [Su]_\alpha \cap [Tu]_\alpha = Fu \cap Gu. \quad (42)$$

□

3.2. Kannan Type Fuzzy Fixed Point Result

Theorem 19. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_\alpha$ and $[Tx]_\alpha$ are nonempty closed bounded subsets of X ; greatest lower bound of $W_x([Ty]_\alpha), W_x([Sy]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned} & \beta d(x, [Sx]_\alpha) + \gamma d(y, [Ty]_\alpha) \\ & + \eta \frac{d(x, [Sx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (43)$$

for all $x, y \in X$ and nonnegative real numbers β, γ , and η with $\beta + \gamma + \eta < 1$. Then there exists some $v \in [Sv]_\alpha \cap [Tv]_\alpha$.

Proof. Let x_0 be an arbitrary point in X . By assumption, we can find $x_1 \in [Sx_0]_\alpha$. So, we have

$$\begin{aligned} & \beta d(x_0, [Sx_0]_\alpha) + \gamma d(x_1, [Tx_1]_\alpha) \\ & + \eta \frac{d(x_0, [Sx_0]_\alpha) d(x_1, [Tx_1]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s([Sx_0]_\alpha, [Tx_1]_\alpha). \end{aligned} \quad (44)$$

By Lemma 6(iii), we have

$$\begin{aligned} & \beta d(x_0, [Sx_0]_\alpha) + \gamma d(x_1, [Tx_1]_\alpha) \\ & + \eta \frac{d(x_0, [Sx_0]_\alpha) d(x_1, [Tx_1]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s(x_1, [Tx_1]_\alpha). \end{aligned} \quad (45)$$

By definition there exists some $x_2 \in [Tx_1]_\alpha$, such that

$$\begin{aligned} & \beta d(x_0, [Sx_0]_\alpha) + \gamma d(x_1, [Tx_1]_\alpha) \\ & + \eta \frac{d(x_0, [Sx_0]_\alpha) d(x_1, [Tx_1]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s(d(x_1, x_2)). \end{aligned} \quad (46)$$

That is,

$$\begin{aligned} d(x_1, x_2) & \leq \beta d(x_0, [Sx_0]_\alpha) + \gamma d(x_1, [Tx_1]_\alpha) \\ & + \eta \frac{d(x_0, [Sx_0]_\alpha) d(x_1, [Tx_1]_\alpha)}{1 + d(x_0, x_1)}. \end{aligned} \quad (47)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_1, x_2) & \leq \beta d(x_0, x_1) + \gamma d(x_1, x_2) \\ & + \eta \frac{d(x_0, x_1) d(x_1, x_2)}{1 + d(x_0, x_1)}, \end{aligned} \quad (48)$$

which implies that

$$\begin{aligned} |d(x_1, x_2)| & \leq \beta |d(x_0, x_1)| + \gamma |d(x_1, x_2)| \\ & + \eta \frac{|d(x_0, x_1)| |d(x_1, x_2)|}{|1 + d(x_0, x_1)|}. \end{aligned} \quad (49)$$

Thus

$$|d(x_1, x_2)| \leq l |d(x_0, x_1)|, \quad (50)$$

where $l = \beta/(1 - \gamma - \eta) < 1$. Inductively, we can construct a sequence $\{x_n\}$ in X such that, for $n = 0, 1, 2, \dots$,

$$|d(x_n, x_{n+1})| \leq l^n |d(x_0, x_1)|, \quad (51)$$

with $l = \beta/(1 - \gamma - \eta) < 1$, for $x_{2n+1} \in [Sx_{2n}]_\alpha$ and $x_{2n+2} \in [Tx_{2n+1}]$. Now for $m > n$, we get

$$\begin{aligned} |d(x_n, x_m)| & \leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \dots \\ & + |d(x_{m-1}, x_m)| \\ & \leq [l^n + l^{n+1} + \dots + l^{m-1}] |d(x_0, x_1)| \\ & \leq \left[\frac{l^n}{1-l} \right] |d(x_0, x_1)|, \end{aligned} \quad (52)$$

and so

$$|d(x_n, x_m)| \leq \frac{l^n}{1-l} |d(x_0, x_1)| \rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \quad (53)$$

