Research Article

Asymptotic Behavior of Solutions of Free Boundary Problem with Logistic Reaction Term

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We study a free boundary problem for a reaction diffusion equation modeling the spreading of a biological or chemical species. In this model, the free boundary represents the spreading front of the species. We discuss the asymptotic behavior of bounded solutions and obtain a trichotomy result: spreading (the free boundary tends to $+\infty$ and the solution converges to a stationary solution defined on $[0, +\infty)$), transition (the free boundary stays in a bounded interval and the solution converges to a stationary solution with positive compact support), and vanishing (the free boundary converges to 0 and the solution tends to 0 within a *finite* time).

1. Introduction

Consider the following free boundary problem:

$$\begin{aligned} u_t &= u_{xx} + u(1-u), \quad 0 < x < h(t), \quad t > 0, \\ u(t, 0) &= u(t, h(t)) = 0, \quad t > 0, \\ h'(t) &= -\mu u_x(t, h(t)) - \mu\alpha, \quad t > 0, \\ h(0) &= h_0, \quad u(0, x) = u_0(x), \quad 0 \leq x \leq h_0, \end{aligned} \quad (1)$$

where $x = h(t)$ is a moving boundary to be determined together with $u(t, x)$ and $\alpha > 0$ is a given constant. The initial function u_0 belongs to $\mathcal{Y}(h_0)$ for some $h_0 > 0$, where

$$\mathcal{Y}(h_0) := \{ \phi \in C^2([0, h_0]) : \phi(0) = \phi(h_0) = 0, \phi(x) \geq (\neq) 0 \text{ in } (0, h_0) \}. \quad (2)$$

Recently, problem (1) with $\alpha = 0$ was studied by [1–3] and so forth. They used this model to describe the spreading of a new or invasive species; they used the free boundary $h(t)$ which represents the expanding front of the species whose density is represented by $u(t, x)$. They obtained a spreading-vanishing dichotomy result; namely, the species either spreads to the whole environment and stabilizes at the

positive state 1 (i.e., $u \rightarrow 1$) or vanishes (i.e., $u \rightarrow 0$) as time goes to *infinity*. Such a result shows that problem (1) with $\alpha = 0$ has advantages comparing with the Cauchy problems (the Cauchy problems have hair-trigger effect: any positive solution which converges to a positive constant; cf. [4, 5]). In the last two years, [6] also studied the corresponding problem of (1) with $\alpha = 0$ in high dimension spaces.

In this paper, we mainly study problem (1) with $\alpha > 0$; such a boundary condition represents that there is a spreading resistant force at the front for some species. Intuitively, the presence of $\alpha > 0$ makes the solution more difficult to spread than the case where $\alpha = 0$. Indeed, $h'(t) > 0$ only if $u_x(t, h(t)) < -\alpha$. This boundary condition is widely used in many biological models. For example, it is often used in protocell models (cf. [7, 8]).

We give the following theorem whose proof is similar to that of [1, 2]. It suffices to repeat their arguments with obvious modification.

Theorem 1. *For any given $\gamma \in (0, 1)$, there is a $T \in (0, +\infty)$ such that free boundary problem (1) has a solution*

$$(u, h) \in C^{((1+\gamma)/2), 1+\gamma}(\overline{D}_T) \times C^{1+\gamma/2}([0, T]), \quad (3)$$

where $D_T := \{(t, x) \in \mathbb{R}^2 : x \in [0, h(t)], t \in (0, T]\}$, and the solution can be extended to some interval $(0, T_0)$ with $T_0 > T$ as long as $\inf_{0 < t < T} h(t) > 0$.

Moreover, as in the proof of [9, Lemma 2.8], one can show that $h_\infty := \lim_{t \rightarrow T} h(t) \in [0, +\infty]$ exist.

