Research Article

q-Extensions for the Apostol Type Polynomials

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The aim of this work is to introduce an extension for q-standard notations. The q-Apostol type polynomials and study some of their properties. Besides, some relations between the mentioned polynomials and some other known polynomials are obtained.

1. Introduction, Preliminaries, and Definitions

Throughout this research we always apply the following notations. \( N \) indicates the set of natural numbers, \( N_0 \) indicates the set of nonnegative integers, \( \mathbb{R} \) indicates the set of all real numbers, and \( \mathbb{C} \) denotes the set of complex numbers. We refer the readers to [1] for all the following q-standard notations. The q-shifted factorial is defined as

\[
(a; q)_0 = 1,
\]

\[
(a; q)_n = \prod_{j=0}^{n-1} (1 - q^ja), \quad n \in \mathbb{N},
\]

\[
(a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^ja),
\]

\[
|q| < 1, \quad a \in \mathbb{C}.
\]

The q-numbers and q-factorials are defined by

\[
[a]_q = \frac{1 - q^a}{1 - q} \quad (q \neq 1);
\]

\[
[0]_q! = 1;
\]

\[
[n]_q! = [1]_q [2]_q \cdot \ldots [n]_q,
\]

\[
n \in \mathbb{N}, \quad a \in \mathbb{C},
\]

respectively. The q-polynomial coefficient is defined by

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.
\]

The q-analogue of the function \((x + y)^n\) is defined by

\[
(x + y)_q^n := \sum_{k=0}^{n} \binom{n}{k}_q q^{(k+1)/2} (-1)^k a^k b^k, \quad n \in \mathbb{N}_0.
\]

The q-binomial formula is known as

\[
(1 - aq^n) = \prod_{j=0}^{n-1} (1 - q^j a) = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k+1)/2} (-1)^k a^k.
\]

In the standard approach to the q-calculus, two exponential functions are used:

\[
e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{j=0}^{\infty} \frac{1}{(1 - (1-q)q^j z)},
\]

\[
0 < |q| < 1, \quad |z| < \frac{1}{1-q}, \quad (6)
\]

\[
E_q(z) = \sum_{k=0}^{\infty} \frac{q^{(k+1)/2} (k-1)! a^k}{[k]_q!} = \prod_{j=0}^{\infty} (1 + (1-q)q^j z),
\]

\[
0 < |q| < 1, \quad z \in \mathbb{C}.
\]
As an immediate result of these two definitions, we have
\( e_q(z)E_q(-z) = 1 \).

Recently, Luo and Srivastava [2] introduced and studied the
generalized Apostol-Bernoulli polynomials \( B_n^{(x;\lambda)} \) and the
generalized Apostol-Euler polynomials \( E_n^{(x;\lambda)} \). Kurt [3] gave the generalization of the Bernoulli polynomials
\( B_{[m-1,\alpha]}(x) \) of order \( \alpha \) and studied their properties. They also studied these polynomials systematically; see [2, 4–9]. There
are numerous recent investigations on this subject by many
other authors; see [3, 10–20]. More recently, Tremblay et al.
[10] further gave the definition of \( B_n^{[m-1,\alpha]}(x;\lambda) \) and studied
their properties. On the other hand, Mahmudov and Kelesh-
teri [21,22] studied various two dimensional \( \alpha \)-polynomials.
Motivated by these papers, we define generalized Apostol
type \( q \)-polynomials as follows.

**Definition 1.** Let \( q, \alpha \in \mathbb{C}, \ m \in \mathbb{N}, \) and \( 0 < |q| < 1 \).
The generalized \( q \)-Apostol-Bernoulli numbers \( B_n^{[m-1,\alpha]} \) and
polynomials \( B_n^{[m-1,\alpha]}(x, y; \lambda) \) in \( x, y \) of order \( \alpha \) are defined, in
a suitable neighborhood of \( t = 0 \), by means of the generating functions:

\[
\left( \frac{t^m}{\lambda e_q(t) - T_{m-1,q}(t)} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(\lambda) \frac{t^n}{[n]_q!},
\]

\[
\left( \frac{t^m}{\lambda e_q(t) - T_{m-1,q}(t)} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x, y; \lambda) \frac{t^n}{[n]_q!}, \tag{8}
\]

where \( T_{m-1,q}(t) = \sum_{k=0}^{m-1} (t^k/[k]_q)! \).