This implies that $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, there exists $v \in X$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$. We now show that $v \in [Tv]_\alpha$ and $v \in [Sv]_\alpha$. From (43), we get

$$\begin{aligned} & \beta d(x_{2n}, [Sx_{2n}]_\alpha) + \gamma d(v, [Tv]_\alpha) \\ & + \eta \frac{d(x_{2n}, [Sx_{2n}]_\alpha) d(v, [Tv]_\alpha)}{1 + d(x_{2n}, v)} \\ & \in s([Sx_{2n}]_\alpha, [Tv]_\alpha). \end{aligned} \quad (54)$$

By Lemma 6 (iii), we have

$$\begin{aligned} & \beta d(x_{2n}, [Sx_{2n}]_\alpha) + \gamma d(v, [Tv]_\alpha) \\ & + \eta \frac{d(x_{2n}, [Sx_{2n}]_\alpha) d(v, [Tv]_\alpha)}{1 + d(x_{2n}, v)} \\ & \in s(x_{2n+1}, [Tv]_\alpha). \end{aligned} \quad (55)$$

By definition there exists some $v_n \in [Tv]_\alpha$ such that

$$\begin{aligned} & \beta d(x_{2n}, [Sx_{2n}]_\alpha) + \gamma d(v, [Tv]_\alpha) \\ & + \eta \frac{d(x_{2n}, [Sx_{2n}]_\alpha) d(v, [Tv]_\alpha)}{1 + d(x_{2n}, v)} \\ & \in s(d(x_{2n+1}, v_n)). \end{aligned} \quad (56)$$

That is,

$$\begin{aligned} d(x_{2n+1}, v_n) & \leq \beta d(x_{2n}, [Sx_{2n}]_\alpha) + \gamma d(v, [Tv]_\alpha) \\ & + \eta \frac{d(x_{2n}, [Sx_{2n}]_\alpha) d(v, [Tv]_\alpha)}{1 + d(x_{2n}, v)}. \end{aligned} \quad (57)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_{2n+1}, v_n) & \leq \beta d(x_{2n}, x_{2n+1}) + \gamma d(v, v_n) \\ & + \eta \frac{d(x_{2n}, x_{2n+1}) d(v, v_n)}{1 + d(x_{2n}, v)}. \end{aligned} \quad (58)$$

Now by using (58) and the triangular inequality, we get

$$\begin{aligned} d(v, v_n) & \leq d(v, x_{2n+1}) + d(x_{2n+1}, v_n) \\ & \leq d(v, x_{2n+1}) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(v, v_n) \\ & + \eta \frac{d(x_{2n}, x_{2n+1}) d(v, v_n)}{1 + d(x_{2n}, v)}, \end{aligned} \quad (59)$$

which implies that

$$\begin{aligned}
 (1-\gamma) |d(v, v_n)| &\leq |d(v, x_{2n+1})| \\
 &+ \beta |d(x_{2n}, x_{2n+1})| \\
 &+ \eta \left| \frac{d(x_{2n}, x_{2n+1}) d(v, v_n)}{1 + d(x_{2n}, v)} \right|, \\
 |d(v, v_n)| &\leq \frac{1}{(1-\gamma)} |d(v, x_{2n+1})| \\
 &+ \frac{\beta}{(1-\gamma)} |d(x_{2n}, x_{2n+1})| \\
 &+ \frac{\eta}{(1-\gamma)} \frac{|d(x_{2n}, x_{2n+1})| |d(v, v_n)|}{|1 + d(x_{2n}, v)|}.
 \end{aligned} \quad (60)$$

By letting $n \rightarrow \infty$ in above inequality, we get

$$|d(v, v_n)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (61)$$

By Lemma 2 [11], we have $v_n \rightarrow v$ as $n \rightarrow \infty$. Since $[Tv]_\alpha$ is closed, so $v \in [Tv]_\alpha$. Similarly, it follows that $v \in [Sv]_\alpha$. Thus there exists some $v \in [Sv]_\alpha \cap [Tv]_\alpha$. $\square$

By setting $\eta = 0$ and $k = \beta = \gamma$ in Theorem 19, we get the following corollary.

