Refinements of Aczél-Type Inequality and Their Applications

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Abstract

Due to the importance of Aczél’s inequality (1), it has received considerable attention by many authors and has motivated a large number of research papers giving it various generalizations, improvements, and applications (see [2–21] and the references therein).

In 1959, Popoviciu [10] first obtained an exponential extension of the Aczél inequality as follows.

Theorem B. Let \( p \geq q > 1 \), \( (1/p) + (1/q) = 1 \), and let \( a_i, b_i \) \( (i = 1, 2, \ldots, n) \) be positive numbers such that \( a_i^p - \sum_{i=2}^{n} a_i^p > 0 \) and \( b_i^q - \sum_{i=2}^{n} b_i^q > 0 \). Then

\[
\left( a_i^p - \sum_{i=2}^{n} a_i^p \right)^{1/p} \left( b_i^q - \sum_{i=2}^{n} b_i^q \right)^{1/q} \leq a_i b_i - \sum_{i=2}^{n} a_i b_i. \tag{2}
\]

Later, in 1982, Vasić and Pečarić [16] established the following reversed version of inequality (2).

Theorem C. Let \( q < 0 \), \( p > 0 \), \( (1/p) + (1/q) = 1 \), and let \( a_i, b_i \) \( (i = 1, 2, \ldots, n) \) be positive numbers such that \( a_i^p - \sum_{i=2}^{n} a_i^p > 0 \) and \( b_i^q - \sum_{i=2}^{n} b_i^q > 0 \). Then

\[
\left( a_i^p - \sum_{i=2}^{n} a_i^p \right)^{1/p} \left( b_i^q - \sum_{i=2}^{n} b_i^q \right)^{1/q} \geq a_i b_i - \sum_{i=2}^{n} a_i b_i. \tag{3}
\]

In another paper, Vasić and Pečarić [15] generalized inequality (2) in the following form.

Theorem D. Let \( a_{ij} > 0 \), \( \beta_j > 0 \), \( a_{ij} - \sum_{r=2}^{n} a_{rj} > 0 \), \( r = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \), and let \( \sum_{j=1}^{m} (1/\beta_j) \geq 1 \). Then

\[
\prod_{j=1}^{m} \left( a_{ij} - \sum_{r=2}^{n} a_{rj} \right)^{1/\beta_j} \leq \prod_{j=1}^{m} a_{ij} - \sum_{j=1}^{m} \prod_{r=2}^{n} a_{rj}. \tag{4}
\]

In 2012, Tian [13] presented the reversed version of inequality (4) as follows.

Theorem E. Let \( a_{ij} > 0 \), \( \beta_j \neq 0 \), \( \beta_j < 0 \) \( (j = 2, 3, \ldots, m) \), \( \sum_{j=1}^{m} (1/\beta_j) \leq 1 \), \( a_{ij}^\beta - \sum_{r=2}^{n} a_{rj}^\beta > 0 \), \( r = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). Then

\[
\prod_{j=1}^{m} \left( a_{ij}^\beta - \sum_{r=2}^{n} a_{rj}^\beta \right)^{1/\beta_j} \geq \prod_{j=1}^{m} a_{ij} - \sum_{j=1}^{m} \prod_{r=2}^{n} a_{rj}. \tag{5}
\]

Moreover, in [13] Tian established an integral type of inequality (5).
Theorem F. Let $\beta_1 > 0$, $\beta_j < 0$ $(j = 2, 3, \ldots, m)$, $\sum_{j=1}^{m}(1/\beta_j) = 1$, let $t_j > 0$ $(j = 1, 2, \ldots, m)$, and let $f_j(x)$ $(j = 1, 2, \ldots, m)$ be positive Riemann integrable functions on $[a, b]$ such that $t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x) \, dx > 0$. Then

$$\prod_{j=1}^{m} \left( t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x) \, dx \right)^{1/\beta_j} \geq \prod_{j=1}^{m} t_j - \int_a^b \prod_{j=1}^{m} f_j(x) \, dx. \quad (6)$$

Remark 1. In fact, the integral form of inequality (4) is also valid; that is, one has the following.

