The ∗Congruence Class of the Solutions to a System of Matrix Equations

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We present the ∗congruence class of the least-square and the minimum norm least-squaresolutionsto the system of complex matrix equation

\[ A X = C, \quad X B = D \]  

by generalized singular value decomposition and canonical correlation decomposition.

1. Introduction

Throughout we denote the complex \( m \times n \) matrix space by \( \mathbb{C}^{m \times n} \). The symbols \( I, A^*, \) and \( \|A\| \) stand for the identity matrix with the appropriate size, the conjugate transpose, and the Frobenius norm of \( A \in \mathbb{C}^{m \times n} \), respectively. Recall that matrices \( X, Y \in \mathbb{C}^{n \times n} \) are in the same ∗congruence class if there is a nonsingular \( P \in \mathbb{C}^{n \times n} \) such that \( X = P^* Y P \) [1].

Investigating the classical system of matrix equations

\[ AX = C, \quad XB = D \]  

has attracted many people’s attention and many results have been obtained about system (1) with various constraints, such as Hermitian, positive definite, positive semidefinite, reflexive, and generalized reflexive solutions (see [2–10]).

Studying the least-square solutions of the system of matrix equations (1) is also a very active research topic (see [11–16]). It is well known that Hermitian, positive definite and positive semidefinite matrices are the special case of ∗congruence. Therefore investigating the ∗congruence class of a solution of the matrix equation (1) is very meaningful.

In 2005, Horn et al. [1] studied the possible ∗congruence class of a square solution when linear matrix equation \( AX = B \) is solvable. In 2009, Zheng et al. [17] describe ∗congruence class of least-square and minimum norm least-square solutions of the equation \( AX = B \) when it is not solvable and discuss a ∗congruence class of the solutions of the system (1) when it is solvable. To our knowledge, so far there has been little investigation of ∗congruence class of the least-square and minimum norm least-square solutions to (1) when it is not solvable.

Motivated by the work mentioned above, we investigate the ∗congruence class of the least-square and the minimum norm least-square solutions to the system of complex matrix equation (1) by generalized singular value decomposition (GSVD) and canonical correlation decomposition (CCD).

2. The ∗Congruence Class of the Solutions to (1)

Lemma 1 (see [4]). Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{p \times n} \). Then the GSVD of \( A \) and \( B^* \) can be expressed as

\[ A = U \Sigma_A P, \quad B^* = V \Sigma_B P, \]  

where \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{p \times p} \) are unitary matrices, \( P \in \mathbb{C}^{n \times n} \) is nonsingular matrix,

\[ \Sigma_A \in \mathbb{C}^{m \times r}, \quad \Sigma_B \in \mathbb{C}^{p \times r}, \quad r = \text{rank} \begin{pmatrix} A \\ B^* \end{pmatrix}, \]

\[ \Sigma_A = \begin{pmatrix} I_A & S_A \\ 0 & O_A \end{pmatrix}, \quad \Sigma_B = \begin{pmatrix} t & s & r-s-t & n-r \end{pmatrix}, \]
\[
\Sigma_B = \begin{pmatrix}
O_B & S_B & I_B \\
t & s & r-s-t & n-r
\end{pmatrix},
\]

where \( I_A \) and \( I_B \) are identity matrices, \( O_A \) and \( O_B \) are zero matrices, and
\[
S_A = \text{diag}(\alpha_1, \ldots, \alpha_s), \quad S_B = \text{diag}(\beta_1, \ldots, \beta_r)
\]
with \( 1 > \alpha_1 \geq \cdots \geq \alpha_s > 0, 0 < \beta_1 \leq \cdots \leq \beta_r < 1 \), and
\( \alpha_i^2 + \beta_i^2 = 1 \) for all \( i \). For convenience, in the following theorem we denote
\[
PXP^* = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
X_{21} & X_{22} & X_{23} & X_{24} \\
X_{31} & X_{32} & X_{33} & X_{34} \\
X_{41} & X_{42} & X_{43} & X_{44}
\end{pmatrix},
\]
(5)

and
\[
U^*CP^* = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} & C_{34} \\
C_{41} & C_{42} & C_{43} & C_{44}
\end{pmatrix},
\]
(6)

and
\[
PDV = \begin{pmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33} \\
D_{41} & D_{42} & D_{43} \\
p-r+t & s & r-s-t
\end{pmatrix},
\]

Theorem 2. Let \( A, C \in \mathbb{C}^{m \times n} \), \( B, D \in \mathbb{C}^{n \times p} \), and the GSVD of \( A \) and \( B^* \) be expressed as (2), and then one has the following.