Corollary 20. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_\alpha$ and $[Tx]_\alpha$ are nonempty closed bounded subsets of X ; greatest lower bound of $W_x([Ty]_\alpha), W_x([Sy]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$k(d(x, [Sx]_\alpha) + d(y, [Ty]_\alpha)) \in s([Sx]_\alpha, [Ty]_\alpha) \quad (62)$$

for all $x, y \in X$ and $0 \leq k < 1/2$. Then there exists some $v \in [Sv]_\alpha \cap [Tv]_\alpha$.

By setting $S = T$ in Theorem 19, we get the following corollary.

Corollary 21. Let (X, d) be a complete complex valued metric space and let T be fuzzy mapping from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Tx]_\alpha$ is nonempty closed bounded subset of X ; greatest lower bound of $W_x([Ty]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned}
 \beta d(x, [Tx]_\alpha) + \gamma d(y, [Ty]_\alpha) \\
 + \eta \frac{d(x, [Tx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \\
 \in s([Tx]_\alpha, [Ty]_\alpha),
 \end{aligned} \quad (63)$$

for all $x, y \in X$ and nonnegative reals β, γ , and η with $\beta + \gamma + \eta < 1$. Then there exists some $v \in [Tv]_\alpha$.

By Definition 11, one can have the following corollaries easily from Theorem 19.

Corollary 22. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ with g.l.b property such that, for each $x, y \in X$ and $\alpha \in (0, 1]$, $[Sx]_\alpha$ and $[Ty]_\alpha$ are nonempty closed bounded subsets of X and

$$\begin{aligned}
 \beta d(x, [Sx]_\alpha) + \gamma d(y, [Ty]_\alpha) \\
 + \eta \frac{d(x, [Sx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \\
 \in s([Sx]_\alpha, [Ty]_\alpha),
 \end{aligned} \quad (64)$$

for all $x, y \in X$ and nonnegative real numbers β, γ , and η with $\beta + \gamma + \eta < 1$. Then there exists some $v \in [Sv]_\alpha \cap [Tv]_\alpha$.

Remark 23. By Definition 11, one can have a host of corollaries of Kannan type contractive fuzzy mappings with g.l.b property easily from Theorem 19.

Corollary 24 (see[20]). Let (X, d) be a complete complex valued metric space and let $F, G : X \rightarrow CB(X)$ be multivalued mappings with g.l.b property such that

$$\begin{aligned}
 \beta d(x, Fx) + \gamma d(y, Gy) \\
 + \eta \frac{d(x, Fx) d(y, Gy)}{1 + d(x, y)} \\
 \in s(Fx, Gy),
 \end{aligned} \quad (65)$$

for all $x, y \in X$ and nonnegative real numbers β, γ , and η with $\beta + \gamma + \eta < 1$. Then there exists some $v \in Fv \cap Gv$.

Proof. Consider a pair of fuzzy mappings $S, T : X \rightarrow \mathcal{F}(X)$ defined by

$$\begin{aligned}
 S(x)(t) &= \begin{cases} \alpha, & t \in Fx \\ 0, & t \notin Fx, \end{cases} \\
 T(x)(t) &= \begin{cases} \alpha, & t \in Gx \\ 0, & t \notin Gx, \end{cases}
 \end{aligned} \quad (66)$$

where $\alpha \in (0, 1]$. Then

$$[Sx]_\alpha = \{t : S(x)(t) \geq \alpha\} = Fx, \quad [Tx]_\alpha = Gx. \quad (67)$$

Thus, Theorem 19 can be applied to obtain $v \in X$ such that

$$v \in [Sv]_\alpha \cap [Tv]_\alpha = Fv \cap Gv. \quad (68)$$

$\square$

3.3. Chatterjea Type Fuzzy Fixed Point Result

Theorem 25. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_\alpha$ and $[Tx]_\alpha$ are nonempty closed bounded subsets

of X ; greatest lower bound of $W_x([Ty]_\alpha), W_x([Sy]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned} & ad(x, [Ty]_\alpha) + bd(y, [Sx]_\alpha) \\ & + c \frac{d(x, [Ty]_\alpha) d(y, [Sx]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (69)$$

for all $x, y \in X$ and nonnegative reals a, b , and c with $a + b + c < 1$. Then there exists some $w \in [Sw]_\alpha \cap [Tw]_\alpha$.