**Definition 2.** Let \( q, \alpha \in \mathbb{C}, \ 0 < |q| < 1, \) and \( m \in \mathbb{N} \). The
generalized \( q \)-Apostol-Euler numbers \( E_n^{[m-1,\alpha]} \) and
polynomials \( E_n^{[m-1,\alpha]}(x, y; \lambda) \) in \( x, y \) of order \( \alpha \) are defined, in
a suitable neighborhood of \( t = 0 \), by means of the generating functions:

\[
\left( \frac{2^m t^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^\alpha = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(\lambda) \frac{t^n}{[n]_q!},
\]

\[
\left( \frac{2^m t^m}{\lambda e_q(t) + T_{m-1,q}(t)} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x, y; \lambda) \frac{t^n}{[n]_q!}. \tag{9}
\]

**Definition 3.** Let \( q, \alpha \in \mathbb{C}, 0 < |q| < 1, \) and \( m \in \mathbb{N} \).
The generalized \( q \)-Apostol-Genocchi numbers \( G_n^{[m-1,\alpha]} \) and
polynomials \( G_n^{[m-1,\alpha]}(x, y; \lambda) \) in \( x, y \) of order \( \alpha \) are defined, in

In this section, we show some basic properties of the generalized \( q \)-polynomials. We only prove the facts for one of them.
Obviously, by applying the similar technique, other ones can be proved.

To prove (11), one has

\[
B_n^{[0,\alpha]}(x, y; \lambda) = B_n^{[0,\alpha]}(x, y), \tag{12}
\]

\[
E_n^{[0,\alpha]}(x, y; \lambda) = E_n^{[0,\alpha]}(x, y), \tag{13}
\]

\[
G_n^{[0,\alpha]}(x, y; \lambda) = G_n^{[0,\alpha]}(x, y). \tag{14}
\]

For \( x = y = 0 \), one has

\[
B_n^{[m-1,\alpha]}(0, 0; \lambda) = B_n^{[m-1,\alpha]}(\lambda), \tag{15}
\]

\[
E_n^{[m-1,\alpha]}(0, 0; \lambda) = E_n^{[m-1,\alpha]}(\lambda), \tag{16}
\]

\[
G_n^{[m-1,\alpha]}(0, 0; \lambda) = G_n^{[m-1,\alpha]}(\lambda). \tag{17}
\]
Proposition 4. The generalized $q$-polynomials $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, and $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ satisfy the following relations:

$$B_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{\alpha=0}^{\infty} (\lambda e_q(t) + T_{m-1-q}(t))^\alpha e_q(0, y; \lambda) t^n$$

$$E_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{\alpha=0}^{\infty} (\lambda e_q(t) + T_{m-1-q}(t))^\alpha e_q(0, y; \lambda) t^n$$

$$G_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{\alpha=0}^{\infty} (\lambda e_q(t) + T_{m-1-q}(t))^\alpha e_q(0, y; \lambda) t^n.$$

Proof. We only prove the second identity. By using Definition 2, we have

$$\sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha+\beta]}(x, y; \lambda) \frac{t^n}{[n]_q!} = \left( \frac{2^m}{\lambda e_q(t) + T_{m-1-q}(t)} \right)^{\alpha+\beta} e_q(tx) E_q(ty)$$

$$= \left( \frac{2^m}{\lambda e_q(t) + T_{m-1-q}(t)} \right)^{\alpha} e_q(tx) E_q(ty)$$

$$\times \left( \frac{2^m}{\lambda e_q(t) + T_{m-1-q}(t)} \right)^{\beta} E_q(ty)$$

$$= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x, 0; \lambda) \frac{t^n}{[n]_q!}$$

$$\times \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\beta]}(0, y; \lambda) \frac{t^n}{[n]_q!}.$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_q E_{k,q}^{[m-1,\alpha]}(x, 0; \lambda) E_{n-k,q}^{[m-1,\beta]}(0, y; \lambda) \frac{t^n}{[n]_q!}.$$

Comparing the coefficients of the term $t^n/[n]_q!$ in both sides gives the result. \(\square\)

Corollary 5. The generalized $q$-polynomials $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, and $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ satisfy the following relations:

$$B_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k},$$

$$E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q E_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k},$$

$$G_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q G_{k,q}^{[m-1,\alpha]}(0, y; \lambda) x^{n-k}.$$

