Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 826173, 9 pages
http://dx.doi.org/10.1155/2014/826173

Research Article

A Class of Volterra-Fredholm Type Weakly Singular Difference Inequalities with Power Functions and Their Applications

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Received 12 July 2014; Accepted 5 August 2014; Published 14 August 2014

Academic Editor: Junjie Wei

We discuss a class of Volterra-Fredholm type difference inequalities with weakly singular. The upper bounds of the embedded unknown functions are estimated explicitly by analysis techniques. An application of the obtained inequalities to the estimation of Volterra-Fredholm type difference equations is given.

1. Introduction

Being an important tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities [1, 2] and their applications have attracted great interests of many mathematicians [3–5]. Some recent works can be found in [6–28].

In 1981, Henry [12] discussed the following linear singular integral inequality:

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1} u(s)\,ds. \tag{1}$$

In 2007, Ye et al. [18] discussed linear singular integral inequality

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1} u(s)\,ds. \tag{2}$$

In 2014, Cheng et al. [28] discussed the following inequalities:

$$\begin{aligned} u^m(t) \le{}& a(t) + b(t)\int_0^t f(s)\,u^n(s)\,ds \\ &+ c(t)\int_0^T g(s)\,u^r(s)\,ds, \\ u^m(t) \le{}& a(t) + b(t)\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{\beta_1-1} s^{\gamma_1-1} f(s)\,u^n(s)\,ds \\ &+ c(t)\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{\beta_2-1} s^{\gamma_2-1} g(s)\,u^r(s)\,ds. \end{aligned} \tag{3}$$

On the other hand, difference inequalities which give explicit bounds on unknown functions provide a very useful and important tool in the study of many qualitative as well as quantitative properties of solutions of nonlinear difference equations. More attentions are paid to some discrete versions of Gronwall-Bellman type inequalities (such as [29–50]).

In 2002, Pachpatte [36] discussed the following difference inequality:

$$u(n) \le c + \sum_{s=\alpha}^{n-1} f(n,s)\,u(s)\,ds + \sum_{s=\alpha}^{\beta} g(n,s)\,u(s), \quad n \in \mathbb{N} \cap [\alpha,\beta]. \tag{4}$$

In 2010, Ma [45] discussed the following difference inequality with two variables:

$$\begin{aligned} u^i(m,n) \le{}& a(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)\,u^j(s,t) \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} g(s,t)\,u^r(s,t). \end{aligned} \tag{5}$$

In 2014, Huang at el. [50] discussed the following linear singular difference inequality:

$$u(n) \le a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{\beta-1} \tau_s w_1(u(s)) \times \left[u(s) + h(s) + \sum_{\sigma=0}^{s-1} (t_s - t_\sigma)^{\beta-1} \tau_\sigma w_2(u(\sigma)) \right]. \tag{6}$$

Motivated by the results given in [6, 11, 28, 36, 45, 49, 50], in this paper, we discuss the following inequalities:

$$u(m,n) \le a(m,n) + b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) u(s,t) + c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t), \tag{7}$$

$$u^i(m,n) \le a(m,n) + b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) u^j(s,t) + c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u^r(s,t), \tag{8}$$

$$u^i(n) \le a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{\beta-1} t_s^{\gamma-1} \tau_s f(s) u^j(s) + c(n) \sum_{s=0}^{N-1} (t_N - t_s)^{\beta-1} t_s^{\gamma-1} \tau_s g(s) u^r(s). \tag{9}$$

2. Difference Inequalities with Two Variables

Throughout this paper, let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, \ldots\}$, and $\Omega_{X,Y} = \{(m,n) : m_0 \le m \le X, n_0 \le n \le Y, m, n, X, Y \in \mathbb{N}\}$. For a function $z(m,n)$, its first-order difference is defined by $\Delta_1 z(m,n) = z(m+1,n) - z(m,n)$. Obviously, the linear difference equation $\Delta z(n) = b(n)$ with the initial condition $z(n_0) = 0$ has the solution $z(n) = \sum_{s=n_0}^{n-1} b(s)$. For convenience, in the sequel, we complementarily define that $\sum_{s=n_0}^{n_0-1} b(s) = 0$.