(a) The system of matrix equation (1) has a solution in \( \mathbb{C}^{m \times n} \) if and only if
\[
C_{3i} = 0, \quad D_{1i} = 0, \quad (i = 1, 2, 3, 4),
\]
(7)

(b) In that case, the general solutions of (1) are
\[
X = P^{-1} \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
S_A^{-1}C_{21} & S_A^{-1}C_{22} & S_A^{-1}C_{23} & S_A^{-1}C_{24} \\
X_{31} & D_{32}S_B^{-1} & D_{33} & X_{34} \\
X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44}
\end{pmatrix}(P^{-1})^*,
\]

where \( X_{31}, X_{41}, X_{34}, \) and \( X_{44} \) are arbitrary.

(c) For arbitrary \( X_{11}, X_{12}, X_{13}, \) and \( X_{44} \), there exists a solution in \( \mathbb{C}^{m \times n} \) of (1) which is congruent to
\[
Y = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{33}S_A^{-1}C_{24} & X_{31} \\
D_{32}S_B^{-1} & D_{33} & X_{34} \\
X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44}
\end{pmatrix}.
\]

(d) There exists a minimum norm solution in \( \mathbb{C}^{m \times n} \) of (1) which is congruent to
\[
Y = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
S_A^{-1}C_{21} & S_A^{-1}C_{22} & D_{33}S_A^{-1}C_{24} & X_{31} \\
D_{32}S_B^{-1} & D_{33} & X_{34} \\
X_{41} & D_{42}S_B^{-1} & D_{43} & X_{44}
\end{pmatrix}.
\]

Proof. Using the GSVD of \( A \) and \( B^* \) given by (2), we get
\[
AX = C \iff U \Sigma_A PX = C \iff \Sigma_A PXP^* = U^* C P^*,
\]
(11)

\[
XB = D \iff XP^* \Sigma_B^* V^* = D \iff PXP^* \Sigma_B^* = PDV.
\]
(12)

By (2) and (5), \( \Sigma_A PXP^* \) and \( PXP^* \Sigma_B^* \) have the following matrix decomposition:
\[
\Sigma_A PXP^* = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
S_A X_{21} & S_A X_{22} & S_A X_{23} & S_A X_{24} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
(12)

\[
PXP^* \Sigma_B^* = \begin{pmatrix}
0 & X_{12}S_B & X_{13} & X_{14} \\
0 & X_{22}S_B & X_{23} & X_{24} \\
0 & X_{32}S_B & X_{33} & X_{34} \\
0 & X_{42}S_B & X_{43} & X_{44}
\end{pmatrix},
\]

and we have that system (1) is equivalent to
\[
\begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
S_A X_{21} & S_A X_{22} & S_A X_{23} & S_A X_{24} \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{23} & C_{24} \\
C_{31} & C_{32} & C_{33} & C_{34}
\end{pmatrix},
\]
(13)

\[
\begin{pmatrix}
0 & X_{12}S_B & X_{13} & X_{14} \\
0 & X_{22}S_B & X_{23} & X_{24} \\
0 & X_{32}S_B & X_{33} & X_{34} \\
0 & X_{42}S_B & X_{43} & X_{44}
\end{pmatrix} = \begin{pmatrix}
D_{11} & D_{12} & D_{13} & D_{14} \\
D_{21} & D_{22} & D_{23} & D_{24} \\
D_{31} & D_{32} & D_{33} & D_{34} \\
D_{41} & D_{42} & D_{43} & D_{44}
\end{pmatrix}.
\]

Therefore, (1) has a solution in \( \mathbb{C}^{m \times n} \) if and only if (7) holds, and a general form of the solutions can be expressed as (8); for arbitrary \( X_{31}, X_{41}, X_{34}, \) and \( X_{44} \), there exists a solution in \( \mathbb{C}^{m \times n} \) of (1) which is congruent to (9), and the part (d) follows from the definition of Frobenius norm.