Proof. Let x_0 be an arbitrary point in X . By assumption, we can find $x_1 \in [Sx_0]_\alpha$. So, we have

$$\begin{aligned} & ad(x_0, [Tx_1]_\alpha) + bd(x_1, [Sx_0]_\alpha) \\ & + c \frac{d(x_0, [Tx_1]_\alpha) d(x_1, [Sx_0]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s([Sx_0]_\alpha, [Tx_1]_\alpha). \end{aligned} \quad (70)$$

By Lemma 6(iii), we have

$$\begin{aligned} & ad(x_0, [Tx_1]_\alpha) + bd(x_1, [Sx_0]_\alpha) \\ & + c \frac{d(x_0, [Tx_1]_\alpha) d(x_1, [Sx_0]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s(x_1, [Tx_1]_\alpha). \end{aligned} \quad (71)$$

By definition there exists some $x_2 \in [Tx_1]_\alpha$, such that

$$\begin{aligned} & ad(x_0, [Tx_1]_\alpha) + bd(x_1, [Sx_0]_\alpha) \\ & + c \frac{d(x_0, [Tx_1]_\alpha) d(x_1, [Sx_0]_\alpha)}{1 + d(x_0, x_1)} \\ & \in s(d(x_1, x_2)). \end{aligned} \quad (72)$$

That is,

$$\begin{aligned} d(x_1, x_2) & \leq ad(x_0, [Tx_1]_\alpha) + bd(x_1, [Sx_0]_\alpha) \\ & + c \frac{d(x_0, [Tx_1]_\alpha) d(x_1, [Sx_0]_\alpha)}{1 + d(x_0, x_1)}. \end{aligned} \quad (73)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_1, x_2) & \leq ad(x_0, x_2) + bd(x_1, x_1) \\ & + c \frac{d(x_0, x_2) d(x_1, x_1)}{1 + d(x_0, x_1)}, \end{aligned} \quad (74)$$

which implies that

$$|d(x_1, x_2)| \leq \frac{a}{1-a} |d(x_0, x_1)|. \quad (75)$$

Similarly from (69), we have

$$\begin{aligned} & ad(x_2, [Tx_1]_\alpha) + bd(x_1, [Sx_2]_\alpha) \\ & + c \frac{d(x_2, [Tx_1]_\alpha) d(x_1, [Sx_2]_\alpha)}{1 + d(x_1, x_2)} \\ & \in s([Tx_1]_\alpha, [Sx_2]_\alpha). \end{aligned} \quad (76)$$

By Lemma 6(iii), we have

$$\begin{aligned} & ad(x_2, [Tx_1]_\alpha) + bd(x_1, [Sx_2]_\alpha) \\ & + c \frac{d(x_2, [Tx_1]_\alpha) d(x_1, [Sx_2]_\alpha)}{1 + d(x_1, x_2)} \\ & \in s(x_2, [Sx_2]_\alpha). \end{aligned} \quad (77)$$

By definition there exists some $x_3 \in [Sx_2]_\alpha$, such that

$$\begin{aligned} & ad(x_2, [Tx_1]_\alpha) + bd(x_1, [Sx_2]_\alpha) \\ & + c \frac{d(x_2, [Tx_1]_\alpha) d(x_1, [Sx_2]_\alpha)}{1 + d(x_1, x_2)} \\ & \in s(d(x_2, x_3)). \end{aligned} \quad (78)$$

That is,

$$\begin{aligned} d(x_2, x_3) & \leq ad(x_2, [Tx_1]_\alpha) + bd(x_1, [Sx_2]_\alpha) \\ & + c \frac{d(x_2, [Tx_1]_\alpha) d(x_1, [Sx_2]_\alpha)}{1 + d(x_1, x_2)}. \end{aligned} \quad (79)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_2, x_3) & \leq ad(x_2, x_2) + bd(x_1, x_3) \\ & + c \frac{d(x_2, x_2) d(x_1, x_3)}{1 + d(x_1, x_2)}, \end{aligned} \quad (80)$$