The main purpose of this paper is to study the asymptotic behavior of bounded solutions of (1) and obtain trichotomy result. We will prove that, for a solution (u, h) of (1), one has either

(i) spreading: $h_\infty = +\infty$ and

$$\lim_{t \rightarrow \infty} u(t, x) = w(x) \quad \text{locally uniformly in } (0, +\infty), \quad (4)$$

where w is the unique positive solution of

$$\begin{aligned} q'' + q(1 - q) &= 0, \quad x > 0, \\ q(0) &= 0, \end{aligned} \quad (5)$$

or

(ii) vanishing: $\lim_{t \rightarrow T} h(t) = 0$ and

$$T < +\infty, \quad \lim_{t \rightarrow T} \max_{0 \leq x \leq h(t)} u(t, x) = 0 \quad (6)$$

or

(iii) transition: $0 < h_\infty < +\infty$ and

$$\lim_{t \rightarrow \infty} u(t, \cdot) = v(\cdot) \quad \text{locally uniformly in } (0, h_\infty), \quad (7)$$

where v is the solution of

$$\begin{aligned} v'' + v(1 - v) &= 0, \quad x \in (0, h_\infty), \\ v(0) &= v(h_\infty) = 0, \quad -v'(h_\infty) = \alpha. \end{aligned} \quad (8)$$

Remark 2. Comparing with the results in [1–3], the phenomenon (iii) is a new one, since it does not happen in case $\alpha = 0$.

Remark 3. (ii) shows that vanishing happens in a finite time and the free boundary converges to the point 0; those phenomena are also new and do not happen in case $\alpha = 0$.

2. Asymptotic Behavior of Solutions

In this section, we study the asymptotic behavior of solutions and obtain trichotomy result when $\alpha < \sqrt{3}/3$; namely, the solution of (1) is either vanishing (Theorem 6) or transition (Theorem 7) or spreading (Theorem 10). Then, we prove that only vanishing happens if $\alpha \geq \sqrt{3}/3$ (Theorem 11) for the completeness of the paper.

We first prepare the following comparison theorems which can be proved similarly as in [2, Lemma 3.5].

Lemma 4. Suppose that $T \in (0, \infty)$, $\bar{h} \in C^1([0, T])$, and $\bar{u} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ with $D_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, 0 < x < \bar{h}(t)\}$ and

$$\begin{aligned} \bar{u}_t &\geq \bar{u}_{xx} + \bar{u}(1 - \bar{u}), \quad 0 < t \leq T, \quad 0 < x < \bar{h}(t), \\ \bar{u}(t, 0) &\geq 0, \quad \bar{u}(t, \bar{h}(t)) = 0, \quad 0 < t \leq T, \\ \bar{h}'(t) &\geq -\mu \bar{u}(t, \bar{h}(t)) - \mu \alpha, \quad 0 < t \leq T. \end{aligned} \quad (9)$$

If $h_0 \leq \bar{h}(0)$ and $u_0(x) \leq \bar{u}(0, x)$ in $[0, h_0]$ and if (u, h) is a solution of (1), then

$$\begin{aligned} h(t) &\leq \bar{h}(t), \quad u(x, t) \leq \bar{u}(x, t) \quad \text{for } t \in (0, T], \\ x &\in (0, h(t)). \end{aligned} \quad (10)$$

Remark 5. The pair $(\bar{u}, \bar{h})$ is usually called an upper solution of problem (1) and one can define a lower solution by revising all the inequalities.

Theorem 6. Let (u, h) be a solution of (1) on $[0, T^*)$. If $\lim_{t \rightarrow T^*} h(t) = 0$, then $T^* < +\infty$ and

$$\lim_{t \rightarrow T^*} \max_{0 \leq x \leq h(t)} u(t, x) = 0. \quad (11)$$

Proof. By [2, 10], one can prove that there exists a constant C_1 such that $u(t, x) \leq C_1$. In order to prove that u converges to 0, we need to construct the function

$$U(t, x) := C_1 \left[2M(h(t) - x) - M^2(h(t) - x)^2 \right] \quad (12)$$

over the region

$$Q := \{(t, x) : 0 < t < T^*, \max\{h(t) - M^{-1}, 0\} < x < h(t)\}, \quad (13)$$

where

$$M := \max \left\{ \frac{\alpha + \sqrt{\alpha^2 + 2}}{2}, \frac{4\|u_0\|_{C^1([-h_0, h_0])}}{3C_1} \right\}. \quad (14)$$