Lemma 1. *Assume that $u(m,n)$, $a(m,n)$, $c(m,n)$, and $g(m,n)$ are nonnegative functions on $\Omega_{M,N} = \{(m,n) : m_0 \le m \le M, n_0 \le n \le N, m, n, M, N \in \mathbb{N}\}$. If $\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} g(s,t)c(s,t) < 1$ and $u(m,n)$ satisfies the following difference inequality:*

$$u(m,n) \le a(m,n) + c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t), \quad \forall (m,n) \in \Omega_{M,N}, \tag{10}$$

then

$$u(m,n) \le a(m,n) + \frac{c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t)}{1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t)}, \quad \forall (m,n) \in \Omega_{M,N}. \tag{11}$$

Proof. Since $\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)u(s,t)$ is a constant. Let $\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)u(s,t) = K$. From (10), we have

$$u(m,n) \le a(m,n) + c(m,n) K, \quad \forall (m,n) \in \Omega_{M,N}. \tag{12}$$

Since $g(m,n)$ is nonnegative, we have

$$g(m,n) u(m,n) \le g(m,n) a(m,n) + c(m,n) g(m,n) K. \tag{13}$$

Let $s = m$ and $t = n$ in (13) and substituting $s = m_0, m_1, m_2, \ldots, M-1$ and $t = n_0, n_1, n_2, \ldots, N-1$, successively, we obtain

$$\begin{aligned} K &= \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t) \\ &\le \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t) \\ &\quad + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t) K. \end{aligned} \tag{14}$$

From (14), we have

$$K \le \frac{\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t)}{1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t)}, \tag{15}$$

where $\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} g(s,t)c(s,t) < 1$. Substituting inequality (15) into (13), we get the explicit estimation (11) for $u(m,n)$. □

Theorem 2. *Assume that $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $f(m,n)$, and $g(m,n)$ are nonnegative functions on $\Omega_{M,N}$ and $a(m,n)$, $b(m,n)$, and $c(m,n)$ are nondecreasing in both m and n. If*

$$\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t) \times \exp\left(b(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f(\tau,\xi) \right) < 1, \quad \forall (m,n) \in \Omega_{M,N}, \tag{16}$$

and $u(m,n)$ satisfies the difference inequality (7), then

$$u(m,n) \le \exp\left(b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right)$$

$$\times \Bigg[a(m,n) + c(m,n) \times \Bigg(\Bigg(\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)\, a(s,t) \times \exp\Bigg(b(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f(\tau,\xi) \Bigg) \Bigg) \times \Bigg(1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)\, c(s,t) \times \exp\Bigg(b(s,t) \times \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f(\tau,\xi) \Bigg) \Bigg)^{-1} \Bigg) \Bigg], \tag{17}$$

for all $(m,n) \in \Omega_{M,N}$.

Proof. Fixing any arbitrary $(X,Y) \in \Omega_{M,N}$, from (7), we have

$$u(m,n) \le a(X,Y) + b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t)\, u(s,t) + c(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)\, u(s,t), \tag{18}$$

for all $(m,n) \in \Omega_{X,Y}$, where $a(m,n)$, $b(m,n)$, and $c(m,n)$ are nondecreasing in both m and n.

Define a function $z(m,n)$ by the right side of (18); that is,

$$z(m,n) := a(X,Y) + b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t)\, u(s,t) + c(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)\, u(s,t), \tag{19}$$

for all $(m,n) \in \Omega_{X,Y}$. Obviously, we have

$$u(m,n) \le z(m,n), \quad \forall (m,n) \in \Omega_{X,Y}, \tag{20}$$

$$z(m_0,n) = a(X,Y) + c(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t)\, u(s,t). \tag{21}$$