Theorem G. Let $\beta_j > 0$ $(j = 1, 2, \ldots, m)$, $\sum_{j=1}^{m}(1/\beta_j) = 1$, let $t_j > 0$ $(j = 1, 2, \ldots, m)$, and let $f_j(x)$ $(j = 1, 2, \ldots, m)$ be positive Riemann integrable functions on $[a, b]$ such that $t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x) \, dx > 0$. Then

$$\prod_{j=1}^{m} \left( t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x) \, dx \right)^{1/\beta_j} \leq \prod_{j=1}^{m} t_j - \int_a^b \prod_{j=1}^{m} f_j(x) \, dx. \quad (7)$$

The main purpose of this work is to give new refinements of inequalities (4) and (5). As applications, new refinements of inequalities (6) and (7) are also given.

2. Refinements of Aczél-Type Inequality

In order to present our main results, we need some lemmas as follows.

Lemma 2 (see [6]). Let $a_i, x_i$ $(i = 1, 2, \ldots, n)$ be real numbers such that $a_i \geq 0$ and $x_i > -1$. If $\sum_{i=1}^{n} a_i \leq 1$, then

$$\prod_{i=1}^{n} (1 + x_i)^{a_i} \leq 1 + \sum_{i=1}^{n} a_i x_i. \quad (8)$$

If either $a_i \geq 1$ $(i = 1, 2, \ldots, n)$ or $a_i \leq 0$ $(i = 1, 2, \ldots, n)$ and if all $x_i$ are positive or negative with $x_i > -1$, then the reverse inequality of (8) holds.

Lemma 3 (see [15]). Let $a_{ij} > 0$ $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$.

(a) If $\lambda_i \geq 0$ and $\sum_{j=1}^{m} \lambda_j \geq 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}^{\lambda_j} \leq \prod_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ij} \right)^{\lambda_j}. \quad (9)$$

(b) If $\lambda_j \leq 0$ $(j = 1, 2, \ldots, m)$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}^{\lambda_j} \geq \prod_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ij} \right)^{\lambda_j}. \quad (10)$$

(c) If $\lambda_1 > 0$, $\lambda_j \leq 0$ $(j = 2, 3, \ldots, m)$, and $\sum_{j=1}^{m} \lambda_j \leq 1$, then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}^{\lambda_j} \geq \prod_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ij} \right)^{\lambda_j}. \quad (11)$$

Lemma 4 (see [18]). Let $0 \leq x < 1, \alpha > 0$. Then

$$(1 - x)^{1/\alpha} \leq 1 - \frac{x}{\max \{\alpha, 1\}}. \quad (12)$$

Lemma 5. Let $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \sum_{j=1}^{m}(1/\beta_j) \geq 1$, $m \geq 2$, let $0 < x_j < 1$ $(j = 1, 2, \ldots, m)$, and let $\xi(m) = \{m/2 \text{ if } m \text{ even } \}$.

Then

$$\prod_{j=1}^{m} \left( 1 - x_j^{\beta_j} \right)^{1/\beta_j} + \sum_{j=1}^{m} x_j \leq 1 - \frac{1}{\xi(m)} \sum_{j=1}^{m} \left( \max \{\beta_j, 1\} \left( x_j^{\beta_j} - x_{j-1}^{\beta_j} \right)^2 \right). \quad (13)$$

Proof. From the assumptions we have that

$$\frac{1}{\beta_j} \geq \frac{1}{\beta_1} \geq \cdots \geq \frac{1}{\beta_m} \geq \frac{1}{m} \geq 0 \quad (j = 1, 2, \ldots, m-1). \quad (14)$$

Case (I) (let $m$ be even). In view of $(1/\beta_1 - 1/\beta_2) + 1/\beta_3 + 1/\beta_4 + (1/\beta_5 - 1/\beta_6) + 1/\beta_7 + 1/\beta_8 + \cdots + (1/\beta_{m-1} - 1/\beta_m) + 1/\beta_m + 1/\beta_m = 1/\beta_1 + 1/\beta_2 + \cdots + 1/\beta_m \geq 1$ by using inequality (9), we get

$$\prod_{j=1}^{m/2} \left( 1 - x_j^{\beta_j} - x_{j-1}^{\beta_j} \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} \left( 1 + x_{j-1}^{\beta_j} \right) \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} \left( 1 + x_j^{\beta_j} \right) \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} \left( 1 + x_{j-1}^{\beta_j} \right) \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} + x_{j-1}^{\beta_j} \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} + x_{j-1}^{\beta_j} \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} + x_{j-1}^{\beta_j} \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} + x_{j-1}^{\beta_j} \right)^{1/\beta_j}$$