which implies that

$$|d(x_2, x_3)| \leq \frac{b}{1-b} |d(x_1, x_2)|. \quad (81)$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that, for $n = 0, 1, 2, \dots$,

$$|d(x_n, x_{n+1})| \leq q^n |d(x_0, x_1)|, \quad (82)$$

with $q = \max\{a/(1-a), b/(1-b)\} < 1$, $x_{2n+1} \in [Sx_{2n}]_\alpha$, and $x_{2n+2} \in [Tx_{2n+1}]_\alpha$. Now for $m > n$, we get

$$\begin{aligned} |d(x_n, x_m)| & \leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| \\ & \quad + \dots + |d(x_{m-1}, x_m)| \\ & \leq [q^n + q^{n+1} + \dots + q^{m-1}] |d(x_0, x_1)| \\ & = \left[\frac{q^n}{1-q} \right] |d(x_0, x_1)|, \end{aligned} \quad (83)$$

and so

$$|d(x_n, x_m)| \leq \frac{q^n}{1-q} |d(x_0, x_1)| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \quad (84)$$

This implies that $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, there exists $w \in X$ such that $x_n \rightarrow w$ as $n \rightarrow \infty$. We now show that $w \in [Tw]_\alpha$ and $w \in [Sw]_\alpha$. From (69), we get

$$\begin{aligned} & ad(x_{2n}, [Tw]_\alpha) + bd(w, [Sx_{2n}]_\alpha) \\ & + c \frac{d(x_{2n}, [Tw]_\alpha) d(w, [Sx_{2n}]_\alpha)}{1 + d(x_{2n}, w)} \\ & \in s([Sx_{2n}]_\alpha, [Tw]_\alpha). \end{aligned} \quad (85)$$

By Lemma 6(iii), we have

$$\begin{aligned} & ad(x_{2n}, [Tw]_\alpha) + bd(w, [Sx_{2n}]_\alpha) \\ & + c \frac{d(x_{2n}, [Tw]_\alpha) d(w, [Sx_{2n}]_\alpha)}{1 + d(x_{2n}, w)} \\ & \in s(x_{2n+1}, [Tw]_\alpha). \end{aligned} \quad (86)$$

By definition there exists some $w_n \in [Tw]_\alpha$, such that

$$\begin{aligned} & ad(x_{2n}, [Tw]_\alpha) + bd(w, [Sx_{2n}]_\alpha) \\ & + c \frac{d(x_{2n}, [Tw]_\alpha) d(w, [Sx_{2n}]_\alpha)}{1 + d(x_{2n}, w)} \\ & \in s(x_{2n+1}, [Tw]_\alpha) \in s(d(x_{2n+1}, w_n)). \end{aligned} \quad (87)$$

That is,

$$\begin{aligned} d(x_{2n+1}, w_n) & \leq ad(x_{2n}, [Tw]_\alpha) + bd(w, [Sx_{2n}]_\alpha) \\ & + c \frac{d(x_{2n}, [Tw]_\alpha) d(w, [Sx_{2n}]_\alpha)}{1 + d(x_{2n}, w)}. \end{aligned} \quad (88)$$

By the meaning of $W_x([Ty]_\alpha)$ and $W_x([Sy]_\alpha)$ for $x, y \in X$, we get

$$\begin{aligned} d(x_{2n+1}, v_n) & \leq ad(x_{2n}, w_n) + bd(w, x_{2n+1}) \\ & + c \frac{d(x_{2n}, w_n) d(w, x_{2n+1})}{1 + d(x_{2n}, w)}. \end{aligned} \quad (89)$$

Now by using the triangular inequality, we get

$$\begin{aligned} d(x_{2n+1}, w_n) & \leq ad(x_{2n}, x_{2n+1}) \\ & + ad(x_{2n+1}, w_n) + bd(w, x_{2n+1}) \\ & + c \frac{d(x_{2n}, w_n) d(w, x_{2n+1})}{1 + d(x_{2n}, w)}, \end{aligned} \quad (90)$$

and it follows that

$$\begin{aligned} d(x_{2n+1}, w_n) & \leq \frac{a}{1-a} d(x_{2n}, x_{2n+1}) + \frac{b}{1-a} d(w, x_{2n+1}) \\ & + \frac{c}{1-a} \frac{d(x_{2n}, w_n) d(w, x_{2n+1})}{1 + d(x_{2n}, w)}. \end{aligned} \quad (91)$$

