In mathematics, to a large extent, control theory addresses the stability of solutions of differential equations, which can describe the behavior of dynamic systems. In this paper, a class of fractional-order nonautonomous systems with multiple time delays modeled by differential equations is considered. A sufficient condition is established for the existence and uniqueness of solutions for such systems involving Caputo fractional derivative, and the uniform stability of solution is studied. At last, two examples are given to demonstrate the applicability of our results.

1. Introduction

Fractional calculus is an ongoing topic for more than 300 years; it is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. The number of literature concerning the application of fractional calculus has been growing rapidly, especially in recent years. In fact, fractional derivatives are more adequate to describe underlying phenomena than traditionally used integral-order derivatives in many cases, because fractional-order derivatives provide an excellent tool for the description of memory and hereditary properties of various processes, in contrast to integral-order derivatives. Nowadays, fractional calculus is a flourishing field of active research [1–9].

Stability is one of the most fundamental and interesting problem in control theory. So far, there have been some advances in control theory of fractional dynamical systems for stability questions such as robust stability [10–12], Mittag-Leffler stability [13], bounded-input bounded-output stability [14, 15], uniform stability [16], finite-time stability [17], and robust controllability [18]. However, it should be noted that it is difficult to evaluate the stability for fractional-order dynamic systems by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. In addition, it is well known that Lyapunov direct method cannot be simply extended and applied to the case of fractional order, although many stability results about integer-order systems are obtained by constructing a suitable Lyapunov functional [19–22]. From the above discussion, it may be seen that study on the stability of fractional-order systems is still meaningful. Moreover, time delay plays an important role in mathematical modeling of many real world phenomena. Time delay can have an effect on the stability of a system and occasionally can cause a system to become unstable. To the best of our knowledge, there are relatively few results on the stability of fractional-order systems with delay, such as Lazarević and Spasić [17], Akbari Moornani and Haeri [23], Kumar and Sukavanam [24], Wang et al. [25], and El-Sayed and Gaafar [26]. In [17], a finite-time stability test procedure is proposed for linear nonhomogeneous fractional-order systems with a pure time delay. In [23], two theorems are given to check the robust BIBO stability of two large classes of fractional-order delay systems (retarded and neutral types), respectively. In [24], sufficient conditions are established for the approximate controllability of a class of semilinear delay control systems of fractional order. A delayed fractional-order financial system is proposed and the complex dynamical behaviors of such a system are discussed by numerical simulations in [25]. The existence of a unique solution and the uniform stability of solution are proved for a class of nonlinear nonautonomous system of Riemann-Liouville fractional differential systems.
with different constant delays and nonlocal condition in [26]. For more details, the reader can refer to [27–30].

Motivated by the above discussions, this paper aims at studying the uniform stability of a class of fractional-order nonautonomous systems with multiple time delays. Besides, a sufficient condition for the existence and uniqueness of solutions is given.

The rest of this paper is organized as follows. In Section 2, some useful definitions, lemmas, and notations are introduced. In Section 3, the existence and uniqueness of solutions and uniform stability problem for fractional-order nonautonomous systems with multiple time delays are studied. In Section 4, two numerical examples are presented to demonstrate the main results. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we present some definitions, lemmas, and notations related to the main results that we will obtain in the following.

**Definition 1** (see [31]). The fractional-order integral of a function \( f(t) \) of order \( \alpha \in \mathbb{R}^+ \) is defined by

\[
I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,
\]

where \( \Gamma(\cdot) \) is the gamma function given as

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.
\]

**Definition 2** (see [32]). The Caputo fractional derivative \( D^\alpha \) of order \( \alpha \) of a function \( f(t) \) is defined as

\[
D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha}} d\tau, \quad (n-1 < \alpha < n),
\]

where \( n = [\alpha], [\alpha] \) is the ceiling function.

From the previous definitions, it is recognized that fractional derivative represents a global property of a function within a given closed interval \([0, t]\), while integral derivative of a function is only related to its nearby value of the independent variable, which is a local property.