Using the difference formula $\Delta_1 z(m,n) = z(m+1,n) - z(m,n)$ and relation (20), from (21), we have

$$\begin{aligned} \Delta_1 z(m,n) &= b(X,Y) \sum_{t=n_0}^{n-1} f(m,t)\, u(m,t) \\ &\le b(X,Y) \sum_{t=n_0}^{n-1} f(m,t)\, z(m,t) \\ &\le b(X,Y)\, z(m,n) \sum_{t=n_0}^{n-1} f(m,t), \end{aligned} \tag{22}$$

where we have used the monotonicity of z in n. From (22), we observe that

$$\frac{\Delta_1 z(m,n)}{z(m,n)} \le b(X,Y) \sum_{t=n_0}^{n-1} f(m,t), \quad \forall (m,n) \in \Omega_{X,Y}. \tag{23}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given integers m, n with $(m+1,n)$, $(m,n) \in \Omega_{X,Y}$, there exists ξ in the open interval $(z(m,n), z(m,n+1))$ such that

$$\begin{aligned} \ln z(m+1,n) - \ln z(m,n) &= \int_{z(m,n)}^{z(m+1,n)} \frac{ds}{s} = \frac{\Delta_1 z(m,n)}{\xi} \\ &\le \frac{\Delta_1 z(m,n)}{z(m,n)}. \end{aligned} \tag{24}$$

From (23) and (24), we have

$$\ln z(m+1,n) - \ln z(m,n) \le b(X,Y) \sum_{t=n_0}^{n-1} f(m,t), \quad \forall (m,n) \in \Omega_{X,Y}. \tag{25}$$

Let $s = m$ and $t = n$ in (25), and substituting $s = m_0, m_1, m_2, \ldots, m-1$ and $t = n_0, n_1, n_2, \ldots, n-1$, successively, we obtain

$$\ln z(m,n) - \ln z(m_0,n) \le b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t), \quad \forall (m,n) \in \Omega_{X,Y}. \tag{26}$$

It implies that

$$z(m,n) \le z(m_0,n) \exp\Bigg(b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \Bigg), \quad \forall (m,n) \in \Omega_{X,Y}. \tag{27}$$

Using (20) and (21), from (27), we have

$$\begin{aligned} u(m,n) &\le \left(a(X,Y)+c(X,Y)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}g(s,t)u(s,t)\right) \\ &\quad\times\exp\left(b(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\right) \\ &= a(X,Y)\exp\left(b(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\right) \\ &\quad+c(X,Y)\exp\left(b(X,Y)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\right) \\ &\quad\times\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}g(s,t)u(s,t), \end{aligned} \tag{28}$$

for all $(m,n)\in\Omega_{X,Y}$. Taking $m=X$ and $n=Y$ in (28), we have

$$\begin{aligned} u(X,Y) &\le a(X,Y)\exp\left(b(X,Y)\sum_{s=m_0}^{X-1}\sum_{t=n_0}^{Y-1}f(s,t)\right) \\ &\quad+c(X,Y)\exp\left(b(X,Y)\sum_{s=m_0}^{X-1}\sum_{t=n_0}^{Y-1}f(s,t)\right) \\ &\quad\times\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}g(s,t)u(s,t). \end{aligned} \tag{29}$$

Since X, Y are chosen arbitrarily, we replace X and Y in (29) with m and n, respectively, and obtain that

$$\begin{aligned} u(m,n) &\le a(m,n)\exp\left(b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\right) \\ &\quad+c(m,n)\exp\left(b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\right) \\ &\quad\times\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}g(s,t)u(s,t), \end{aligned} \tag{30}$$

for all $(m,n)\in\Omega_{M,N}$. Applying the result of Lemma 1 to inequality (30), we obtain desired estimation (17). □

Lemma 3 (see [39]). *Let $a\ge 0$, $i\ge j\ge 0$, and $i\ne 0$. Then,*

$$a^{j/i}\le\frac{j}{i}K^{(i-j)/i}a+\frac{i-j}{i}K^{j/i},\quad\forall K>0. \tag{31}$$