By using again triangular inequality, we get

$$\begin{aligned} d(w, w_n) & \leq d(w, x_{2n+1}) + d(x_{2n+1}, w_n) \\ & \leq d(w, x_{2n+1}) + \frac{a}{1-a} d(x_{2n}, x_{2n+1}) \\ & + \frac{b}{1-a} d(w, x_{2n+1}) \\ & + \frac{c}{1-a} \frac{d(x_{2n}, w_n) d(w, x_{2n+1})}{1 + d(x_{2n}, w)}, \end{aligned} \quad (92)$$

and it follows that

$$\begin{aligned} |d(w, w_n)| & \leq |d(w, x_{2n+1})| + \frac{a}{1-a} |d(x_{2n}, x_{2n+1})| \\ & + \frac{b}{1-a} |d(w, x_{2n+1})| \\ & + \frac{c}{1-a} \frac{|d(x_{2n}, w_n)| |d(w, x_{2n+1})|}{|1 + d(x_{2n}, w)|}. \end{aligned} \quad (93)$$

By letting $n \rightarrow \infty$ in above inequality, we get $|d(w, w_n)| \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2 [11], we have $w_n \rightarrow w$ as $n \rightarrow \infty$. Since $[Tw]_\alpha$ is closed, so $w \in [Tw]_\alpha$. Similarly, it follows that $w \in [Sw]_\alpha$. Thus there exists some $w \in [Sw]_\alpha \cap [Tw]_\alpha$. $\square$

Corollary 26. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$, such that, for each $x \in X$, $[Sx]_\alpha$ and $[Tx]_\alpha$ are nonempty closed bounded subsets of X ; greatest lower bound of $W_x([Ty]_\alpha), W_x([Sy]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$h(d(x, [Ty]_\alpha) + d(y, [Sx]_\alpha)) \in s([Sx]_\alpha, [Ty]_\alpha), \quad (94)$$

for all $x, y \in X$ and $0 \leq h < 1/2$. Then there exists some $w \in [Sw]_\alpha \cap [Tw]_\alpha$.

By taking $S = T$ in Theorem 25, we get the following corollary.

Corollary 27. Let (X, d) be a complete complex valued metric space and let T be fuzzy mapping from X into $\mathcal{F}(X)$. Assume that there exists some $\alpha \in (0, 1]$ such that, for each $x \in X$, $[Tx]_\alpha$ is nonempty closed bounded subset of X ; greatest lower bound of $W_x([Ty]_\alpha)$ exists in $\mathbb{C}$ for all $y \in X$ and

$$\begin{aligned} & ad(x, [Ty]_\alpha) + bd(y, [Tx]_\alpha) \\ & + c \frac{d(x, [Ty]_\alpha) d(y, [Tx]_\alpha)}{1 + d(x, y)} \\ & \in s([Tx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (95)$$

for all $x, y \in X$ and nonnegative reals a, b , and c with $a+b+c < 1$. Then there exists some $w \in [Tw]_\alpha$.

Corollary 28. Let (X, d) be a complete complex valued metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ with g.l.b property such that, for each $x, y \in X$ and $\alpha \in (0, 1]$, $[Sx]_\alpha$ and $[Ty]_\alpha$ are nonempty closed bounded subsets of X and

$$\begin{aligned} & ad(x, [Ty]_\alpha) + bd(y, [Sx]_\alpha) \\ & + c \frac{d(x, [Ty]_\alpha) d(y, [Sx]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (96)$$

for all $x, y \in X$ and nonnegative reals a, b , and c with $a+b+c < 1$. Then there exists some $w \in [Sw]_\alpha \cap [Tw]_\alpha$.

Remark 29. By Definition 11, one can have a host of corollaries of Chatterjea type contractive fuzzy mappings with g.l.b property easily from Theorem 25.