Remark 3. In 2009, Zheng et al. [17] discuss a congruence class of the solutions of the system (1) when it is solvable. Our result in Theorem 2 is different with the result mentioned above.
3. The *Congruence Class of the Least-Square Solutions to (1)

Lemma 4 (see [18]). Let the CCD of matrix pair \([A, C]\) with \(A \in \mathbb{C}^{m \times r}, C \in \mathbb{C}^{n \times k}\), rank \(A = g, \) and rank \(C = h\) be given as

\[
A = U \left( \Sigma_A, 0 \right) E_A^{-1}, \quad C = U \left( \Sigma_C, 0 \right) E_C^{-1},
\]

(15)

where \(U\) is a unitary matrix and

\[
\Sigma_A = \begin{pmatrix}
I_i & 0 \\
\Lambda_j & \bar{0} \\
0 & \Delta_j \\
I_i
\end{pmatrix}, \quad \Sigma_C = \begin{pmatrix}
I_h \\
0
\end{pmatrix}
\]

(16)

are nonsingular matrices with the same row partitioning, and \(g = i + j + t\),

\[
\Lambda_j = \text{diag} \left( \lambda_{i+1}, \ldots, \lambda_{i+j} \right), \quad 1 > \lambda_{i+1} \geq \cdots \geq \lambda_{i+j} > 0,
\]

\[
\Delta_j = \text{diag} \left( \Delta_{i+1}, \ldots, \Delta_{i+j} \right), \quad 0 > \Delta_{i+1} \geq \cdots \geq \Delta_{i+j} > 1,
\]

\[
U = \begin{pmatrix}
u_i & u_2 & u_3 & u_4 & u_5 & u_6 \\
i & j & h-i-j & m-h-j-t & j & t
\end{pmatrix}.
\]

(17)

Lemma 5 (see [18]). Given \(E, F \in \mathbb{C}^{m \times n}\), then there exists a unique matrix \(S \in \mathbb{C}^{m \times n}\) such that

\[
\|S - E\|^2 + \|S - F\|^2 = \min,
\]

(18)

and \(S\) can be expressed as

\[
S = E + F.
\]

(19)

Lemma 6 (see [10]). Given \(E, F \in \mathbb{C}^{m \times r}, \Omega_1 = \text{diag}(a_1, \ldots, a_m), \Omega_2 = \text{diag}(b_1, \ldots, b_n), a_i > 0 (i = 1, \ldots, m), \) and \(b_j > 0 (j = 1, \ldots, n),\) then there exists a unique matrix \(S \in \mathbb{C}^{m \times n}\) such that

\[
\|\Omega_1 S - E\|^2 + \|\Omega_2 S - F\|^2 = \min,
\]

(20)

and \(S\) can be expressed as

\[
S = \Phi \ast \left( \Omega_1 E + F \Omega_2 \right),
\]

(21)

where

\[
\Phi = \left( \frac{1}{a^2 + b^2} \right) \in \mathbb{C}^{m \times n}.
\]

(22)

Using Lemmas 5 and 6, we can easily obtain the following.