Clearly $0 \leq U \leq C_1$ in Q . By the definitions of U and M , we have

$$U_t - U_{xx} - U(1 - U) \geq C_1 (2M^2 - 2M\alpha - 1) \geq 0 \quad \text{in } Q. \quad (15)$$

Moreover,

$$\begin{aligned} U(t, h(t)) &= u(t, h(t)) = 0 \quad \text{for } t \in (0, T^*), \\ U(t, 0) &> 0 = u(t, 0) \quad \text{when } h(t) < M^{-1}. \end{aligned} \quad (16)$$

Therefore, $u(t, x) \leq U(t, x)$ in Q by the comparison principle Lemma 4. Note that $\lim_{t \rightarrow T^*} h(t) = 0$; then there exists $T_1 < T^*$ such that $h(t) - M^{-1} < 0$ for $t > T_1$. Therefore, $u(t, x) \leq U(t, x)$ for $t > T_1$ and $x \in [0, h(t)]$. For such t and x , we have

$$U(t, x) \leq 2MC_1 h(t) \rightarrow 0 \quad \text{as } t \rightarrow T^*; \quad (17)$$

it follows that

$$\|u(t, \cdot)\|_{L^\infty([0, h(t)])} \rightarrow 0 \quad \text{as } t \rightarrow T^*. \quad (18)$$

We now prove that $T^* < +\infty$. By $\lim_{t \rightarrow T^*} h(t) = 0$, there is some $L_* > 0$ such that

$$h(t) \leq L_* \quad \text{for } t \in [0, T^*]. \quad (19)$$

Set $L := 2(1 + L_*)$ and

$$\xi_0(x) := \frac{2\varepsilon}{L^2} (L^2 - x^2), \quad (20)$$

where $\varepsilon > 0$ is small such that

$$8(\alpha + \sqrt{\alpha^2 + 2})\varepsilon \leq \alpha, \quad 32\varepsilon \leq \alpha. \quad (21)$$

Consider the problem

$$\begin{aligned} \xi_t &= \xi_{xx} + 2\xi \left(1 - \frac{\xi}{2\varepsilon}\right), \quad 0 < x < \bar{h}(t), \quad t > 0, \\ \xi(t, 0) &= \xi(t, \bar{h}(t)) = 0, \quad t > 0, \\ \bar{h}'(t) &= -\mu\xi_x(t, \bar{h}(t)) - \mu\alpha, \quad t > 0, \\ \bar{h}(0) &= L, \quad \xi(0, x) = \xi_0(x), \quad 0 \leq x \leq L. \end{aligned} \quad (22)$$

It is obvious that $\xi(t, x) \leq 2\varepsilon$ for all $t \geq 0$. Construct a function

$$U^\varepsilon(t, x) := 2\varepsilon \left[2M(\bar{h}(t) - x) - M^2(\bar{h}(t) - x)^2 \right] \quad (23)$$

over $\bar{Q} := \{(t, x) : t > 0, \max\{0, \bar{h}(t) - M^{-1}\} \leq x \leq \bar{h}(t)\}$, where $M := \max\{\alpha + \sqrt{\alpha^2 + 2}, 4\}$. Then $U^\varepsilon(t, x)$ is an upper solution of (22) over $\bar{Q}$ and so

$$-\xi_x(t, \bar{h}(t)) \leq -U_x^\varepsilon(t, \bar{h}(t)) = 4M\varepsilon \leq \frac{\alpha}{2}. \quad (24)$$

Therefore, $\bar{h}'(t) \leq -\alpha\mu/2$. Thus, $\bar{h}(t) \rightarrow 0$ as $t \rightarrow \bar{T}^* \leq 2L/\alpha\mu$.