Corollary 30 (see [20]). Let (X, d) be a complete complex valued metric space and let $F, G : X \rightarrow CB(X)$ be multivalued mappings with g.l.b property such that

$$ad(x, Gy) + bd(y, Fx) + c \frac{d(x, Gy) d(y, Fx)}{1 + d(x, y)} \in s(Fx, Gy), \quad (97)$$

for all $x, y \in X$ and nonnegative reals a, b , and c with $a+b+c < 1$. Then there exists some $w \in Fw \cap Gw$.

Proof. Consider a pair of fuzzy mappings $S, T : X \rightarrow \mathcal{F}(X)$ defined by

$$\begin{aligned} S(x)(t) &= \begin{cases} \alpha, & t \in Fx \\ 0, & t \notin Fx, \end{cases} \\ T(x)(t) &= \begin{cases} \alpha, & t \in Gx \\ 0, & t \notin Gx, \end{cases} \end{aligned} \quad (98)$$

where $\alpha \in (0, 1]$. Then

$$[Sx]_\alpha = \{t : S(x)(t) \geq \alpha\} = Fx, \quad [Tx]_\alpha = Gx. \quad (99)$$

Thus, Theorem 25 can be applied to obtain $w \in X$ such that

$$w \in [Sw]_\alpha \cap [Tw]_\alpha = Fw \cap Gw. \quad (100)$$

□

Remark 31. Consider the following.

- (i) By setting ζ, κ , and ς in Corollary 18, β, γ , and η in Corollary 24, and a, b , and c in Corollary 30 with different combinations, one can get corresponding results in [19, 20] as corollaries.
- (ii) By Remark 7 and Corollaries 18, 24, and 30, one can easily get the results of [13, 18–21].

Example 32. Let $X = [0, 1]$ and let $\mathbb{C}$ be the set of complex numbers; define $d : X \times X \rightarrow \mathbb{C}$ as follows:

$$d(x, y) = |x - y| e^{i\theta}, \quad \text{where } \theta = \text{Arg}(z), \quad (101)$$

$$z = x + iy, \quad \text{and } |\cdot| \text{ is modulus function.}$$

Then (X, d) is a complete complex valued metric space. Define a pair of mappings $S, T : X \rightarrow (X)$, for $\alpha \in (0, 1]$ as follows.

For $x, y \in X$ with $x \leq y$, we have

$$S(x)(t) = \begin{cases} \alpha & \text{if } 0 \leq t \leq \frac{x}{40} \\ \frac{\alpha}{2} & \text{if } \frac{x}{40} < t \leq \frac{x}{30} \\ \frac{\alpha}{3} & \text{if } \frac{x}{30} < t \leq \frac{x}{20} \\ \frac{\alpha}{5} & \text{if } \frac{x}{20} < t \leq 1, \end{cases} \quad (102)$$

$$T(x)(t) = \begin{cases} \alpha & \text{if } 0 \leq t \leq \frac{x}{20} \\ \frac{\alpha}{3} & \text{if } \frac{x}{20} < t \leq \frac{x}{10} \\ \frac{\alpha}{4} & \text{if } \frac{x}{10} < t \leq \frac{x}{5} \\ \frac{\alpha}{7} & \text{if } \frac{x}{5} < t \leq 1, \end{cases} \quad (103)$$

such that

$$[Tx]_\alpha = \left[0, \frac{x}{20}\right], \quad [Sx]_\alpha = \left[0, \frac{x}{40}\right], \quad (104)$$

and then

$$\begin{aligned} W_x([Ty]_\alpha) &= \left\{d(x, u) : u \in \left[0, \frac{y}{20}\right]\right\}, \\ W_y([Sx]_\alpha) &= \left\{d(y, v) : v \in \left[0, \frac{x}{40}\right]\right\}. \end{aligned} \quad (105)$$

Denote $d(x, [Tx]_\alpha)$ and $d(x, [Sx]_\alpha)$ by the greatest lower bounds of $W_x([Tx]_\alpha)$ and $W_x([Sx]_\alpha)$. Then

$$d(x, [Ty]_\alpha)(z) = \begin{cases} 0 & \text{if } x < \frac{y}{20} \\ \left(x - \frac{y}{20}\right) e^{i\theta} & \text{if } x > \frac{y}{20}, \end{cases} \quad (106)$$

$$d(y, [Sx]_\alpha)(z) = \begin{cases} \left(y - \frac{x}{40}\right) e^{i\theta}, & \text{as } y > \frac{x}{40}, \end{cases}$$

and also

$$\begin{aligned} d(y, [Ty]_\alpha)(z) &= \left(\frac{19y}{20}\right) e^{i\theta}, \\ d(x, [Sx]_\alpha)(z) &= \left(\frac{39x}{40}\right) e^{i\theta}. \end{aligned} \quad (107)$$

