New Exact Solutions for High Dispersive Cubic-Quintic Nonlinear Schrödinger Equation

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We study a class of high dispersive cubic-quintic nonlinear Schrödinger equations, which describes the propagation of femtosecond light pulses in a medium that exhibits a parabolic nonlinearity law. Applying bifurcation theory of dynamical systems and the Fan sub-equations method, more types of exact solutions, particularly solitary wave solutions, are obtained for the first time.

1. Introduction

Propagation of short pulses in optical fibers is governed by the well-known nonlinear Schrödinger equation (NLS) [1]. In recent years, there have been extensive study and application of NLS. The main purpose of this paper is to discuss the traveling wave solutions for a class of high dispersive cubic-Quintic nonlinear Schrödinger equations describing the ultrashort light pulse propagation as in the following:

\[ E_z = -i\beta_2 E_{tt} + i\gamma_1 |E|^2 E + \beta_3 E_{ttt} + i\beta_4 E_{tttt} - i\gamma_2 |E|^4 E, \]  

(1)

where \( E(z, t) \) is the slowly varying envelope of the electric field, \( \beta_2 \) is the parameter of the group velocity dispersion, \( \beta_3 \) and \( \beta_4 \) are, respectively, the third-order and fourth-order dispersions, and \( \gamma_1 \) and \( \gamma_2 \) are the nonlinearity coefficients. When the higher order terms are ignored, we obtain the NLS. However, for femtosecond light pulses, whose duration is shorter than 10 fs, the last three terms are not ignored. Equation (1) was derived by Palacios and Fernández-Díaz [2]. Azzouzi et al. [3] by using the extended hyperbolic auxiliary equation method in getting the exact explicit solutions to (1). He et al. [4] find the exact bright, dark, and gray analytical nonautonomous soliton solutions of the generalized CQNLSE with spatially inhomogeneous group velocity dispersion (GVD) and amplification or attenuation by the similarity transformation method under certain parametric conditions.

We will study (1) by using the improved Fan subequation method. As a result, more types of exact solutions to (1) are obtained, which include solitons, kink solutions, and Jacobian elliptic function solutions with double periods. The rest of this paper is organized as follows. In Section 2, we give the mathematical framework of the improved method. In Section 3, we apply it to the generalized equation (1) for finding more exact solutions. Finally, some conclusions are given.

2. The Ansatz Solution and Fan Subequation Method

The integrability of a nonlinear equation can be studied by applying the Painleve analysis. It is widely believed that possession of the Painleve property is a sufficient criterion for integrability. Moreover, there exists another technique which basically consists of expressing the solution in terms of an amplitude and a phase function as an approach to find exact solutions of nonlinear evolution equations. We will make use of this formalism looking for exact solution of (1) such as

\[ E(z, t) = e^{i(wz - wt)} \varphi(\xi), \]  

(2)
where \( \varphi(\xi) \) is a real function and \( \xi = \nu_0 z - vt \). By inserting the expressions (2) into (1), and separating real and imaginary parts, we obtain

\[
\begin{align*}
I_1 \varphi' (\xi) + I_2 \varphi''' (\xi) &= 0, \\
I_0 \varphi (\xi) + I_2 \varphi'' (\xi) + I_4 \varphi''' (\xi) + \gamma_1 \varphi^3 (\xi) - \gamma_2 \varphi^5 (\xi) &= 0,
\end{align*}
\]

(3a)

(3b)

where

\[
\begin{align*}
l_0 &= \frac{1}{2} w^2 \left( \beta_2 + \frac{1}{3} \beta_3 w + \frac{1}{12} \beta_4 w^2 \right) - w_0, \\
l_1 &= \nu \nu \left( \beta_2 + \frac{1}{2} \beta_3 w + \frac{1}{6} \beta_4 w^2 \right) - \nu_0, \\
l_2 &= -\nu^2 \left( \frac{1}{2} \beta_2 + \frac{1}{2} \beta_3 w + \frac{1}{4} \beta_4 w^2 \right), \\
l_3 &= -\frac{1}{6} (\beta_3 + \beta_4 w)^2, \\
l_4 &= \frac{1}{24} \beta_4 \nu^4.
\end{align*}
\]