**Lemma 3** (see [32]). Let \( n \) be a positive integer such that \( n-1 < \alpha < n \); if \( f(t) \in C^{n-1}[t_0, t] \), then

\[
I_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{m=0}^{n-1} \frac{f^{(m)}(t_0)}{m!} (t-t_0)^m.
\]

In particular, if \( 0 < \alpha < 1 \) and \( f(t) \in C[t_0, t] \), then

\[
I_t^\alpha D_t^\alpha f(t) = f(t) - f(t_0).
\]

The following notations will be used throughout the paper. Let \( C([a, b], \mathbb{R}^n) \) denote the class of all continuous column \( n \)-vectors function defined on \([a, b] \); for \( x \in C([a, b], \mathbb{R}^n) \), the norm is \( ||x(t)|| = \sum_{i=1}^n \sup_{\in \mathbb{R}^+} |e^{-Kt}| x_i(t) || \), where \( K \) is a large enough constant. In addition, we define the norm of \( A = (a_{ij})_{n \times n} \) by \( \|A\| = \sum_{i=1}^n a_i = \sum_{i=1}^n \sup_{t \in [a, b]} |a_{ij}(t)| \).

3. Main Results

The differential equation describing the dynamic behavior of a class of nonautonomous systems with multiple time delays can be represented as follows:

\[
D^\alpha x_i(t) = \sum_{j=1}^n a_{ij}(t) x_j(t) + \sum_{j=1}^n b_{ij}(t) f_j(x_j(t))
\]

\[
+ \sum_{j=1}^n c_{ij}(t) f_j(x_j(t - r_{ij})),
\]

\[
+ u_i(t), \quad t \in [0, T], \quad i \in \{1, 2, \ldots, n\},
\]

where \( T < +\infty \); \( D^\alpha \) denotes Caputo fractional-order derivative of order \( \alpha \) \((0 < \alpha < 1) \) with \( t_0 = 0 \); \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) represents the state vector at time \( t \); \( A(t) = (a_{ij})_{n \times n}, B(t) = (b_{ij})_{n \times n}, C(t) = (c_{ij})_{n \times n}, \) and \( u(t) = (u_i(t))_{n \times 1} \) are given matrices whose elements are absolutely continuous; \( f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))) \) corresponds to a vector function; \( r_{ij} \) is a constant and denotes the corresponding time delay.

Accompanying the system (6) is an initial condition of the form:

\[
x_i(t) = \phi_i(t), \quad t \in [-\tau, 0],
\]

for all \( i \in \mathbb{N} \), where \( \tau = \max_{i \in \mathbb{N}} \{r_{ij}\} \) and \( \phi_i(t) \) is continuous on \([-\tau, 0]\).

In order to prove the main results, we make the following assumption:

(H1) \( f_j (j = 1, 2, \ldots, n) \) is Lipschitz continuous with Lipschitz constant \( L_j \); that is,

\[
|f_j(x) - f_j(y)| \leq L_j |x - y|.
\]

3.1. Existence and Uniqueness of Solutions. In this section, we will give a sufficient condition for the existence and uniqueness of solutions of system (6).

**Theorem 4.** Assume that (H1) holds; then, system (6) exists a unique solution \( x(t) \) which satisfies \( x(t) \in C([0, T], \mathbb{R}^n) \) on \([0, T]\) and coincides with \( \phi \) for \( t \in [-\tau, 0] \).
Proof. Equation (6) is equivalent to the equation
\[ x_i(t) = \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^n c_{ij}(s) f_j(x_j(s-\tau_{ij})) \right) ds, \quad t \in [0,T]. \] (9)

Now, construct a mapping \( T_i \), defined by
\[ T_i x_i(t) = \phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^n c_{ij}(s) f_j(x_j(s-\tau_{ij})) \right) + u_i(s) \right] ds, \quad t \in [0,T], \] (10)
where \( T x = (T_1 x_1, T_2 x_2, \ldots, T_n x_n)^T \).

