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Research Article

Modified Nonradial Supper Efficiency Models

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Ranking Efficient Decision Making Units (DMUs) are an important issue in Data Envelopment Analysis (DEA). This is one of the main areas for the researcher. Different methods for this purpose have been suggested. Appearing nonzero slack in optimal solution makes the method problematic. In this paper, we modify the nonradial supper efficiency model to remove this difficulty. Some numerical examples are solved by modified model.

1. Introduction

Data Envelopment Analysis is a mathematical programming technique which evaluates the relative efficiency of DMUs. DEA classifies the DMUs into two different classes, called set of efficient DMUs and the set of inefficient DMUs.

Conventional DEA model cannot differentiate the efficient DMUs whose efficiency value is one. Toward this end, different methods are suggested; see [1–3]. One of the important models is AP-Model which was proposed by Andersen and Petersen [4] and also see [5, 6].

This model is widely used and the results are almost satisfactory. The main deficiencies of AP model are being as follows:

(1) infeasible for some kind of data,

(2) unstable in the sense that a small variation in data causes big increase (degrease) in the result,

(3) not taking into account the nonzero slacks which appear in optional solution (projection of omitted DMU is weak efficient in new PPS).

For removing these difficulties, many researches have suggested different models; for example, see [7–9].

Tone [7] suggested the nonradial supper efficiency model. This model fails to remove the 3rd deficiency. In this paper, we modify the nonradial supper efficiency model that takes into account nonzero slack that appears in optimal solution.

The rest of the paper is organized as follows. In Sections 2 and 3, AP-model and Tone's model are discussed. Section 4 contains the modified model. In Sections 5 and 6, input and output oriented models are proposed. Discussion and conclusion cover Section 7.

2. Anderson Peterson (AP) Model

Consider n decision making units DMU_j $(j = 1, \ldots, n)$ which consumes $0 \neq x_j \geq 0$ and $x_j \in R^m$ $(j = 1, \ldots, n)$ vector as input to produce output vector $0 \neq y_j \geq 0$ and $y_j \in R^s$ $(j = 1, \ldots, n)$. The supper efficiency model may be written as follows:

$$\begin{aligned}
\theta_p^* = {} & \operatorname{Min} \theta \\
& \text{s.t.} \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip} \quad i = 1, \ldots, m \\
& \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad r = 1, \ldots, s \\
& \lambda_j \geq 0 \quad j = 1, \ldots, n,\ j \neq p \\
& s_i^- \geq 0 \quad i = 1, \ldots, m \\
& s_r^+ \geq 0 \quad r = 1, \ldots, s.
\end{aligned} \tag{1}$$

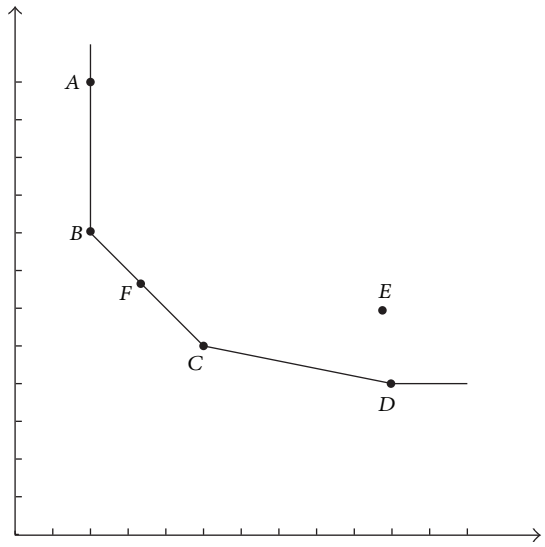

Figure 1

The following example which has been taken from [7] shows the deficiency in case of having nonzero slacks in optimal solution.

Example 1. Consider the data given by Table 1.