Let \( l_1 = l_3 = 0 \), and we get

\[
\begin{align*}
w &= -\frac{\beta_3}{\beta_4}, \\
v_0 &= \frac{\nu \beta_3 (\beta_2 - 3 \beta_2 \beta_4)}{3 \beta_4^2}, \\
l_0 &= -w_0 - \frac{\beta_3^2 (\beta_2 - 4 \beta_2 \beta_4)}{8 \beta_4^3}, \\
l_2 &= \nu^2 \left( \frac{\beta_2 - 2 \beta_2 \beta_4}{4 \beta_4} \right), \\
l_4 &= \frac{1}{24} \beta_4 \nu^4.
\end{align*}
\]

(5)

Then, (3a) and (3b) become

\[
l_0 \varphi (\xi) + l_2 \varphi'' (\xi) + l_4 \varphi''' (\xi) + \gamma_1 \varphi^3 (\xi) - \gamma_2 \varphi^5 (\xi) = 0. \quad (6)
\]

We introduce auxiliary equation:

\[
\varphi' (\xi) = \epsilon \sqrt{c_0 + c_1 \varphi (\xi) + c_2 \varphi^2 (\xi) + c_3 \varphi^3 (\xi) + c_4 \varphi^4 (\xi)}, \quad (7)
\]

where \( \epsilon = \pm 1 \), which is known as Fan subequation method and proposed by Fan in [5]. This method is proposed to seek more types of exact solutions of nonlinear partial differential equations. Obviously, (7) is equivalent to the two-dimensional systems as follows:

\[
\frac{d \varphi}{d \xi} = y, \quad \frac{d y}{d \xi} = \frac{1}{2} \left( c_1 + 2 c_2 \varphi + 3 c_3 \varphi^2 + 4 c_4 \varphi^3 \right), \quad (8)
\]

which has the Hamiltonian function:

\[
H (\varphi, y) = y^2 - \left( c_1 \varphi + c_2 \varphi^2 + c_3 \varphi^3 + c_4 \varphi^4 \right) = c_0. \quad (9)
\]

One can easily find that \( c_0 \) corresponds to the Hamiltonian constant and (7) is equivalent to the Hamiltonian system (8). Thus, in order to search the exact solutions of (7) we need only to discuss (8). For a fixed \( c_0 \), (9) determines a set of orbits of (8). As \( c_0 \) varies, (9) defines different families of orbits of (8) which have different dynamical behavior. Below we will first study the bifurcation of phase portraits of (8) by making use of bifurcation method of dynamical systems and with the aid of the computer symbolic system Mathematica. Then according to the obtained bifurcation and the Hamiltonian function (9), we will gain many exact solutions of (7) for all possible parameters \( c_i \). [6].

Substituting (7) into (6), we have

\[
\begin{align*}
(-\gamma_2 + 4 l_4 c_4^3) \varphi'^4 + 30 l_4 c_3 c_4 \varphi^4 \\
+ \left( 2 l_2 c_4 + l_4 \left( 20 c_2 c_4 + \frac{15}{2} \beta_3^2 \right) + \gamma_1 \right) \varphi^3 \\
+ \left( \frac{3}{2} l_2 c_5 + l_4 \left( \frac{15}{2} c_2 c_4 + 1 \right) \frac{c_1 c_4}{\beta_4} \right) \varphi^2 \\
+ \left( l_0 + l_4 c_2 + l_4 \left( \frac{9}{2} c_2 c_3 + 1 \right) c_3 c_4 + c_5^2 \right) \varphi \\
+ \frac{1}{2} l_4 c_1 + l_4 \left( 3 c_6 c_3 + \frac{1}{2} c_4 c_2 \right) = 0.
\end{align*}
\]

(10)