For any two different \( x(t), y(t) \in C([0,T], \mathbb{R}^n) \), we have
\[ |T_i x_i(t) - T_i y_i(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n |a_{ij}(s)| |x_j(s) - y_j(s)| + \sum_{j=1}^n (|b_{ij}(s)| + |c_{ij}(s)|) \right) ds \]
\[ \leq \sup_{t, \forall j} |a_{ij}(t)| \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n e^{-K(t-s)} |x_j(s) - y_j(s)| \right) ds \]
from which it follows that
\[ e^{-Kt} |T_i x_i(t) - T_i y_i(t)| \leq \frac{1}{\Gamma(\alpha)} e^{-Kt} \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n |a_{ij}(s)| |x_j(s) - y_j(s)| + \sum_{j=1}^n (|b_{ij}(s)| + |c_{ij}(s)|) \right) ds \]
\[ \leq \frac{1}{\Gamma(\alpha)} e^{-Kt} \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n e^{-K(t-s)} |x_j(s) - y_j(s)| + \sum_{j=1}^n (|b_{ij}(s)| + |c_{ij}(s)|) \right) ds \]
\[ \times e^{-Ks} \left| x_j(s) - y_j(s) \right| ds \]
\[ \leq \frac{1}{\Gamma(\alpha)} \sup_{l, \forall j} |a_{ij}(t)| \int_0^t (t-s)^{\alpha-1} \left( \sum_{j=1}^n e^{-K(t-s)} e^{-Ks} |x_j(s) - y_j(s)| \right) ds \]
(11)
\[
\|Tx(t) - Ty(t)\| \\
= \sum_{i=1}^{n} \sup_{t} \left\{ e^{-Kt_i} \left| T_i x_i(t) - T_i y_i(t) \right| \right\} \\
< \sum_{i=1}^{n} a_i + (b_i + c_i) L \frac{\|x(t) - y(t)\|}{K^\alpha} \\
= \frac{\|A\| + (\|B\| + \|C\|) L}{K^\alpha} \|x(t) - y(t)\|. 
\]

For \( K \) is a large enough constant, here we can choose \( K \) such that \( \|A\| + (\|B\| + \|C\|) L < K^\alpha \); then, we have
\[
\|Tx(t) - Ty(t)\| < \|x(t) - y(t)\|. 
\]

From the above proof, we know that \( T \) is a contraction mapping and it has a unique fixed point \( x = Tx \), so that system (6) exists a unique solution.

### 3.2. Stability of Nonautonomous Systems

In this section, we study the stability of solution of system (6).

**Definition 5** (see [33]). The solution of (6) is called stable if for any \( \varepsilon > 0 \) and \( t_0 \geq 0 \), there exists \( \delta(\varepsilon, t_0) \) such that \( t \geq t_0 \geq 0 \), \( \|\phi(t) - \psi(t)\| < \delta(\varepsilon, t_0) \) imply \( \|y(t, t_0, \phi) - x(t, t_0, \psi)\| < \varepsilon \) for any two solutions \( x(t, t_0, \phi) \) and \( y(t, t_0, \psi) \). And the solution of (6) is called uniformly stable if \( \delta(\varepsilon, t_0) \) can be chosen independently of \( t_0 : \delta(\varepsilon, t_0) = \delta(\varepsilon) \).

**Theorem 6.** Under assumption (H1), the solution of system (6) is uniformly stable.
\[
\Gamma(\alpha) \times \int_0^t (t-s)^{\alpha-1} e^{-Ks} \left[ x_j(s) - y_j(s) \right] ds.
\]

It immediately follows that
\[
e^{-Kt} |x_i(t) - y_i(t)|
\leq e^{-Kt} |\phi_i(0) - \varphi_i(0)| + \frac{1}{\Gamma(\alpha)} e^{-Kt}
\times \int_0^t (t-s)^{\alpha-1}
\times \left[ \sum_{j=1}^n |a_{ij}(s)| \left| x_j(s) - y_j(s) \right| + \sum_{j=1}^n |b_{ij}(s)| L_j \left| x_j(s) - y_j(s) \right| + \sum_{j=1}^n |c_{ij}(s)| L_j \times \left| x_j(s - \tau_{ij}) - y_j(s - \tau_{ij}) \right| \right] ds
\leq e^{-Kt} |\phi_i(0) - \varphi_i(0)| + \frac{1}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1}
\times \left[ \sum_{j=1}^n |a_{ij}(s)| e^{-K(t-s)} e^{-Kt} \left| x_j(s) - y_j(s) \right| + \sum_{j=1}^n |b_{ij}(s)| L_j e^{-K(t-s)} e^{-Kt} \left| x_j(s) - y_j(s) \right| + \sum_{j=1}^n |c_{ij}(s)| L_j e^{-K(t-s-\tau_{ij})} e^{-K(t-\tau_{ij})} \times \left| x_j(s - \tau_{ij}) - y_j(s - \tau_{ij}) \right| \right] ds
\leq e^{-Kt} |\phi_i(0) - \varphi_i(0)| + \frac{\sup_{\tau_{ij}} |a_{ij}(\tau)|}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} e^{-K(t-s)} ds
\]
\[ + b_j L \sum_{j=1}^{n} \sup_t \left\{ e^{-K_t} \left| x_j(t) - y_j(t) \right| \right\} \frac{1}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} e^{-K(t-s)} ds \]