On the other hand, (18) implies that there exists some $T_0 \in (0, T^*)$ such that $u(t, x) \leq \varepsilon$ for all $x \in [0, h(t)]$ and $t > T_0$. Clearly $\xi_0(x) \geq u(T_0, x)$ for $x \in [0, h(T_0)]$. By the comparison principle, we have $h(t + T_0) \leq \bar{h}(t)$, and so T^* cannot be ∞ . $\square$

Theorem 7. Assume that $0 < \alpha < \sqrt{3}/3$. Let (u, h) be a solution of (1). If $0 < h_\infty < +\infty$, then

$$\begin{aligned} h_\infty &= L_\alpha, \\ \lim_{t \rightarrow \infty} u(t, \cdot) &= v_\alpha(\cdot) \quad \text{locally uniformly in } (0, h_\infty), \end{aligned} \quad (25)$$

where v_α is a unique positive solution of

$$\begin{aligned} v'' + v(1 - v) &= 0, \quad 0 < x < L_\alpha, \\ v(0) &= v(L_\alpha) = 0, \quad v'(0) = -v'(L_\alpha) = \alpha, \end{aligned} \quad (26)$$

where

$$L_\alpha := 2 \int_0^B \frac{dr}{\sqrt{\alpha^2 - r^2 + (2/3)r^3}} \quad (27)$$

with $B \in (0, 1)$ given by $\alpha^2 = 2 \int_0^B s(1 - s)ds$.

Remark 8. This is a new phenomenon. It never happens when $\alpha = 0$. Moreover, by the phase plane method, one can prove that $v_\alpha \rightarrow 0$ and $L_\alpha \rightarrow \pi$ as $\alpha \rightarrow 0$. This conclusion gives an explanation of Lemma 3.1 in [2]; that is, vanishing happens if $h_\infty \leq \pi$.

Remark 9. It is easily seen that (26) has no positive solution when $\alpha \geq 2 \int_0^1 s(1 - s)ds = \sqrt{3}/3$.

Proof of Theorem 7. For any $\varepsilon > 0$, there exists $t^* > 0$ such that $h_\infty - \varepsilon < h(t) < h_\infty + \varepsilon$ for $t > t^*$. Let $\bar{u}_0(x)$ be a function defined on $(0, h_\infty + \varepsilon)$ and satisfies

$$\begin{aligned} \bar{u}_0(x) &\geq u(t^*, x) \quad \text{for } x \in (0, h_\infty), \\ \bar{u}_0(0) &= \bar{u}_0(h_\infty + \varepsilon) = 0. \end{aligned} \quad (28)$$

By the comparison principle we have $u(t, x) \leq \bar{u}(t, x)$ in $(t^*, \infty) \times (0, h(t))$, where $\bar{u}(t, x)$ is the solution of

$$\begin{aligned} \bar{u}_t &= \bar{u}_{xx} + \bar{u}(1 - \bar{u}), \quad t > t^*, \quad 0 < x < h_\infty + \varepsilon, \\ \bar{u}(t, 0) &= \bar{u}(t, h_\infty + \varepsilon) = 0, \quad t > t^*, \\ \bar{u}(t^*, x) &= \bar{u}_0(x), \quad 0 < x < h_\infty + \varepsilon. \end{aligned} \quad (29)$$

It is well known that

- (i) $\bar{u} \rightarrow 0$ as $t \rightarrow \infty$ if $h_\infty + \varepsilon \leq \pi$; or
- (ii) $\bar{u} \rightarrow \bar{u}_\varepsilon^*$ as $t \rightarrow \infty$ if $h_\infty + \varepsilon > \pi$,

where $\bar{u}_\varepsilon^*$ is a positive function. More precisely, when $h_\infty + \varepsilon > \pi$, it follows from [11, Corollary 3.4] that $\bar{u}_\varepsilon^*$ is the unique positive solution of

$$\begin{aligned} (\bar{u}_\varepsilon^*)'' + \bar{u}_\varepsilon^*(1 - \bar{u}_\varepsilon^*) &= 0, \quad 0 < x < h_\infty + \varepsilon, \\ \bar{u}_\varepsilon^*(h_\infty + \varepsilon) &= \bar{u}_\varepsilon^*(0) = 0. \end{aligned} \quad (30)$$