Setting all the coefficients of \( \varphi^i \) (\( i = 0, 1, \ldots, 5 \)) to zero, and solving the obtained algebraic equations, we find the following sets of solutions (I)

\[
\begin{align*}
c_1 &= c_3 = 0, \\
c_2 &= \frac{1}{5 \nu^2} \left( - \frac{3 \left( \beta_2^2 - 2 \beta_2 \beta_4 \right)}{\beta_4^2} - \frac{6 \gamma_1 \sqrt{\gamma_2 / \beta_4}}{5 \gamma_2} \right), \\
c_4 &= \frac{1}{5 \nu^2} \sqrt{\frac{\gamma_2}{\beta_4}}, \\
c_0 &= \left( 50 u_0 \gamma_2 \beta_4 - 3 \gamma_2 \beta_4^3 - 52 \gamma_2 \beta_2 \beta_4 \right) \\
&\quad + 27 \gamma_2 \beta_2 \beta_4 \beta_4 + 13 \gamma_2 \beta_3^2 \beta_4 \\
&\quad \times \sqrt{\frac{\gamma_2}{\beta_4}} + 12 \gamma_1 \gamma_2 \beta_4 \left( \beta_2^2 - 2 \beta_2 \beta_4 \right) \\
&\quad \times \left( 25 \beta_2^2 \beta_4^2 \right)^{-1}.
\end{align*}
\]

(11)

and (II)

\[
\begin{align*}
c_1 &= c_3 = 0, \\
c_2 &= \frac{1}{5 \nu^2} \left( - \frac{3 \left( \beta_2^2 - 2 \beta_2 \beta_4 \right)}{\beta_4^2} + \frac{6 \gamma_1 \sqrt{\gamma_2 / \beta_4}}{\gamma_2} \right), \\
c_4 &= -\frac{1}{5 \nu^2} \sqrt{\frac{\gamma_2}{\beta_4}},
\end{align*}
\]

(12)
\begin{equation}
\psi_{\pm1} = \pm \sqrt{\frac{c_2 + \sqrt{\Delta}}{c_4}} \left( \sqrt{\frac{c_2 + \sqrt{\Delta}}{2}} + \sqrt{\frac{c_2}{c_4} + \Delta} \right).
\end{equation}

where sn(x, k) and below cn(x, k) are Jacobian elliptic functions with modulus k [10]. The profiles of periodic solutions are shown in Figure 2.

Thus, we obtain the following solutions of (1):

\begin{equation}
E(z,t) = \pm e^{i(w_0z + \beta_2 t/\beta_4)} \sqrt{\frac{c_2}{c_4} \text{sech} \left( \sqrt{2} \xi \right)}.
\end{equation}

The profiles of solutions of (22) are shown in Figure 3.
Remark 2. By the expression of \( c_0 \), there always exists a \( \omega_0 \) such that \( c_0 = 0 \) if \( (c_2, c_4) \in D4 \).

Because of the limitation of length, we omit the expression of \( E(z, t) \), beginning from here.

(c) For \( c_0 > 0 \), (6) has periodic solution:

\[
\varphi_3 = \sqrt{\frac{c_2 + \sqrt{\Delta}}{-2c_4}} \, \text{cn} \left( \sqrt{\Delta} \xi, \sqrt{\frac{c_2 + \sqrt{\Delta}}{2}} \right). \tag{23}
\]

(2) Suppose that \( (c_2, c_4) \in D2 \).

(a) For \( c_0 \in (0, c_2^2/(4c_4)) \), system (14) has periodic solution:

\[
\varphi_{44} = \pm \sqrt{-c_2 - \sqrt{\Delta}} \, \text{sn} \left( \sqrt{-c_2 - \sqrt{\Delta}} \, \xi, \sqrt{\frac{-c_2 + \sqrt{\Delta}}{2}} \, \text{sn} \left( \sqrt{-c_2 + \sqrt{\Delta}} \xi, \sqrt{\frac{-c_2 + \sqrt{\Delta}}{2}} \right) \right). \tag{24}
\]

(b) For \( c_0 = c_2^2/(4c_4) \), there exists smooth kink wave solution of (14) as follows:

\[
\varphi_{45} = \pm \sqrt{-c_2} \, \text{tanh} \left( \sqrt{-c_2} \, \xi \right). \tag{25}
\]
(3) If \((c_2, c_4) \in D_3\), for \(c_0 > 0\), we have the following periodic solution of (14):
\[
\varphi_{±0} = \pm \sqrt{\frac{-c_2 - \sqrt{\Delta}}{2c_4}} \operatorname{cn} \left( \frac{c_2 + \sqrt{\Delta}}{2 \sqrt{\Delta}}, \frac{c_2}{2} \right). \tag{26}
\]

(4) If \((c_2, c_4) \in D_1\), it is observed from Figure 1(a) that there is no bounded solution of system (14).