By using AP-Model,

$$\begin{aligned}
&\text{Sup } A = \theta_A^* = 1 \qquad S_2^{-*} = 4\\
&\text{Sup } B = \theta_B^* = 1.260 \quad \text{All slacks are zero}\\
&\text{Sup } C = \theta_C^* = 1.133 \quad \text{All slacks are zero}\\
&\text{Sup } D = \theta_D^* = 1.250 \qquad S_1^{-*} = 7.5\\
&\text{Sup } E = \theta_E^* = 0.750\\
&\text{Sup } F = \theta_F^* = 1.
\end{aligned} \tag{2}$$

In other words, the supper efficiencies are as follows.

$$\begin{aligned}
\theta_A^* &= 1 - 4\varepsilon\,(*)\\
\theta_B^* &= 1.260\\
\theta_C^* &= 1.133\\
\theta_D^* &= 1.25 - 7.5\varepsilon\,(*)\\
\theta_E^* &= 0.750\\
\theta_F^* &= 1\,(*)\,.
\end{aligned} \tag{3}$$

Note. These values are not correct in [7], because the result shown in the paper has taken from original paper in which F is not included. From Figure 1 it can be seen that the radial supper Efficiency model is not able to rank F which is non-extreme efficient and also is not able to rank D and A, with nonzero slack in optimal solutions.

Table 1

DMU	A	B	C	D	E	F
Input 1	2	2	5	10	10	3.5
Input 2	12	8	5	4	6	6.5
Output 1	1	1	1	1	1	1

Table 2

DMU	A	B	C	D	E	F	G	H
Input 1	4	7	8	4	2	10	12	10
Input 2	3	3	1	2	4	1	1	1.5
Output 1	1	1	1	1	1	1	1	1

3. Nonradial Supper Efficiency Model (Tone's Model)

In 2002, Tone proposed the following nonradial supper efficiency model:

$$\begin{aligned}
\rho_p^* = \ &\text{Min } \frac{(1/m)\sum_{i=1}^{m}\left(\overline{x}_i/x_{ip}\right)}{(1/s)\sum_{r=1}^{s}\left(\overline{y}_r/y_{rp}\right)}\\
&\text{s.t.} \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} \le \overline{x}_i \quad i = 1,\ldots,m\\
&\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} \ge \overline{y}_r \quad r = 1,\ldots,s\\
&\overline{x}_i \ge x_{ip} \quad i = 1,\ldots,m\\
&\overline{y}_r \le y_{rp} \quad r = 1,\ldots,s\\
&\overline{y}_r \ge 0 \quad r = 1,\ldots,s\\
&\lambda_j \ge 0 \quad j = 1,\ldots,n,\ j\neq p.
\end{aligned} \tag{4}$$

Based on the SBM model (4), the nonradial supper efficiency model should be as follows:

$$\begin{aligned}
\gamma_p^* = \ &\text{Min } \frac{1 + (1/m)\sum_{i=1}^{m}\left(s_i^+/x_{ip}\right)}{1 - (1/s)\sum_{r=1}^{s}\left(s_r^-/y_{rp}\right)}\\
&\text{s.t.} \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} - s_i^+ = x_{ip} \quad i = 1,\ldots,m\\
&\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} + s_r^- = y_{rp} \quad r = 1,\ldots,s\\
&\lambda_j \ge 0 \quad j = 1,\ldots,n, j\neq p\\
&s_i^+ \ge 0 \quad i = 1,\ldots,m\\
&s_r^- \ge 0 \quad r = 1,\ldots,s.
\end{aligned} \tag{5}$$

We will show that (4) and (5) are not equivalent, in the sense that their optimal solutions are different.

Example 2. Consider the eight Decision Making Units (DMU), with two inputs and one output (see Table 2).

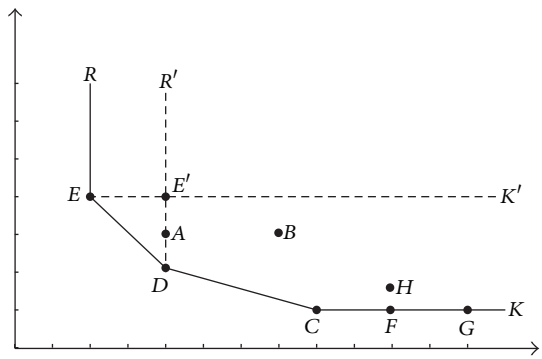

Figure 2

First the following are defined:

$$PPS = \left\{(x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j, \; y \leq \sum_{j=1}^{n} \lambda_j y_j, \; \lambda_j \geq 0, j = 1, \ldots, n\right\},$$
$$\overline{PPS} = \left\{(x, y) \mid x \geq \sum_{j=1, j \neq p}^{n} \lambda_j x_j, \; y \leq \sum_{j=1, j \neq p}^{n} \lambda_j y_j, \; \lambda_j \geq 0, j = 1, \ldots, n\right\},$$
$$\overline{\overline{PPS}} = \overline{PPS} \cap \left\{(x, y)\, x \geq x_p, \; y \leq y_p\right\}. \tag{6}$$