Moreover, if $w_{yx} \in \mathbb{C}$ such that

$$w_{yx} = \left| \frac{y}{20} - \frac{x}{40} \right| e^{i\theta}, \quad (108)$$

then

$$s([Ty]_\alpha, [Sx]_\alpha) = \{w \in \mathbb{C} : w_{xy} \leq w\}. \quad (109)$$

For $x > y/20$, we have

$$\begin{aligned} & \frac{d(y, [Ty]_\alpha) + d(x, [Sx]_\alpha)}{3}(z) \\ &= \frac{1}{3} \left(\frac{19y}{20} + \frac{39x}{40} \right) e^{i\theta} \\ &= \frac{1}{3} \left(y - \frac{y}{20} + x - \frac{x}{40} \right) e^{i\theta} \\ &= \frac{1}{3} \left(y - \frac{x}{40} + x - \frac{y}{20} \right) e^{i\theta} \\ &= \frac{1}{3} \left(y - \frac{x}{40} \right) e^{i\theta} + \left(x - \frac{y}{20} \right) e^{i\theta} \\ &= \frac{d(y, [Sx]_\alpha) + d(x, [Ty]_\alpha)}{3}(z) \\ &= \frac{1}{3} \left(\frac{6y}{20} + \frac{13y}{20} + \frac{6x}{40} + \frac{33x}{40} \right) e^{i\theta} \\ &= \frac{1}{3} \left(\frac{6y}{20} + \frac{6x}{40} \right) e^{i\theta} + \left(\frac{13y}{20} + \frac{33x}{40} \right) e^{i\theta} \\ &> \frac{1}{3} \left(\frac{6y}{20} + \frac{6x}{40} \right) e^{i\theta} \\ &= \left(\frac{2y}{20} + \frac{2x}{40} \right) e^{i\theta} > \left(\frac{y}{20} + \frac{x}{40} \right) e^{i\theta} \\ &> \left| \frac{y}{20} - \frac{x}{40} \right| e^{i\theta} = w_{xy}, \end{aligned} \quad (110)$$

also as

$$\frac{1}{2}d(x, y) = \frac{1}{2}|x - y|e^{i\theta} \geq \left| \frac{y}{20} - \frac{x}{40} \right| e^{i\theta}. \quad (111)$$

It follows that, with $\zeta = \kappa = 1/3$, $\varsigma \neq 0$, such that $\zeta + \kappa + \varsigma < 1$, we have

$$\begin{aligned} & \zeta d(x, [Sx]_\alpha) + \kappa d(y, [Ty]_\alpha) \\ &+ \varsigma \frac{d(x, [Sx]_\alpha) d(y, [Ty]_\alpha)}{1 + d(x, y)} \end{aligned} \quad (112)$$

$$\begin{aligned} & \in s([Sx]_\alpha, [Ty]_\alpha), \\ & \zeta d(x, [Ty]_\alpha) + \kappa d(y, [Sx]_\alpha) \\ &+ \varsigma \frac{d(x, [Ty]_\alpha) d(y, [Sx]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha), \end{aligned} \quad (113)$$

and for $\zeta = 1/2$ with $\kappa \neq 0$ and $\varsigma \neq 0$, such that $\zeta + \kappa + \varsigma < 1$, we have

$$\begin{aligned} & \zeta d(x, y) \\ &+ \frac{\kappa d(x, [Sx]_\alpha) d(y, [Ty]_\alpha) + \varsigma d(y, [Sx]_\alpha) d(x, [Ty]_\alpha)}{1 + d(x, y)} \\ & \in s([Sx]_\alpha, [Ty]_\alpha). \end{aligned} \quad (114)$$

Hence T and S satisfy all the conditions of our main Theorem 12 to obtain $0 \in [S0]_\alpha \cap [T0]_\alpha$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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