Theorem 4. *Assume that $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $f(m,n)$, and $g(m,n)$ are defined as in Theorem 2 and that $i\ge j>0$ and $i\ge r>0$. If*

$$\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}G(s,t)C(s,t)\exp\left(B(s,t)\sum_{\tau=m_0}^{s-1}\sum_{\xi=n_0}^{t-1}F(\tau,\xi)\right)<1,\quad\forall(m,n)\in\Omega_{M,N}, \tag{32}$$

and $u(m,n)$ satisfies difference inequality (8), then

$$\begin{aligned} u(m,n) &\le\Bigg\{a(m,n)+\exp\left(B(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}F(s,t)\right) \\ &\quad\times\Bigg[A(m,n)+C(m,n) \\ &\quad\times\Bigg(\left(\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}G(s,t)A(s,t)\right. \\ &\qquad\left.\times\exp\left(B(s,t)\sum_{\tau=m_0}^{s-1}\sum_{\xi=n_0}^{t-1}F(\tau,\xi)\right)\right) \\ &\quad\times\Bigg(1-\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}G(s,t)C(s,t) \\ &\qquad\times\exp\Bigg(B(s,t) \\ &\qquad\times\sum_{\tau=m_0}^{s-1}\sum_{\xi=n_0}^{t-1}F(\tau,\xi)\Bigg)\Bigg)^{-1}\Bigg)\Bigg]\Bigg\}^{1/i}, \end{aligned} \tag{33}$$

for all $(m,n)\in\Omega_{M,N}$, where

$$\begin{aligned} A(m,n) &:= b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f(s,t)\left(\frac{j}{i}K_1^{(i-j)/i}a(s,t)+\frac{i-j}{i}K_1^{j/i}\right) \\ &\quad+c(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}g(s,t)\left(\frac{r}{i}K_2^{(i-r)/i}a(s,t)\right. \\ &\qquad\left.+\frac{i-r}{i}K_2^{r/i}\right), \end{aligned} \tag{34}$$

$$B(m,n):=\frac{jb(m,n)}{i},\qquad C(m,n):=\frac{rc(m,n)}{i}, \tag{35}$$

$$F(m,n):=f(s,t)K_1^{(i-j)/i},\qquad G(m,n):=g(s,t)K_2^{(i-r)/i}, \tag{36}$$

and K_1, K_2 are arbitrary constants.

Proof. Define a function $v(m,n)$ by

$$v(m,n) = b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)\,u^j(s,t) + c(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} g(s,t)\,u^r(s,t), \tag{37}$$

for all $(m,n)\in\Omega_{M,N}$. Then, from (8), we have

$$u(m,n) \le (a(m,n)+v(m,n))^{1/i}, \quad \forall (m,n)\in\Omega_{M,N}. \tag{38}$$

Applying Lemma 3 to (38), we obtain

$$\begin{aligned} u^j(m,n) &\le (a(m,n)+v(m,n))^{j/i} \\ &\le \frac{j}{i}K_1^{(i-j)/i}(a(m,n)+v(m,n)) + \frac{i-j}{i}K_1^{j/i}, \\ u^r(m,n) &\le (a(m,n)+v(m,n))^{r/i} \\ &\le \frac{r}{i}K_2^{(i-r)/i}(a(m,n)+v(m,n)) + \frac{i-r}{i}K_2^{r/i}, \end{aligned} \tag{39}$$

for all $(m,n)\in\Omega_{M,N}$. Substituting (39) into (37), we obtain

$$\begin{aligned} v(m,n) &\le b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t) \times\left(\frac{j}{i}K_1^{(i-j)/i}(a(s,t)+v(s,t)) + \frac{i-j}{i}K_1^{j/i}\right) \\ &\quad + c(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} g(s,t) \times\left(\frac{r}{i}K_2^{(i-r)/i}(a(s,t)+v(s,t)) + \frac{i-r}{i}K_2^{r/i}\right) \\ &= b(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)\left(\frac{j}{i}K_1^{(i-j)/i}a(s,t) + \frac{i-j}{i}K_1^{j/i}\right) \\ &\quad + c(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} g(s,t) \times\left(\frac{r}{i}K_2^{(i-r)/i}a(s,t) + \frac{i-r}{i}K_2^{r/i}\right) \\ &\quad + \frac{jb(m,n)}{i}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f(s,t)\,K_1^{(i-j)/i}v(s,t) \\ &\quad + \frac{rc(m,n)}{i}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} g(s,t)\,K_2^{(i-r)/i}v(s,t) \\ &= A(m,n) + B(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s,t)\,v(s,t) \\ &\quad + C(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} G(s,t)\,v(s,t), \end{aligned} \tag{40}$$