Proposition 6. The generalized $q$-polynomials $B_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, $E_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$, and $G_{n,q}^{[m-1,\alpha]}(x, y; \lambda)$ satisfy the following relations:

$$\lambda B_{n,q}^{[m-1,\alpha]}(1, y; \lambda) - B_{n,q}^{[m-1,\alpha]}(0, y; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q}^{[m-1,\alpha]}(0, y; \lambda) B_{n-k,q}^{[0-1]}(\lambda), \text{ for } n \geq 1,$$

$$\lambda E_{n,q}^{[m-1,\alpha]}(1, y; \lambda) + E_{n,q}^{[m-1,\alpha]}(0, y; \lambda)$$

$$= 2 \sum_{k=0}^{n} \binom{n}{k}_q E_{k,q}^{[m-1,\alpha]}(0, y; \lambda) E_{n-k,q}^{[0-1]}(\lambda),$$

$$\lambda G_{n,q}^{[m-1,\alpha]}(1, y; \lambda) + G_{n,q}^{[m-1,\alpha]}(0, y; \lambda)$$

$$= 2 \sum_{k=0}^{n} \binom{n}{k}_q G_{k,q}^{[m-1,\alpha]}(0, y; \lambda) G_{n-k,q}^{[0-1]}(\lambda), \text{ for } n \geq 1.$$
Comparing the coefficients of the term $t^n/[n_q]!$ in both sides gives the result. \( \Box \)

### 3. $q$-Analogue of the Luo-Srivastava Addition Theorem

In this section, we state and prove a $q$-generalization of the Luo-Srivastava addition theorem.

**Theorem 7.** The following relation holds between generalized $q$-Apostol-Euler and $q$-Apostol-Bernoulli polynomials:

$$E_{n_k}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{[n+1]}{[n+1]_q} \right)_q \left( \frac{\lambda}{[n+1]} \sum_{j=0}^{n-k} \frac{[n-k+1]}{[n-k+1]_q} B_{j,q}(x,0;\lambda) \right)$$

(21)

Proof. We take aid of the following identity to prove (21):

$$\frac{\lambda}{\lambda e_q(t) - 1} e_q(t) e_q(t) - \frac{\lambda}{\lambda e_q(t) - 1} e_q(t) x$$

(22)

That is,

$$x^n = \frac{1}{[n+1]_q} \left( \sum_{k=0}^{n} \frac{[n+1]}{[n+1]_q} \left( \frac{\lambda}{[n+1]} \sum_{j=0}^{n-k} \frac{[n-k+1]}{[n-k+1]_q} B_{j,q}(x,0;\lambda) - B_{n-k+1,q}(x,0;\lambda) \right) \right).$$

(25)

Substituting (25) into the right hand side of (16), we obtain

$$E_{n_k}^{[m-1,\alpha]}(x, y; \lambda)$$

(26)

As we know that

$$\left( \begin{array}{c} m \\ l \end{array} \right)_q \left( \begin{array}{c} n \\ l \end{array} \right)_q = \left( \begin{array}{c} m \\ m-n \end{array} \right)_q \left( \begin{array}{c} m-n \\ m-l \end{array} \right)_q,$$

(28)
we can continue as

\[ I_1 = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{\lambda}{[n+1]_q} \binom{n+1}{j} \binom{n-j+1}{k} \binom{1}{q} B_{j,q}(x,0) \]

\[ \times E_{[n-1,a]}^{[m-1,a]}(0,y;\lambda) \]

\[ = \sum_{j=0}^{n} \frac{\lambda}{[n+1]_q} \binom{n+1}{j} B_{j,q}(x,0) \]

\[ \times \left( E_{[n-1,a]}^{[m-1,a]}(1,y;\lambda) - E_{[n-j+1,a]}^{[m-1,a]}(0,y;\lambda) \right). \quad (29) \]

On the other hand, for \( I_2 \), we can write

\[ I_2 = \sum_{k=0}^{n} \frac{1}{[n+1]_q} \binom{n+1}{k} E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ = \sum_{k=0}^{n+1} \frac{1}{[n+1]_q} \binom{n+1}{k} B_{n+k+1,q}(x,0) \]

\[ \times \left( \frac{\lambda}{[n-k+1]_q} B_{n-k+1,q}(x,0) \right) \]

\[ = \sum_{j=0}^{n} \frac{1}{[n+1]_q} \binom{n+1}{j} \lambda \frac{1}{[n+1]_q} B_{n-k+1,q}(x,0) \]

\[ \times E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ - \frac{\lambda-1}{[n+1]_q} B_{0,q}(x,0) E_{k,q}^{[m-1,a]}(0,y;\lambda). \quad (30) \]

and, as \( B_{0,q}(x,0;\lambda) = 0 \), we have

\[ I_2 = \sum_{j=0}^{n+1} \frac{1}{[n+1]_q} \binom{n+1}{j} \lambda \frac{1}{[n+1]_q} B_{n-k+1,q}(x,0) \]

\[ \times E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ = \sum_{j=0}^{n} \frac{1}{[n+1]_q} \binom{n+1}{j} \lambda \frac{1}{[n+1]_q} B_{n-k+1,q}(x,0) \]

\[ \times E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ + \frac{\lambda-1}{[n+1]_q} B_{n,q}(x,0) E_{0,q}^{[m-1,a]}(0,y;\lambda). \quad (31) \]