Lemma 7. Given \(E, F, G \in \mathbb{C}^{m \times n}, \Omega_1 = \text{diag}(a_1, \ldots, a_m), \Omega_2 = \text{diag}(b_1, \ldots, b_n), I_n = \text{diag}(i_1, \ldots, i_n), a_i > 0 (i = 1, \ldots, m), b_j > 0 (j = 1, \ldots, m), \) and \(i_k = 1 (k = 1, \ldots, n),\) then there exist unique matrices \(S\) and \(W\) such that

\[
\|\Omega_1 S + \Omega_2 W - E\|^2 + \|S - F\|^2 + \|W - G\|^2 = \min,
\]

(23)

and \(S\) and \(W\) can be expressed as

\[
S = F, \quad W = \Phi \ast \left( \Omega_1 (F_E - E) + G \right),
\]

(24)

where

\[
\Phi = \left( \frac{1}{b^2 + i^2} \right) \in \mathbb{C}^{m \times n}.
\]

(25)

Lemma 8. Given \(E, F \in \mathbb{C}^{m \times n}, \Omega_1 = \text{diag}(a_1, \ldots, a_m), \Omega_2 = \text{diag}(b_1, \ldots, b_n), a_i > 0 (i = 1, \ldots, m), \) and \(b_j > 0 (j = 1, \ldots, n),\) then there exist unique matrices \(S\) and \(W\) such that

\[
\|\Omega_1 S + \Omega_2 W - E\|^2 = \min,
\]

(26)

and \(S\) and \(W\) can be expressed as

\[
S = 0, \quad W = \Omega_2^{-1} E.
\]

(27)

Let \(A, C \in \mathbb{C}^{m \times r}, B \in \mathbb{C}^{r \times l}, \) and rank \(A = p \geq \) rank \(B = q.\) According to Lemma 4, there exist a unitary matrix \(U \in \mathbb{C}^{m \times r}\) and nonsingular matrices \(R_A \in \mathbb{C}^{m \times p}\) and \(R_B \in \mathbb{C}^{r \times l},\)

\[
A^* = U \left( \Sigma_A, 0 \right) R_A^{-1}, \quad B = U \left( \Sigma_B, 0 \right) R_B^{-1},
\]

(28)

where \(\Sigma_A \in \mathbb{C}^{m \times p}, \Sigma_B \in \mathbb{C}^{r \times q},\)

\[
\Sigma_A = \begin{pmatrix}
I_r & 0 & 0 \\
0 & G_s & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & S_q & 0 \\
0 & 0 & I_t
\end{pmatrix}, \quad \Sigma_B = \begin{pmatrix}
I_r & 0 & 0 \\
0 & I_t & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

(29)

where \(p = r + s + t,\)

\[
G_s = \text{diag} \left( g_{r+1}, \ldots, g_{r+s} \right), \quad 1 > g_{r+1} \geq \cdots \geq g_{r+s} > 0,
\]

\[
S_q = \text{diag} \left( w_{r+1}, \ldots, w_{r+s} \right), \quad 0 > w_{r+1} \geq \cdots \geq w_{r+s} > 1,
\]

\[
G_s^2 + S_q^2 = I_t,
\]

(30)

Using Lemmas 5 and 6, we can easily obtain the following.

Theorem 9. Let \(A, C \in \mathbb{C}^{m \times r}, B \in \mathbb{C}^{r \times l}, \) and the CCD of matrix pair \([A^*, B]\) be expressed as (28), and then one has the following.

(a) The least-square solutions to the system (1) are

\[
S = F, \quad W = \Phi \ast \left( \Omega_1 (F_E - E) + G \right),
\]

(24)
where $X_{34}, X_{35}, X_{36}, X_{44}, X_{45},$ and $X_{46}$ are arbitrary, $Y_{5i} = \Phi \ast (S(GD_{2i} - C_{2i}) + D_{3i}), i = 1, 2, 3, \Phi = (1/(w_{r+j} + e_k^2)) \in C^{x^x}$, and $e_k = 1, j = 1, \ldots, s, k = 1, \ldots, s.$

(b) For arbitrary $X_{34}, X_{35}, X_{36}, X_{44}, X_{45},$ and $X_{46}$, there exists a least-square solution in $C^{m \times n}$ of (1) which is congruent to

\[
Y = \begin{pmatrix}
C_{11} + D_{11} & C_{12} + D_{12} & C_{13} + D_{13} & C_{14} & C_{15} & C_{16} \\
D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\
D_{31} & D_{32} & D_{33} & X_{34} & X_{35} & X_{36} \\
D_{41} & D_{42} & D_{43} & X_{44} & X_{45} & X_{46} \\
Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{34} & S^{-1}C_{35} & S^{-1}C_{36} \\
D_{31} + D_{61} & D_{32} + D_{62} & D_{33} + D_{63} & C_{44} & C_{45} & C_{46}
\end{pmatrix}
\]