for all $(m,n)\in\Omega_{M,N}$, where A, B, C and F, G are defined by (34), (35), and (36), respectively. Since $a(m,n)$, $b(m,n)$, and $c(m,n)$ are nonnegative and nondecreasing in both m and n and by (34), (35), and (36), $A(m,n)$, $B(m,n)$, and $C(m,n)$ are also nonnegative and nondecreasing in both m and n. Using Theorem 2, from (40), we obtain

$$\begin{aligned} v(m,n) &\le \exp\left(B(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s,t)\right) \\ &\quad \times\Bigg[A(m,n) + C(m,n) \\ &\quad \times\Bigg(\left(\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} G(s,t)\,A(s,t) \times \exp\left(B(s,t)\sum_{\tau=m_0}^{s-1}\sum_{\xi=n_0}^{t-1} F(\tau,\xi)\right)\right) \\ &\quad \times\left(1 - \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} G(s,t)\,C(s,t) \times \exp\left(B(s,t)\sum_{\tau=m_0}^{s-1}\sum_{\xi=n_0}^{t-1} F(\tau,\xi)\right)\right)^{-1}\Bigg)\Bigg], \end{aligned} \tag{41}$$

for all $(m,n)\in\Omega_{M,N}$. Substituting (41) into (38), we get our required estimation (33) of unknown function in (8). □

3. Difference Inequality with Weakly Singular

For the reader's convenience, we present some necessary Lemmas.

Lemma 5 (discrete Jensen inequality [47]). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers, $k > 1$ a real number, and n a natural number. Then,*

$$\left(A_1 + A_2 + \cdots + A_n\right)^k \le n^{k-1}\left(A_1^k + A_2^k + \cdots + A_n^k\right). \quad (42)$$

Lemma 6 (discrete Hölder inequality [48]). *Let a_i, b_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers and p, q positive numbers such that $(1/q) + (1/p) = 1$. Then,*

$$\sum_{i=0}^{n-1} a_i b_i \le \left(\sum_{i=0}^{n-1} a_i^p\right)^{1/p}\left(\sum_{i=0}^{n-1} b_i^q\right)^{1/q}. \quad (43)$$

Lemma 7 (see [15, 49]). *Let $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, and $\sup_{s\in\mathbb{N}, 0\le s\le n-1}\{\tau_s, s \in \mathbb{N}\} = \tau$. If $\beta \in (0.5, 1)$, $\gamma > 1.5 - \beta$, and $p = 1/\beta$, then*

$$\begin{aligned}&\sum_{s=0}^{n-1}(t_n - t_s)^{p(\beta-1)} t_s^{p(\gamma-1)}\tau_s \\ &\quad \le t_n^{\theta}\mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right],\end{aligned} \quad (44)$$

where $\theta = p(\beta+\gamma-2)+1 > 0$ and $\mathscr{B}(\xi,\eta) := \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ is the well-known $\mathscr{B}$-function.

Now, we consider the weakly singular difference inequality (9).

Theorem 8. *Let $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s\in\mathbb{N}, 0\le s\le n-1}\{\tau_s, s \in \mathbb{N}\} = \tau$, $\beta \in (0.5, 1)$, and $\gamma > 1.5 - \beta$. Assume that $i \ge j > 0$, $i \ge r > 0$, $u(n)$, $a(n)$, $b(n)$, $c(n)$, $f(n)$, and $g(n)$ are nonnegative functions on $\mathbb{N}_0$ and $a(n)$, $b(n)$, and $c(n)$ are nondecreasing. If*