$$= \prod_{j=1}^{m/2} \left( \left( 1 - x_j^{\beta_j} \right) + x_j^{\beta_j} + x_{j-1}^{\beta_j} + x_j^{\beta_j} + x_{j-1}^{\beta_j} \right)^{1/\beta_j}.$$
\[ \times \left( 1 - x^{\beta_{m-1}}_m + x^{\beta_{m-1}}_m \right)^{1/\beta_{m-1} - 1/\beta_m} \geq \frac{m}{2} \prod_{j=1}^{m/2} \left( 1 - x^{\beta_{2j-1}}_{2j-1} \frac{1}{\beta_{2j-1}} \right)^{1/\beta_{2j-1}} \times \left( 1 - x^{\beta_{2j-1}}_{2j-1} \frac{1}{\beta_{2j}} \right)^{1/\beta_{2j}} \]

\[ + \prod_{j=1}^{m/2} \left( x^{\beta_{2j}}_{2j} \frac{1}{\beta_{2j}} \right)^{1/\beta_{2j}} \left( 1 - x^{\beta_{2j-1}}_{2j-1} \frac{1}{\beta_{2j}} \right)^{1/\beta_{2j}} \]

\[ = \prod_{j=1}^{m/2} \left( 1 - x^{\beta_{2j}}_{2j} \right)^{1/\beta_{2j}} + \prod_{j=1}^{m} x_j. \]

(15)

On the other hand, applying Lemma 4 and the arithmetic-geometric means inequality we obtain

\[ \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \]

\[ \leq \prod_{j=1}^{m/2} \left[ 1 - \frac{1}{\max \{ \beta_{2j}, 1 \}} \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right] \leq \prod_{j=1}^{(m-1)/2} \left( 1 - x^{\beta_{2j}}_{2j} \right)^{1/\beta_{2j}} \left( 1 - x^{\beta_{2j-1}}_{2j-1} \frac{1}{\beta_{2j}} \right)^{1/\beta_{2j-1}} \left( x_m^{\beta_m} \right)^{1/\beta_m} \]

\[ = \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \times \left( 1 - x^{\beta_{2j}}_{2j} \right)^{1/\beta_{2j}} \times \left( 1 - x^{\beta_{2j-1}}_{2j-1} \frac{1}{\beta_{2j-1}} \right)^{1/\beta_{2j-1}} \left( x_m^{\beta_m} \right)^{1/\beta_m} \]

(16)

Applying Lemma 4 again, we get

\[ \left\{ 1 - \frac{2}{m} \sum_{j=1}^{m/2} \frac{1}{\max \{ \beta_{2j}, 1 \}} \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right\}^{m/2} \]

\[ \leq 1 - \frac{2}{m} \sum_{j=1}^{m/2} \frac{1}{\max \{ \beta_{2j}, 1 \}} \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \]

(17)

Combining (15), (16), and (17) yields immediately inequality (13).

Case (II) (let \( m \) be odd). In view of (1/\( \beta_1 - 1/\beta_2 \) + 1/\( \beta_2 \) + 1/\( \beta_3 - 1/\beta_4 \) + 1/\( \beta_4 \) + \( \cdots \) + (1/\( \beta_{m-2} - 1/\beta_{m-1} \) + 1/\( \beta_{m-1} + 1/\beta_{m-1} + 1/\beta_m = 1/\beta_1 + 1/\beta_2 + \cdots + 1/\beta_m \geq 1 \), by using inequality (9), we have

\[ \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \]

\[ = \left\{ \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \right\} \]

\[ \times \left( 1 - x^{\beta_{2j}}_{2j} + x^{\beta_{2j-1}}_{2j-1} \right)^{1/\beta_{2j}} \]

\[ \times \left( 1 - x^{\beta_{2j-1}}_{2j-1} + x^{\beta_{2j-1}}_{2j-1} \right)^{1/\beta_{2j-1}} \left( x_m^{\beta_m} \right)^{1/\beta_m} \]

(18)