(c) There exists a minimum norm least-square solution in $C^{m \times n}$ of (1) which is congruent to

\[
Y = \begin{pmatrix}
C_{11} + D_{11} & C_{12} + D_{12} & C_{13} + D_{13} & C_{14} & C_{15} & C_{16} \\
D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\
D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\
D_{41} & D_{42} & D_{43} & 0 & 0 & 0 \\
Y_{51} & Y_{52} & Y_{53} & S^{-1}C_{34} & S^{-1}C_{35} & S^{-1}C_{36} \\
D_{31} + D_{61} & D_{32} + D_{62} & D_{33} + D_{63} & C_{44} & C_{45} & C_{46}
\end{pmatrix}
\]

Then,

\[
\|AX - C\|^2 + \|XB - D\|^2
\]

Proof. It follows from (28) that

\[
AX = C \iff (R_A^{-1})^* \left( \begin{smallmatrix} \Sigma_A^* \\ 0 \end{smallmatrix} \right) U^* X = C
\]

\[
\iff \left( \begin{smallmatrix} \Sigma_A^* \\ 0 \end{smallmatrix} \right) U^* X = (R_A)^* C,
\]

\[
XB = D \iff X U (\Sigma_{\beta}, 0) R_{\beta}^{-1} = D \iff X U (\Sigma_{\beta}, 0) = D R_{\beta},
\]

(34)
Assume that

\[
U^* X U = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\
X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\
X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\
X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\
X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\
X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66}
\end{pmatrix},
\]

(36)

\[
(R_A)^* C U = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}
\end{pmatrix},
\]

(37)

\[
U^* D R_B = \begin{pmatrix}
D_{11} & D_{12} & D_{13} & D_{14} \\
D_{21} & D_{22} & D_{23} & D_{24} \\
D_{31} & D_{32} & D_{33} & D_{34} \\
D_{41} & D_{42} & D_{43} & D_{44} \\
D_{51} & D_{52} & D_{53} & D_{54} \\
D_{61} & D_{62} & D_{63} & D_{64}
\end{pmatrix},
\]

and then

By Lemmas 5, 7, and 8, a general form of the least-square solutions can be expressed as (31); for arbitrary \(X_{34}, X_{55}, X_{66}, X_{44}, X_{45},\) and \(X_{66}\), there exists a least-square solution in \(C_{mn}^{\infty}\) of (1) which is * congruent to (32), and the part (c) follows from the definition of Frobenius norm.

\[\square\]

4. An Algorithm and Numerical Examples

Based on the main results of this paper, we in this section propose an algorithm for finding the least-square solutions to the system (1). All the tests are performed by MATLAB 6.5 which has a machine precision of around \(10^{-16}\).

Algorithm 1. (1) Input \(A \in C_{mn}^{\infty}\) and \(B \in C_{ml}^{\infty}\), and compute \(U \in C_{mn}^{\infty}, R_A^{-1} \in C_{mn}^{\infty}, R_B^{-1} \in C_{ml}^{\infty}, S_A, S_B \in C_{mn}^{\infty}, S, G \in C_{ml}^{\infty}\) by the CCD of matrix pair \([A^*, B]\).

(2) Input \(C \in C_{mn}^{\infty}, D \in C_{ml}^{\infty}\), and compute \(C_{ij} (i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5, 6)\) and \(D_{lk} (l = 1, 2, 3, 4, 5, 6; k = 1, 2, 3, 4)\) according to (37).

(3) Compute the least-square solutions of the system (1) by (31).

(4) Compute the * congruence class of the least-square and the minimum norm least-square solutions to the system (1) according to (32) and (33).