Adding \( I_2 \) to \( I_1 \) we obtain

\[ E_{[n,a]}^{[m-1,a]}(x,y;\lambda) \]

\[ = I_1 + I_2 \]

\[ = \sum_{j=0}^{n} \left[ \frac{\lambda}{[n+1]_q} \binom{n+1}{j} B_{j,q}(x,0;\lambda) \right. \]

\[ \times \left( E_{n-1,a}^{[m-1,a]}(1,y;\lambda) - E_{n-j+1,a}^{[m-1,a]}(0,y;\lambda) \right) \]

\[ + \frac{\lambda-1}{[n+1]_q} B_{n+1,q}(x,0;\lambda) E_{0,q}^{[m-1,a]}(0,y;\lambda). \]

Consequently,

\[ E_{[n,a]}^{[m-1,a]}(x,y;\lambda) \]

\[ = \sum_{j=0}^{n} \frac{1}{[n+1]_q} \binom{n+1}{j} \lambda \frac{1}{[n+1]_q} B_{n-k+1,q}(x,0) \]

\[ \times \left( \frac{\lambda}{[n-k+1]_q} B_{n-k+1,q}(x,0) \right) \]

\[ \times E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ \times \left( \frac{\lambda}{[n-k+1]_q} B_{n-k+1,q}(x,0) \right) \]

\[ \times E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ - \frac{\lambda-1}{[n+1]_q} B_{0,q}(x,0) E_{k,q}^{[m-1,a]}(0,y;\lambda) \]

\[ \times \left( \frac{\lambda}{[n+1]_q} B_{n,q}(x,0) \right) \]

\[ \times E_{0,q}^{[m-1,a]}(0,y;\lambda). \quad (32) \]

\[ \square \]
Taking \( m = 1 \) in Theorem 7, we get a \( q \)-generalization of the Luo-Srivastava addition theorem [2].

**Corollary 8.** The following relation holds between generalized \( q \)-Apostol-Euler and \( q \)-Apostol-Bernoulli polynomials:

\[
E_{n,q}^{(\alpha)}(x, y; \lambda) = 2 \sum_{j=0}^{n} \binom{n}{j} q \left( \frac{1}{n+1} q \right)^{j} \left( E_{n-j,q}^{(\alpha)}(0, y; \lambda) - E_{n-j,q}^{(\alpha)}(0, \lambda) \right) 
\]

(34)

\[
\times B_{n-j,q}(0, \lambda) + \frac{\lambda - 1}{n+1} \left( \frac{2}{\lambda + 1} \right)^{\alpha} B_{n+1,q}(0, \lambda). 
\]

Corollary 8. The following relation holds between generalized \( q \)-Apostol-Euler and \( q \)-Apostol-Bernoulli polynomials:

\[
E_{n,q}^{(\alpha)}(x, y; \lambda) = 2 \sum_{j=0}^{n} \binom{n}{j} q \left( \frac{1}{n+1} q \right)^{j} E_{n-j,q}^{(\alpha)}(0, y; \lambda) - E_{n-j,q}^{(\alpha)}(0, \lambda) 
\]

(34)

\[
\times B_{n-j,q}(0, \lambda) + \frac{\lambda - 1}{n+1} \left( \frac{2}{\lambda + 1} \right)^{\alpha} B_{n+1,q}(0, \lambda). 
\]

Letting \( q \uparrow 1 \), we get the Luo-Srivastava addition theorem (see [12]):

\[
E_{n}^{(\alpha)}(x + y; \lambda) = \sum_{j=0}^{n} \binom{n}{j} q \left( \frac{1}{n+1} q \right)^{j} \left( E_{n-j}^{(\alpha)}(y; \lambda) - E_{n-j}^{(\alpha)}(\lambda) \right) 
\]

(35)

\[
\times B_{n-j}(x; \lambda) + \frac{\lambda - 1}{n+1} \left( \frac{2}{\lambda + 1} \right)^{\alpha} B_{n+1}(x; \lambda). 
\]

Next theorem gives relationship between \( E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \) and \( G_{n,q}(x, 0) \).