On the other hand, applying Lemma 4 and the arithmetic-geometric means inequality we obtain

\[ \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \]

\[ \leq \prod_{j=1}^{(m-1)/2} \left[ 1 - \left( x^{\beta_{2j}}_{2j} - x^{\beta_{2j-1}}_{2j-1} \right)^2 \right]^{1/\beta_{2j}} \times \left( 1 - x^{\beta_{2j}}_{2j} \right)^{1/\beta_{2j}} \times \left( 1 - x^{\beta_{2j-1}}_{2j-1} \right)^{1/\beta_{2j-1}} \left( x_m^{\beta_m} \right)^{1/\beta_m} \]
\[ \left\{ \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[ 1 - \frac{1}{\max \{\beta_{2j}, 1\}} \times \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2 \right] \right\}^{(m-1)/2} \]

Applying Lemma 4 again, we have

\[ \left\{ 1 - \frac{2}{m-1} \sum_{j=1}^{(m-1)/2} \left[ \frac{1}{\max \{\beta_{2j}, 1\}} \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2 \right] \right\}^{(m-1)/2} \]

Hence, combining (18), (19), and (20) yields immediately inequality (13).

\[ \text{Lemma 6.} \quad \text{Let } \beta_1 > 0, 0 > \beta_2 \geq \beta_3 \geq \cdots \geq \beta_m, \sum_{j=1}^{m} 1/(\beta_j) \leq 1, m \geq 2, \text{ and let } \xi(m) = \left\{ \begin{array}{ll} \frac{(m-2)/2}{m} & \text{if } m \text{ even} \\ \frac{(m-1)/2}{m} & \text{if } m \text{ odd} \end{array} \right. \]

Then

\[ m \prod_{j=1}^{m} \left( 1 - x_j^{\beta_j} \right)^{1/\beta_j} + m \prod_{j=1}^{m} x_j \geq 1 - \sum_{j=1}^{m} \frac{\xi(m) \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \]

Using the same methods as in Lemma 6, we get the following Lemma.

\[ \text{Lemma 7.} \quad \text{Let } 0 > \beta_1 \geq \beta_2 \geq \cdots \geq \beta_m, m \geq 2, \text{ and let } \xi(m) = \left\{ \begin{array}{ll} \frac{m/2}{(m-1)/2} & \text{if } m \text{ even} \\ \frac{(m-1)/2}{m} & \text{if } m \text{ odd} \end{array} \right. \]

Then

\[ m \prod_{j=1}^{m} \left( 1 - x_j^{\beta_j} \right)^{1/\beta_j} + m \prod_{j=1}^{m} x_j \geq 1 - \sum_{j=1}^{m} \frac{\xi(m) \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \]

Now, we present some new refinements of inequalities (4) and (5).

\[ \text{Theorem 8.} \quad \text{Let } a_{ij} > 0, r = 1, 2, \ldots, n, j = 1, 2, \ldots, m, j \geq 2, n \geq 2, \text{ and let } a_{ij} \geq 0, a_{ij} - \sum_{r=2}^{n} a_{ij}^{\beta_{2j}} > 0, \text{ and let } \xi(m) = \left\{ \begin{array}{ll} \frac{m/2}{(m-1)/2} & \text{if } m \text{ even} \\ \frac{(m-1)/2}{m} & \text{if } m \text{ odd} \end{array} \right. \]

Then

\[ \prod_{j=1}^{m} \left( \frac{a_{ij}}{\sum_{r=2}^{n} a_{ij}^{\beta_{2j}}} \right)^{1/\beta_j} \]

\[ \leq \prod_{j=1}^{m} a_{ij} - \prod_{j=1}^{m} a_{ij} - \sum_{r=2}^{n} a_{ij}^{\beta_{2j}} \sum_{r=2}^{n} a_{ij}^{\beta_{2j-1}} \frac{\beta_{2j}}{\beta_{2j-1}} \]

\[ \times \left( \frac{\beta_{2j}}{\beta_{2j-1}} \right)^{1/\beta_j} \]

\[ = 1 - \frac{\xi(m)}{\max \{\beta_{2j}, 1\}} \sum_{j=1}^{m} \frac{\xi(m) \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \]