$$\sum_{s=0}^{N-1}\widetilde{G}(s)\widetilde{C}(s)\exp\left(\widetilde{B}(s)\sum_{\tau=0}^{s-1}\widetilde{F}(\tau)\right) < 1, \quad n \in \mathbb{N}_0,\ n < N, \quad (45)$$

and $u(n)$ satisfies (9), then

$$\begin{aligned}u(n) &\le \Bigg\{a(n) + \exp\left(\widetilde{B}(n)\sum_{s=0}^{n-1}\widetilde{F}(s)\right) \\ &\quad\times\Bigg[\widetilde{A}(n) + \widetilde{C}(n) \\ &\quad\times\Bigg(\Bigg(\sum_{s=0}^{N-1}\widetilde{G}(s)\widetilde{A}(s) \\ &\quad\times\exp\left(\widetilde{B}(s)\sum_{\tau=0}^{s-1}\widetilde{F}(\tau)\right)\Bigg) \\ &\quad\times\Bigg(1 - \sum_{s=0}^{N-1}\widetilde{G}(s)\widetilde{C}(s) \\ &\quad\times\exp\left(\widetilde{B}(s)\sum_{\tau=0}^{s-1}\widetilde{F}(\tau)\right)\Bigg)^{-1}\Bigg)\Bigg]\Bigg\}^{1/i}, \\ &\qquad n \in \mathbb{N}_0,\ n < N,\end{aligned} \quad (46)$$

where

$$\begin{aligned}\widetilde{A}(n) &:= \widetilde{b}(n)\sum_{s=0}^{n-1} f^q(s)\left(\frac{j}{i}K_1^{(i-j)/i}\widetilde{a}(s) + \frac{i-j}{i}K_1^{j/i}\right) \\ &\quad + \widetilde{c}(n)\sum_{s=n_0}^{N-1} g^q(s)\left(\frac{r}{i}K_2^{(i-r)/i}\widetilde{a}(s) + \frac{i-r}{i}K_2^{r/i}\right), \\ \widetilde{B}(n) &:= \frac{j\widetilde{b}(n)}{i}, \qquad \widetilde{C}(n) := \frac{r\widetilde{c}(n)}{i}, \\ \widetilde{F}(n) &:= f^q(n)K_1^{(i-j)/i}, \qquad \widetilde{G}(n) := g^q(n)K_2^{(i-r)/i}, \\ \widetilde{a}(n) &:= 3^{q-1}a^q(n), \\ \widetilde{b}(n) &:= 3^{q-1}b^q(n)\tau \\ &\quad\times\left(t_n^{\theta}\mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p}, \\ \widetilde{c}(n) &:= 3^{q-1}c^q(n)\tau \\ &\quad\times\left(t_N^{\theta}\mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p}, \\ \theta &:= p(\beta+\gamma-2)+1,\end{aligned} \quad (47)$$

and $p = 1/\beta$, $q = 1/(1-\beta)$, and K_1, K_2 are arbitrary constants.

Proof. Applying Lemma 6 with $p = 1/\beta$, $q = 1/(1-\beta)$ to (8), we obtain that

$$\begin{aligned}u^i(n) &\le a(n) + b(n)\tau^{(p-1)/p} \\ &\quad\times\left(\sum_{s=n_0}^{n-1}(t_n - t_s)^{p(\beta-1)}t_s^{p(\gamma-1)}\tau_s\right)^{1/p} \\ &\quad\times\left(\sum_{s=n_0}^{n-1} f^q(s)u^{qj}(s)\right)^{1/q} \\ &\quad + c(n)\tau^{(p-1)/p} \\ &\quad\times\left(\sum_{s=n_0}^{N-1}(t_N - t_s)^{p(\beta-1)}t_s^{p(\gamma-1)}\tau_s\right)^{1/p} \\ &\quad\times\left(\sum_{s=n_0}^{n-1} g^q(s)u^{qr}(s)\right)^{1/q},\end{aligned} \quad (48)$$

for all $n \in \mathbb{N}_0$, $n < N$, where $\tau_s < \tau$ is used. Applying Lemma 5 to (48), we have