Hence,

$$\lim_{t \rightarrow \infty} u(t, x) = 0, \quad \text{or } \limsup_{t \rightarrow \infty} u(t, x) \leq \bar{u}_\varepsilon^*. \quad (31)$$

Similarly,

$$\liminf_{t \rightarrow \infty} u(t, x) \geq \underline{u}_\varepsilon^*(x) \quad \text{when } h_\infty - \varepsilon > \pi, \quad (32)$$

where $\underline{u}_\varepsilon^*(x)$ is a positive solution of

$$\begin{aligned} (\underline{u}_\varepsilon^*)'' + \underline{u}_\varepsilon^*(1 - \underline{u}_\varepsilon^*) &= 0, \quad 0 < x < h_\infty - \varepsilon, \\ \underline{u}_\varepsilon^*(h_\infty - \varepsilon) &= \underline{u}_\varepsilon^*(0) = 0. \end{aligned} \quad (33)$$

We conclude from (31) and (32) that

$$\lim_{t \rightarrow \infty} u(t, x) = 0 \quad \text{if } h_\infty \leq \pi, \quad (34)$$

or when $h_\infty > \pi$,

$$\lim_{t \rightarrow \infty} u(t, x) = u^*(x) \quad \text{locally uniformly in } (0, h_\infty), \quad (35)$$

where $u^*(x)$ is the unique positive solution of

$$\begin{aligned} (u^*)'' + u^*(1 - u^*) &= 0, \quad 0 < x < h_\infty, \\ u^*(h_\infty) &= u^*(0) = 0. \end{aligned} \quad (36)$$

We now show that $\lim_{t \rightarrow \infty} u(t, x) = 0$ is impossible when $h_\infty > 0$. Suppose that this does not hold; there exists L_0 such that $h(t) \leq L_0$. Then using the approach of proving $T^* < +\infty$ in Theorem 7, we can show that $\lim_{t \rightarrow T} h(t) = 0$ for some $0 < T < +\infty$; this contradicts the assumption $h_\infty > 0$. Hence, $\lim_{t \rightarrow \infty} u(t, x) = u^*(x)$, locally uniformly in $(0, h_\infty)$; we next prove that $u^*(x) = v_\alpha(x)$.

Make a change of the variable x to reduce $[0, h(t)]$ to the fixed interval $[0, h_0]$ and use L^p estimates as well as Sobolev embedding theorems on the reduced equation with Dirichlet boundary conditions to conclude that

$$\|u(t, \cdot) - u^*(\cdot)\|_{C^{1+(\gamma/2)}([0, h(t)])} \rightarrow 0 \quad (t \rightarrow \infty) \quad (37)$$

for some $\gamma > 0$. It follows that $h'(t) = -\mu u_x(t, h(t)) - \mu\alpha \rightarrow -\mu(u^*)'(h_\infty) - \mu\alpha$ as $t \rightarrow \infty$. Hence, we conclude that $(0, h_\infty)$ is not a finite interval unless $-(u^*)'(h_\infty) = \alpha$. $\square$

Theorem 10. Let (u, h) be a solution of (1). If $h_\infty = +\infty$, then

$$\lim_{t \rightarrow \infty} u(t, x) = w(x) \quad \text{locally uniformly in } [0, +\infty), \quad (38)$$

where w is the unique positive solution of

$$\begin{aligned} q'' + q(1 - q) &= 0, \quad x > 0, \\ q(0) &= 0. \end{aligned} \quad (39)$$

Proof. Choose a bounded continuous function $W_0(x) \geq u_0(x)$ for $x \in [0, h_0]$ and $W_0 \geq 0$ for $x \in [0, +\infty)$. Let $W(t, x)$ be the unique solution of

$$\begin{aligned} W_t &= W_{xx} + W(1 - W), \quad t > 0, \quad x > 0, \\ W(t, 0) &= 0, \quad t > 0, \\ W(0, x) &= W_0(x), \quad x > 0. \end{aligned} \quad (40)$$