Thus, by using Lemma 5 with a substitution \( x_j \rightarrow ((a_{ij} - \sum_{r=2}^{n} a_{ij}^{\beta_{2j}})/a_{ij})^{1/\beta_j} \) in (13), we obtain

\[ \prod_{j=1}^{m} \left( \frac{\sum_{r=2}^{n} a_{ij}^{\beta_{2j}}}{a_{ij}} \right)^{1/\beta_j} + \prod_{j=1}^{m} \left( \frac{a_{ij} - \sum_{r=2}^{n} a_{ij}^{\beta_{2j}}}{a_{ij}} \right)^{1/\beta_j} \]

\[ \leq 1 - \frac{\xi(m)}{\max \{\beta_{2j}, 1\}} \sum_{j=1}^{m} \frac{\xi(m) \left( \chi_{2j}^{\beta_{2j}} - \chi_{2j-1}^{\beta_{2j-1}} \right)^2}{\beta_{2j}}. \]

(23)
which implies
\[
\prod_{j=1}^{m}\left(\frac{a_{ij}}{r-\sum_{r=2}^{n}a_{ij}}\right)^{1/\beta_j}
\leq \prod_{j=1}^{m}a_{ij} - \prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{ij}\right)^{1/\beta_j}
- \frac{a_{11}a_{12} \cdots a_{1m}}{\xi(m)}
\times \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\max\{\beta_{2j},1\}} \right\}^{2}.
\]
(26)

On the other hand, we get from Lemma 3 that
\[
\prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{ij}\right)^{1/\beta_j} \geq \prod_{r=2}^{n}a_{ij}.
\]
(27)

Combining (26) and (27) yields immediately the desired inequality (23).

\[\square\]

**Theorem 9.** Let \(a_{ij} > 0, 0 < \beta_1 \geq \beta_2 \geq \cdots \geq \beta_m, a_{ij}^\beta j - \sum_{r=2}^{n}a_{ij}^\beta j > 0, r = 1, 2, \ldots, n, j = 1, 2, \ldots, m, \) let \(m \geq 2, n \geq 2,\) and let \(\xi(m) = \left\{ \begin{array}{ll}
\frac{m^{1/2}}{2} & \text{if } m \text{ even} \\
\frac{(m-1)/2}{2} & \text{if } m \text{ odd}.
\end{array} \right.\)

Then
\[
\prod_{j=1}^{m}\left(\frac{a_{ij}}{r-\sum_{r=2}^{n}a_{ij}}\right)^{1/\beta_j} \geq \prod_{j=1}^{m}a_{ij} - \prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{ij}\right)^{1/\beta_j}
- \frac{a_{11}a_{12} \cdots a_{1m}}{\xi(m)}
\times \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\beta_{2j}} \left(\frac{\sum_{r=2}^{n}a_{ij}^\beta j}{\sum_{r=2}^{n}a_{ij}^\beta j(2j-1)} - \frac{\sum_{r=2}^{n}a_{ij}^\beta j-1}{\sum_{r=2}^{n}a_{ij}^\beta j-1(2j-1)}\right)\right\}^{2}.
\]
(28)

Inequality (28) is also valid for \(\beta_1 > 0, 0 < \beta_2 \geq \beta_3 \geq \cdots \beta_m, \sum_{j=1}^{m}1/\beta_j \leq 1.\)

\[\square\]

**Proof.** The proof of Theorem 9 is similar to the one of Theorem 8, and we omit it.

3. Applications

In this section, we show two applications of the inequalities newly obtained in Section 2.

Firstly, we present a new refinement of inequality (6) by using Theorem 9.

**Theorem 10.** Let \(t_j > 0 (j = 1, 2, \ldots, m), \beta_j > 0, 0 > \beta_2 \geq \beta_3 \geq \cdots \geq \beta_m, \sum_{j=1}^{m}1/\beta_j = 1, \) let \(f_j(x) (j = 1, 2, \ldots, m)\) be positive integrable functions defined on \([a, b]\) with \(t_j^\beta j - \int_{a}^{b} f_j^\beta j(x)dx > 0, \) and let \(\xi(m) = \left\{ \begin{array}{ll}
m^{1/2} & \text{if } m \text{ even} \\
(m-1)/2 & \text{if } m \text{ odd}.
\end{array} \right.\)