$$\begin{aligned}u^i(n) &\le a(n) + b(n)\tau^{(p-1)/p} \\ &\quad\times\left(t_n^{\theta}\mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{1/p} \\ &\quad\times\left(\sum_{s=n_0}^{n-1} f^q(s)u^{qj}(s)\right)^{1/q} \\ &\quad + c(n)\tau^{(p-1)/p} \\ &\quad\times\left(t_N^{\theta}\mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{1/p} \\ &\quad\times\left(\sum_{s=n_0}^{n-1} g^q(s)u^{qr}(s)\right)^{1/q},\end{aligned} \quad (49)$$

for all $n \in \mathbb{N}_0$, $n < N$. By discrete Jensen inequality (42) with $n = 2$, $k = q$, from (49), we obtain that

$$\begin{aligned} u^{qi}(n) &\le 3^{q-1}a^q(n) + 3^{q-1}b^q(n)\tau^{q(p-1)/p} \\ &\quad \times \left(t_n^\theta \mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p} \\ &\quad \times \sum_{s=n_0}^{n-1} f^q(s)u^{qj}(s) + 3^{q-1}c^q(n)\tau^{q(p-1)/p} \\ &\quad \times \left(t_N^\theta \mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p} \\ &\quad \times \sum_{s=n_0}^{n-1} g^q(s)u^{qr}(s) \\ &= 3^{q-1}a^q(n) + 3^{q-1}b^q(n)\tau \\ &\quad \times \left(t_n^\theta \mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p} \\ &\quad \times \sum_{s=n_0}^{n-1} f^q(s)u^{qj}(s) + 3^{q-1}c^q(n)\tau \\ &\quad \times \left(t_N^\theta \mathscr{B}\left[p(\gamma-1)+1, p(\beta-1)+1\right]\right)^{q/p} \\ &\quad \times \sum_{s=n_0}^{n-1} g^q(s)u^{qr}(s) \\ &= \tilde{a}(n) + \tilde{b}(n)\sum_{s=n_0}^{n-1} f^q(s)u^{qj}(s) \\ &\quad + \tilde{c}(n)\sum_{s=n_0}^{n-1} g^q(s)u^{qr}(s), \\ &\qquad n \in \mathbb{N}_0,\ n < N. \end{aligned} \tag{50}$$

Applying Theorem 4 to (50), we have

$$\begin{aligned} &u(n) \\ &\le \Bigg\{ a(n) + \exp\left(\tilde{B}(n)\sum_{s=0}^{n-1}\tilde{F}(s)\right) \\ &\quad \times \Bigg[\tilde{A}(n) + \tilde{C}(n) \\ &\quad \times \Bigg(\Bigg(\sum_{s=0}^{N-1}\tilde{G}(s)\tilde{A}(s) \\ &\quad \times \exp\left(\tilde{B}(s)\sum_{\tau=0}^{s-1}\tilde{F}(\tau)\right)\Bigg) \\ &\quad \times \Bigg(1 - \sum_{s=0}^{N-1}\tilde{G}(s)\tilde{C}(s) \\ &\quad \times \exp\left(\tilde{B}(s)\sum_{\tau=0}^{s-1}\tilde{F}(\tau)\right)\Bigg)^{-1}\Bigg)\Bigg]\Bigg\}^{1/i}, \\ &\qquad n \in \mathbb{N}_0,\ n < N. \end{aligned} \tag{51}$$

This is our required estimation (46) of unknown function in (9). □

4. Applications

In this section, we apply our results to discuss the boundedness of solutions of an iterative difference equation with a weakly singular kernel.