3.2. The Case (III). For this case, the Hamiltonian system (8) becomes
\[
\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = f(\varphi) = \frac{1}{2} \left( c_1 + 2c_2 \varphi + 3c_3 \varphi^2 \right) \tag{27}
\]
and the Hamiltonian function (9) reduces to
\[
H_1(\varphi, y) = y^2 - c_1 \varphi - c_2 \varphi^2 - c_3 \varphi^3 = c_0. \tag{28}
\]

Similar to the previous discussion, we have the following proposition.

**Proposition 3.**

1. For \(c_2^2 - 3c_1c_3 > 0\), (27) has two equilibria at \(E_{21}(\psi_{21}, 0)\) and \(E_{22}(\psi_{22}, 0)\), where
\[
\psi_{21} = -\frac{c_2 - \sqrt{c_2^2 - 3c_1c_3}}{3c_3}, \quad \psi_{22} = -\frac{c_2 + \sqrt{c_2^2 - 3c_1c_3}}{3c_3}. \tag{29}
\]

2. For \(c_2^2 - 3c_1c_3 = 0\), (27) has a unique equilibrium at \(E_{20}(\psi_{20}, 0)\), where \(\psi_{20} = -c_2/(3c_3)\).

3. For \(c_2^2 - 3c_1c_3 < 0\), (27) has no equilibrium.

Let \(h_{2i} = H_i(\psi_{2i}, 0), i = 0, 1, 2\), and notice that we need only to consider the case \(c_3 \geq 0\) because of the invariance of (27) under the transformations \(\varphi \to -\varphi, y \to -y\), and \(c_3 \to -c_3\).

1. \(c_2^2 - 3c_1c_3 > 0\) and \(c_0 \in (h_{21}, h_{22})\). In this case,
\[
c_2 \varphi^3 + c_2 \varphi^2 + c_1 \varphi + c_0 = 0 \tag{30}
\]
has three mutually different real roots \(\varphi_m < \varphi_l < \varphi_M\); thus,
\[
c_2 \varphi^3 + c_2 \varphi^2 + c_1 \varphi + c_0 = c_3 (\varphi - \varphi_m) (\varphi - \varphi_l) (\varphi - \varphi_M). \tag{31}
\]
Equation (7) has periodic wave solutions as follows:
\[
\varphi_0 = \varphi_M - \left( (\varphi_M - \varphi_l) (\varphi_M - \varphi_m) \right) \times \left( \varphi_M - \varphi_m - (\varphi_l - \varphi_m) \operatorname{sn}^2 \right) \times \left( \frac{\sqrt{c_3} (\varphi_M - \varphi_m)}{2} \xi, \frac{\varphi_l - \varphi_m}{\varphi_M - \varphi_m} \right)^{-1}, \tag{32}
\]
\[
\varphi_0 = \varphi_M + (\varphi_l - \varphi_m) \operatorname{sn}^2 \left( \frac{\sqrt{c_3} (\varphi_M - \varphi_m)}{2} \xi, \frac{\varphi_l - \varphi_m}{\varphi_M - \varphi_m} \right). \tag{33}
\]

2. \(c_2^2 - 3c_1c_3 > 0\) and \(c_0 = h_{22}\). In this case, \(\varphi_{22}\) is double root of (30); suppose that \(\varphi_l\) is other root of the equation,
obviously $\varphi_1 < \varphi_{22}$, and we have a solitary wave solution of peak type of (7) as follows:

$$
\varphi_{10} = \varphi_1 + (\varphi_{22} - \varphi_1) \tanh \left( \frac{1}{2} \sqrt{c_3} (\varphi_{22} - \varphi_1) \xi \right).
$$

(33)

4. Conclusions

In this study, we apply bifurcation theory of dynamical systems and the Fan subequation method to investigate (1), and many new exact solutions have been obtained; most
importantly, under more general conditions than [3], to the best of our knowledge, these solutions have not been reported in the literature. This method can help us find exact solutions of other types of nonlinear dispersion partial differential equations.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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