**Theorem 9.** The following relation holds between generalized \( q \)-Apostol-Euler and \( q \)-Apostol-Genocchi polynomials:

\[
E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} q \left( \frac{1}{n+1} q \right)^{k} \left( E_{n-k,q}^{[m-1,\alpha]}(y; \lambda) - E_{n-k,q}^{[m-1,\alpha]}(\lambda) \right) 
\]

(36)

\[
\times \sum_{j=0}^{n-k} \binom{n-k}{j} q \left( \frac{1}{n+1} q \right)^{j} E_{n-k-j,q}^{[m-1,\alpha]}(0, y; \lambda) 
\]

\[
\times G_{n+k+1,q}(x, 0). 
\]

Proof. The proof follows from the following identity:

\[
\left( \frac{2}{\lambda e_{q}(t) + T_{m-1,q}(t)} \right)^{\alpha} e_{q}(tx) E_{q}(ty) 
\]

(37)

\[
= \left( \frac{2}{\lambda e_{q}(t) + T_{m-1,q}(t)} \right)^{\alpha} E_{q}(ty) \frac{2t}{e_{q}(t) + 1} e_{q}(tx) - \frac{1}{2t}. 
\]

**Theorem 10.** The following relation holds between generalized \( q \)-Apostol-Euler and \( q \)-Stirling polynomials \( S_{q}(i, j) \) of the second kind:

\[
E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n-j}{k} q \left( \frac{1}{n+1} q \right)^{k} E_{n-k,j,q}^{[m-1,\alpha]}(0, y; \lambda) 
\]

(38)

\[
\times S_{q}(j, k) x_{k}(x). 
\]

Proof. The \( q \)-Stirling polynomials \( S_{q}(n, k) \) of the second kind are defined by means of the following generating function:

\[
x^{n} = \sum_{k=0}^{n} S_{q}(n, k) x_{k}(x), 
\]

(39)

where \( x_{k}(x) = x(x - [1]_{q})(x - [2]_{q}) \cdots (x - [k-1]_{q}) \); see [23]. Replacing identity (39) in the right hand side of (16), we have

\[
E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-k}{j} q \left( \frac{1}{n+1} q \right)^{j} E_{n-k-j,q}^{[m-1,\alpha]}(0, y; \lambda) 
\]

(40)

\[
\times S_{q}(j, k) x_{k}(x). 
\]

**Theorem 11.** The relationship

\[
E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) = \sum_{k=0}^{n} \sum_{j=0}^{n-2k} \binom{n-2k}{j} q \left( \frac{1}{n+1} q \right)^{j} E_{n-2k-j,q}^{[m-1,\alpha]}(0, y; \lambda) 
\]

(41)

\[
\times S_{q}(j, k) H_{n-2k-j,q}(x) 
\]

holds between the polynomials \( E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \) and the \( q \)-Hermite polynomials defined by (see [24])

\[
e_{q}(tx) E_{q}^{\frac{i^{2}}{[2]_{q}}} = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{i^{n}}{[n]_{q}!}. 
\]

(42)
Proof. Indeed,
\[\sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(x, y; \lambda) \frac{t^n}{[n]_q!} = \left( \frac{2^m}{\lambda e_t(t) + T_{m-1,q}(t)} \right)^\alpha e_q(tx) E_q(ty)\]
\[= \left( \frac{2^m}{\lambda e_t(t) + T_{m-1,q}(t)} \right)^\alpha E_q(ty) e_q(tx) \times E_q^2\left( - \frac{t^2}{[2]_q} \right) e_q^2\left( \frac{t^2}{[2]_q} \right)\]
\[= \sum_{n=0}^{\infty} E_{n,q}^{[m-1,\alpha]}(0, y; \lambda) \frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \frac{t^{2n}}{[2]_q^j [n]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{j=0}^{n} E_{j,q}^{[m-1,\alpha]}(0, y; \lambda) H_{n-j,q}(x) \frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{t^{2n}}{[2]_q^j [n]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^n}{[2]_q^n [n]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^n}{[2]_q^n [k]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{j=0}^{n-2k} \frac{t^n}{[n]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n-2k} \frac{t^n}{[k]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n-2k} \frac{t^n}{[k]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n-2k} \frac{t^n}{[n]_q!}\]
\[= \sum_{n=0}^{\infty} \sum_{k=0}^{n-2k} \frac{t^n}{[n]_q!}\]

(43)

\[\square\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References