Example 9. Suppose that $u(n)$ satisfies the difference equation

$$\begin{aligned} x^3(n) &= a(n) + b(n)\sum_{s=0}^{n-1}(t_n - t_s)^{-0.3}t_s^{-0.1}\tau_s f(s)x^2(s) \\ &\quad + c(n)\sum_{s=0}^{N-1}(t_N - t_s)^{-0.3}t_s^{-0.1}\tau_s g(s)x(s), \end{aligned} \tag{52}$$

where $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s\in\mathbb{N}, 0\le s\le n-1}\{\tau_s, s \in \mathbb{N}\} = \tau$, $u(n)$, $a(n)$, $b(n)$, $c(n)$, $f(n)$, and $g(n)$ are nonnegative functions on $\mathbb{N}_0$, and $a(n)$, $b(n)$, and $c(n)$ are nondecreasing. From (52), we have

$$\begin{aligned} |x(n)|^3 &\le a(n) + b(n)\sum_{s=0}^{n-1}(t_n - t_s)^{-0.3}t_s^{-0.1}\tau_s f(s)|x(s)|^2 \\ &\quad + c(n)\sum_{s=0}^{N-1}(t_N - t_s)^{-0.3}t_s^{-0.1}\tau_s g(s)|x(s)|. \end{aligned} \tag{53}$$

Let $p = 10/7$, $q = 10/3$, and K_1, K_2 are arbitrary constants, and

$$\begin{aligned} &\theta := \frac{3}{7}, \qquad \tilde{a}(n) := 3^{7/3}a^{10/3}(n), \\ &\tilde{b}(n) := 3^{7/3}b^{10/3}(n)\tau\left(t_n^\theta \mathscr{B}\left[\frac{6}{7}, \frac{4}{7}\right]\right)^{7/3}, \\ &\tilde{c}(n) := 3^{7/3}c^{10/3}(n)\tau\left(t_N^\theta \mathscr{B}\left[\frac{6}{7}, \frac{4}{7}\right]\right)^{7/3}, \\ &\tilde{A}(n) := \tilde{b}(n)\sum_{s=0}^{n-1} f^{10/3}(s)\left(\frac{2}{3}K_1^{1/3}\tilde{a}(s) + \frac{1}{3}K_1^{2/3}\right) \\ &\qquad + \tilde{c}(n)\sum_{s=n_0}^{N-1} g^{10/3}(s)\left(\frac{1}{3}K_2^{2/3}\tilde{a}(s) + \frac{2}{3}K_2^{1/3}\right), \\ &\tilde{B}(n) := \frac{2\tilde{b}(n)}{3}, \qquad \tilde{C}(n) := \frac{\tilde{c}(n)}{3}, \\ &\tilde{F}(n) := f^{10/3}(n)K_1^{1/3}, \qquad \tilde{G}(n) := g^{10/3}(n)K_2^{2/3}. \end{aligned} \tag{54}$$

If

$$\begin{aligned} &\sum_{s=0}^{N-1}\tilde{G}(s)\tilde{C}(s)\exp\left(\tilde{B}(s)\sum_{\tau=0}^{s-1}\tilde{F}(\tau)\right) < 1, \\ &\qquad n \in \mathbb{N}_0,\ n < N. \end{aligned} \tag{55}$$

Applying Theorem 8 to (53), we obtain the estimation of the solutions of difference equation (52)

$$
\begin{aligned}
&|x(n)| \\
&\quad \le \left\{ a(n) + \exp\left(\widetilde{B}(n) \sum_{s=0}^{n-1} \widetilde{F}(s) \right) \right. \\
&\qquad \times \left[\widetilde{A}(n) + \widetilde{C}(n) \right. \\
&\qquad \times \left(\left(\sum_{s=0}^{N-1} \widetilde{G}(s) \widetilde{A}(s) \right. \right. \\
&\qquad\qquad \left. \left. \times \exp\left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \right) \right. \\
&\qquad \times \left(1 - \sum_{s=0}^{N-1} \widetilde{G}(s) \widetilde{C}(s) \right. \\
&\qquad\qquad \left. \left. \left. \times \exp\left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \right)^{-1} \right) \right] \left. \right\}^{1/i}, \\
&\qquad\qquad n \in \mathbb{N}_0, \ n < N.
\end{aligned} \tag{56}
$$

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Project no. 11161018), the Guangxi Natural Science Foundation of China (Projects nos. 2012GXNSFAA053009, 2013GXNSFAA019022), and the Scientific Research Foundation of the Education Department of Guangxi Autonomous Region (no. 2013YB243).

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