Example 1. Suppose

\[
A = \begin{bmatrix}
-1.625 & 0 & -0.6875i & 0.875i & 0.3438 & 0 \\
-2 & 0 & -0.5i & 0.875i & 0.25 & 0 \\
-0.75 & 0 & -0.125i & 0.25i & 0.0625 & 0 \\
2.625 & 0 & 0.6875i & -0.875i & -0.3438 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
2 & -2 & 0 & -1 \\
-5i & 6i & 0 & 2i \\
31i & -37i & i & -15i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
\(C = \begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 5 \\ 4 & i & i & -i & 3 & 1 \\ 5 & 6 & 7 & 4 & 3 & 1 \\ 2 & 1 & 1 & 4 & 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} i & -i & -i & 1 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & 2 & 3 \\ 5 & 7 & 6 & 9 \\ 9 & 7 & 5 & 6 \\ 1 & 2 & 3 & 2 \end{bmatrix}.\)

Applying Algorithm 1, we obtain the following:

\[U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \end{bmatrix}, \quad \Sigma_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.\]

\(\Sigma_A = \begin{bmatrix} 5.03 - 0.05i & 0.02 + 2.95i \\ -0.51i & -0.27i \\ 0.43i & 0.21i \\ 0.41i & 0.26i \\ 2.02 - 0.06i & 0.864 - 0.366i \\ -0.84i & -0.5i \end{bmatrix}, \quad \Sigma_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.\)

\((R_A)^*CU = \begin{bmatrix} 5 & 3i & -3i & -9i & 4 & 5i \\ 14 & -1 + 6i & 1 - 6i & -19i & 11 & 1 + 10i \\ 32 & -2 + 26i & 2 - 23i & -22i & 25 & 2 + 25i \\ 53 & -1 + 57i & 1 - 50i & -34i & 38 & 1 + 46i \end{bmatrix}, \quad U^*DR_B = \begin{bmatrix} 0.03 - 0.05i & 0.02 - 0.05i & 0.04 - 0.02i & 0.02 - 0.04i \\ -0.51i & -0.27i & -0.33i & -0.25i \\ 0.43i & 0.21i & 0.29i & 0.23i \\ 1.35 & 0.67 & 0.78 & 0.62 \\ -1.27i & -0.71i & -0.82i & -0.65i \end{bmatrix}.\)

The least-square solutions to the system (1) are

\[X = U \begin{bmatrix} 5.03 - 0.05i & 0.02 + 2.95i \\ -0.51i & -0.27i \\ 0.43i & 0.21i \\ 0.41i & 0.26i \\ 2.02 - 0.06i & 0.864 - 0.366i \\ -0.84i & -0.5i \end{bmatrix} U^*, \quad (41)\]

where \(X_{34}, X_{35}, X_{36}, X_{44}, X_{45},\) and \(X_{46}\) are arbitrary.

For arbitrary \(X_{34}, X_{35}, X_{36}, X_{44}, X_{45},\) and \(X_{46}\), there exists a least-square solution in \(\mathbb{C}^{6 \times 6}\) of (1) which is * congruent to

\[Y = \begin{bmatrix} 5.03 - 0.05i & 0.02 + 2.95i & 0.04 - 3.02i & -9i & 4 & 5i \\ -0.51i & -0.27i & 0.29i & X_{34} & X_{35} & X_{36} \\ 0.43i & 0.21i & 0.25i & X_{44} & X_{45} & X_{46} \\ 2.02 - 0.06i & 0.864 - 0.366i & 0.498 + 1.448i & 76i & 44 & 4 + 40i \\ -0.84i & -0.5i & -0.53i & -22i & 25 & 2 + 25i \end{bmatrix}. \quad (42)\]

There exists a minimum norm least-square solution in \(\mathbb{C}^{6 \times 6}\) of (1) which is * congruent to

\[Y = \begin{bmatrix} 5.03 - 0.05i & 0.02 + 2.95i \\ -0.51i & -0.27i \\ 0.43i & 0.21i \\ 0.41i & 0.26i \\ 2.02 - 0.06i & 0.864 - 0.366i \\ -0.84i & -0.5i \end{bmatrix}. \quad (43)\]
**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


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