\[ + c_i L \sum_{j=1}^{n} \sup_t \left\{ e^{-K_t} \left| x_j(t) - y_j(t) \right| \right\} \frac{1}{\Gamma(\alpha)} \times \int_0^{t-t_j} (t-t_j-\theta)^{\alpha-1} e^{-K(t-\theta)} d\theta \]

\[ + \sum_{j=1}^{n} \sum_{i \in \mathbb{T}} e^{-K_i} \left| \phi_i(t) - \phi_i(t) \right| \frac{1}{\Gamma(\alpha)} \times \int_0^0 (t-t_j-\theta)^{\alpha-1} e^{-K(t-\theta)} d\theta \]

\[ < \sup_{t \in [-\tau, 0]} \left\{ e^{-K_t} \left| \phi_i(t) - \phi_i(t) \right| \right\} \]

\[ + \frac{a_i}{K^\alpha} \sum_{j=1}^{n} \sup_t \left\{ e^{-K_t} \left| x_j(t) - y_j(t) \right| \right\} \]

\[ + \frac{b_j L}{K^\alpha} \sum_{j=1}^{n} \sup_t \left\{ e^{-K_t} \left| x_j(t) - y_j(t) \right| \right\} \]

\[ + \frac{c_i L}{K^\alpha} \sum_{j=1}^{n} \sup_t \left\{ e^{-K_t} \left| x_j(t) - y_j(t) \right| \right\} \]

\[ + \frac{c_i L}{K^\alpha} \left\| \phi(t) - \phi(t) \right\| + \frac{a_i + b_j L + c_i L}{K^\alpha} \left\| x(t) - y(t) \right\| , \]

which implies

\[ \left\| x(t) - y(t) \right\| = \sum_{i=1}^{n} \left\{ e^{-K_t} \left| x_i(t) - y_i(t) \right| \right\} \]

\[ < \sum_{i=1}^{n} \left\{ e^{-K_t} \left| \phi_i(t) - \phi_i(t) \right| \right\} \]

\[ + \frac{c_i L}{K^\alpha} \left\| \phi(t) - \phi(t) \right\| + \frac{a_i + b_j L + c_i L}{K^\alpha} \left\| x(t) - y(t) \right\| . \]

(17)

Because \( K \) is a large enough constant, now we choose \( K \) large enough constant such that \( \|A\| + (\|B\| + \|C\|)L < K^\alpha \); then, from (18), we know that

\[ \left\| x(t) - y(t) \right\| < \frac{K^\alpha + \|C\| L}{K^\alpha - (\|A\| + (\|B\| + \|C\|)L)} \left\| \phi(t) - \phi(t) \right\| . \]

(19)

Therefore, for any \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) = ((K^\alpha - (\|A\| + (\|B\| + \|C\|)L))(K^\alpha + \|C\| L))\varepsilon > 0 \) such that \( \|x(t) - y(t)\| < \varepsilon \) when \( \|\phi(t) - \phi(t)\| < \delta \). According to Definition 5, the solution of system (6) is uniformly stable.

Remark 7. Reference [16] investigated the uniform stability and the existence and uniqueness of solutions of a class of fractional-order nonautonomous systems (or differential equations), respectively, but without considering the time delay.

Remark 8. To the best of our knowledge, the theoretical result on stability analysis of fractional-order nonlinear systems with multiple time delays has not been seen yet.

4. Illustrative Examples

In this section, we consider two examples to illustrate the obtained results.