Then
\[
\prod_{j=1}^{m}\left(t_j^\beta j - \int_{a}^{b} f_j^\beta j(x)dx\right)^{1/\beta_j} \geq \prod_{j=1}^{m}t_j - \prod_{j=1}^{m}\left(\int_{a}^{b} f_j(x)dx\right)^{1/\beta_j}
- t_1t_2 \cdots t_m
\times \sum_{j=1}^{\xi(m)} \left\{ \frac{1}{\beta_{2j}} \left(\frac{\sum_{r=2}^{n}f_{2j}^\beta j(x)}{\sum_{r=2}^{n}f_{2j}^\beta j(2j-1)} - \frac{\sum_{r=2}^{n}f_{2j}^\beta j-1}{\sum_{r=2}^{n}f_{2j}^\beta j-1(2j-1)}\right)dx\right\}^{2}.
\]
(29)

**Proof.** For any positive integer \(n,\) we choose an equidistant partition of \([a, b]\) as

\[a < a + \frac{b-a}{n} < \cdots < a + \frac{b-a}{n} < b, \]
(30)

\[x_i = a + \frac{b-a}{n} i, \quad i = 0, 1, \ldots, n, \]
\[\Delta x_k = a + \frac{b-a}{n}, \quad k = 1, 2, \ldots, n. \]
(31)

Since \(t_j^\beta j - \int_{a}^{b} f_j^\beta j(x)dx > 0 (j = 1, 2, \ldots, m),\) it follows that

\[t_j^\beta j - \lim_{n \to \infty} \sum_{k=1}^{n} f_j^\beta j\left(a + \frac{k(b-a)}{n}\right) = \frac{b-a}{n} > 0 \quad (j = 1, 2, \ldots, m). \]
(32)

Therefore, there exists a positive integer \(N\) such that

\[t_j^\beta j - \sum_{k=1}^{N} f_j^\beta j\left(a + \frac{k(b-a)}{n}\right) = \frac{b-a}{n} > 0, \]
(33)

for all \(n > N\) and \(j = 1, 2, \ldots, m.\)
Moreover, for any \( n > N \), it follows from Theorem 9 that
\[
\prod_{j=1}^{m} t_j^{\beta_j} - \sum_{k=1}^{n} t_j^{\beta_j} \left( a + \frac{k (b-a)}{n} \right) b - \frac{a}{n} \biggr]^{1/\beta_j} \geq \prod_{j=1}^{m} t_j^{\beta_j} - \sum_{k=1}^{n} \left( m \prod_{j=1}^{m} f_j \left( a + \frac{k (b-a)}{n} \right) \right) \left( b - \frac{a}{n} \right) \]

Next, we give a new refinement of inequality (7) by using Theorem 8.

Theorem 11. Let \( t_j > 0 \), \( 1 \leq j \leq m \), \( 0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \), \( \sum_{j=1}^{m} (1/\beta_j) = 1 \), \( m \geq 2 \), and let \( f_j(x) \) (\( j = 1, 2, \ldots, m \)) be positive integrable functions defined on \([a, b]\) with \( t_j^{\beta_j} \int_a^b f_j^{\beta_j}(x) \, dx > 0 \), and let \( \xi(m) = \left\{ \begin{array}{ll} \frac{m}{2} & \text{if } m \text{ even} \\
\frac{m-1}{2} & \text{if } m \text{ odd} \end{array} \right. \).

Then
\[
\prod_{j=1}^{m} t_j^{\beta_j} - \int_a^b f_j^{\beta_j}(x) \, dx \biggr]^{1/\beta_j} \geq \prod_{j=1}^{m} t_j^{\beta_j} - \int_a^b t_j \prod_{j=1}^{m} f_j(x) \, dx \leq \prod_{j=1}^{m} t_j^{\beta_j} - \int_a^b \frac{m}{\xi(m)} \sum_{j=1}^{m} \left\{ \frac{1}{\beta_j} \right\} \left[ \frac{f_j^{\beta_j}(x)}{t_j^{\beta_j}} - \frac{f_j^{\beta_j-1}(x)}{t_j^{\beta_j-1}} \right] \, dx \biggr]^{1/\beta_j}.
\]

Proof. The proof of Theorem 11 is similar to the one of Theorem 10, and we omit it.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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