Then the comparison principle theorem shows that $u(t, x) \leq W(t, x)$ for $t > 0, x > 0$. Using [11, Lemma 3.4], we see that

$$\limsup_{t \rightarrow \infty} u(t, x) \leq \lim_{t \rightarrow \infty} W(t, x) = w(x) \quad \text{for } x \in [0, +\infty). \quad (41)$$

On the other hand, since $h_\infty = +\infty$, for any large $l > \pi$, there is $\tau > 0$ such that $h(\tau) = l$ and $h(t) \geq l$ for all $t > \tau$. Let $\underline{u}_l(t, x)$ be the solution of the following problem:

$$\begin{aligned} \underline{u}_t &= \underline{u}_{xx} + \underline{u}(1 - \underline{u}), \quad t > \tau, \quad 0 < x < l, \\ \underline{u}(t, 0) &= \underline{u}(t, l) = 0, \quad t > \tau, \\ \underline{u}(0, x) &= \psi(x), \quad 0 < x < l, \end{aligned} \quad (42)$$

where ψ is a nonnegative continuous function satisfying $\psi(x) \leq u(\tau, x)$ for $0 < x < l$. The comparison principle implies

$$\underline{u}_l(t, x) \leq u(t, x) \quad \text{for } t > \tau, \quad 0 \leq x \leq l. \quad (43)$$

By [11], one can obtain

$$\lim_{t \rightarrow \infty} \underline{u}_l(t, x) = v_l(x) \quad \text{uniformly in } [0, l], \quad (44)$$

where v_l is the positive solution of

$$\begin{aligned} v'' + v(1 - v) &= 0, \quad 0 < x < l, \\ v(0) &= v(l) = 0, \end{aligned} \quad (45)$$

It is well known that $\lim_{l \rightarrow \infty} v_l(x) = w(x)$. Combining this with (43) and (44), we have

$$w(x) \leq \liminf_{t \rightarrow \infty} u(t, x). \quad (46)$$

By (41) and (46), we have

$$\lim_{t \rightarrow \infty} u(t, x) = w(x). \quad (47)$$

$\square$

Theorem 11. Suppose that $\alpha \geq \sqrt{3}/3$ and (u, h) is a solution of (1) defined on some maximal existence interval $[0, T_*)$; then $T_* < +\infty$, u converges to 0 as $t \rightarrow T_*$, and $\lim_{t \rightarrow T_*} h(t) = 0$.

Proof. The proof of this theorem is similar to [10]; it suffices to repeat their arguments with obvious modification. $\square$

3. Example

In this section, we give some sufficient conditions for vanishing, spreading, and transition.

Example 1. Suppose that $\alpha < \sqrt{3}/3$. Let $h_0 > 0$ and $u_0(x) \in \mathcal{H}(h_0)$; then the following properties hold:

- (i) vanishing happens when $u_0(x) < v_\alpha(x)$;
- (ii) spreading happens if $u_0(x) > v_\alpha(x)$ for $x \in [0, h_0]$;
- (iii) transition happens if $u_0(x) \equiv v_\alpha(x)$ for $x \in [0, h_0]$.

Proof. (i) By [1], we see that $v_{\alpha_1}(x) < v_{\alpha_2}(x)$ for $\alpha_1 < \alpha_2$. Since $u_0(x) < v_\alpha(x)$, there is $\beta < \alpha$ such that $u_0(x) < v_\beta(x)$, by the comparison principle that $u(t, x) < v_\beta(x)$, so $h_\infty \neq +\infty$ and $h_\infty \neq L_\alpha$. It then follows from Theorem 6 that vanishing happens.

(ii) Let (u, h) be a solution of (1) with initial data $u_0(x)$; by the phase plane analysis, there is $\gamma > \alpha$ such that $u_0(x) > v_\gamma(x)$. It then follows from the comparison principle that $u(t, x) > v_\gamma(x)$, so Theorem 10 implies that $h_\infty = +\infty$ and spreading happens.

(iii) It follows from the comparison principle Lemma 4 that $u(t, x) \equiv v_\alpha(x)$ and $h(t) \equiv L_\alpha$ for all $t > 0$. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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