Example 1. Consider the following fractional system with time-invariant coefficients and multiple time delays:

\[ D^\alpha x_1(t) = -1.3x_1(t) + 0.9x_2(t) - 0.5 \tanh(x_1(t)) \]

\[ + \tanh(x_2(t)) - 0.5 \tanh(x_1(t - 0.01)) \]

\[ + 0.6 \tanh(x_2(t - 0.02)) - 1.8, \]

\[ D^\alpha x_2(t) = 1.2x_1(t) - 1.6x_2(t) - 0.4 \tanh(x_1(t)) \]

\[ + 0.1 \tanh(x_2(t)) - \tanh(x_1(t - 0.02)) \]

\[ - 1.9 \tanh(x_2(t - 0.01)) + 2, \]

where \( \alpha = 0.9, T < +\infty \), with an associated function of the initial state:

\[ \Phi_1(t) = 2 \cos t, \quad \Phi_2(t) = \sin t - 1, \quad t \in [-0.02, 0]. \]

(20)

Since function \( \tanh(x) \) is Lipschitz continuous \( (L_1 = L_2 = 1) \), then the condition of the theorems is satisfied. Hence, by Theorems 4 and 6, we conclude that system (20) has a unique solution, which is uniformly stable.
In fact, system (20) has a unique fixed point \( x^* = (x^*_1, x^*_2)^T \), which satisfies

\[
-1.3x^*_1 + 0.9x^*_2 - \tanh(x^*_1) + 1.6 \tanh(x^*_2) - 1.8 = 0,
1.2x^*_1 - 1.6x^*_2 - 1.5 \tanh(x^*_1) - 1.5 \tanh(x^*_2) + 2 = 0.
\] (22)

By calculation, the unique fixed point \( x^* = (x^*_1, x^*_2)^T \) of system (20) is \((-0.088, 0.670)\). Figure 1 shows that the solution of system (20) converges to the fixed point \( x^* \).

Example 2. Consider the following fractional delay system with time-varying coefficients:

\[
D^\alpha x_1(t) = \frac{1}{5}e^{-t} x_1(t) - \frac{1}{6} \sin t \times x_2(t)
+ \frac{3}{10} \tanh(x_1(t)) - \frac{1 + \cos t}{8} \tanh(x_2(t))
- \frac{1}{8} \tanh(x_1(t - 0.03))
+ \frac{1}{9} \tanh(x_2(t - 0.01)) + \sin^2 t,
\]

\[
D^\alpha x_2(t) = \frac{1}{10} \cos t \times x_1(t) + \frac{1}{10} e^{-2t} x_2(t)
- \frac{1}{10} \tanh(x_1(t)) + \frac{1}{12} e^{-t} \tanh(x_2(t))
+ \frac{1}{6} \sin t \times \tanh(x_1(t - 0.01))
+ \frac{1}{8} \tanh(x_2(t - 0.02)) - \cosh(t),
\] (23)

where \( 0 < \alpha < 1, T < +\infty \), with an associated function of the initial state:

\[
\phi_1(t) = \left| x + \frac{3}{2} \right| - \left| x - \frac{1}{2} \right|, \quad \phi_2(t) = 1 - x^2,
\] (24)

\[ t \in [-0.03, 0]. \]

From (23), one can obtain

\[
A(t) = \left( \frac{1}{5} e^{-t} - \frac{1}{6} \sin t \right)
\]

\[
B(t) = \left( \frac{3}{10} - \frac{1 + \cos t}{8} \right)
\]

\[
C(t) = \left( -\frac{1}{8} \frac{1}{9} \sin t \frac{1}{8} \right)
\]

\[
u(t) = \left( \frac{\sin^2 t}{-\cosh(t)} \right).
\] (25)

Since function \( \tanh(x) \) is a Lipschitz continuous function, then according to Theorems 4 and 6, we conclude that system (23) has a unique uniformly stable solution.

5. Conclusion

Uniform stability problem of a class of fractional-order nonautonomous systems with multiple time delays is discussed in this paper. Moreover, the existence and uniqueness of solutions under certain conditions are proven by using Banach fixed point principle. At last, two examples are given for illustration.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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