In above-mentioned example and shown in Figure 2,

$$\begin{aligned} PPS &= REDCK = S \\ \overline{PPS} &= R'E'DCK = \overline{S} \\ \overline{\overline{R'E'K'}} &= \overline{\overline{S}}. \end{aligned} \tag{7}$$

We can see that $\overline{\overline{S}}$ is a proper subset of $\overline{S}$ and $\overline{S}$ is a proper subset of S; that is,

$$\overline{\overline{S}} \subset \overline{S} \subset S. \tag{8}$$

In this example, the optimum value of objective function in (4) is

$$\rho_E^* = \frac{1}{2}\left(\frac{4}{2} + \frac{4}{4}\right) = 1.5 \tag{9}$$

and the optimal value of objective function in (5) is

$$\gamma_{\overline{S}}^* = \frac{1}{2}\left(\frac{4}{2} + \frac{2}{4}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4} = 1.25, \tag{10}$$

so (4) and (5) are not equivalent. In other words, the constraints $\overline{x} \geq x_p$ and $\overline{y} \leq y_p$ and $\overline{y} \geq 0$ should be omitted from (4). Now we show that the nonradial supper efficiency model has the same difficulty as radial supper efficiency model in treating nonzero slacks in optimal solution. Using (5), the supper efficiency of E, D, C may be evaluated as follows:

$$\begin{aligned} \gamma_E^* = \text{Min} \quad & \left(1 + \frac{1}{2}\sum_{i=1}^{2} \frac{s_i^+}{x_{iE}}\right) \\ \text{s.t.} \quad & 4\lambda_A + 7\lambda_B + 8\lambda_C + 4\lambda_D + 10\lambda_F + 12\lambda_G \\ & + 10\lambda_H - s_1^+ = 2 \\ & 3\lambda_A + 3\lambda_B + \lambda_C + 2\lambda_D + \lambda_F + \lambda_G + 1.5\lambda_H \\ & - s_2^+ = 4 \\ & \lambda_A + \lambda_B + \lambda_C + \lambda_D + \lambda_F + \lambda_G + \lambda_H + s^- = 1 \\ & \lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_F, \lambda_G, \lambda_H, s_1^+, s_2^+, s^- \geq 0, \\ & \gamma_E^* = 1 + \frac{1}{2}\left(\frac{2}{2} + \frac{0}{4}\right) = 1.5. \end{aligned} \tag{11}$$

The projection of E is on weak frontier

$$\begin{aligned} \gamma_D^* = \text{Min} \quad & \left(1 + \frac{1}{2}\sum_{i=1}^{2} \frac{s_i^+}{x_{iD}}\right) \\ \text{s.t.} \quad & 4\lambda_A + 7\lambda_B + 8\lambda_C + 2\lambda_E + 10\lambda_F + 12\lambda_G \\ & + 10\lambda_H - s_1^+ = 2 \\ & 3\lambda_A + 3\lambda_B + \lambda_C + 4\lambda_E + \lambda_F + \lambda_G + 1.5\lambda_H \\ & - s_2^+ = 4 \\ & \lambda_A + \lambda_B + \lambda_C + \lambda_E + \lambda_F + \lambda_G + \lambda_H + s^- = 1 \\ & \lambda_A, \lambda_B, \lambda_C, \lambda_E, \lambda_F, \lambda_G, \lambda_H, s_1^+, s_2^+, s^- \geq 0, \\ & \gamma_D^* = 1 + \frac{1}{2}\left(\frac{2}{4} + \frac{0}{3}\right) = 1.25. \end{aligned} \tag{12}$$

The projection is on strong frontier

$$\begin{aligned} \gamma_C^* = \text{Min} \quad & \left(1 + \frac{1}{2}\sum_{i=1}^{2} \frac{s_i^+}{x_{iC}}\right) \\ \text{s.t.} \quad & 4\lambda_A + 7\lambda_B + 4\lambda_D + 2\lambda_E + 10\lambda_F \\ & + 12\lambda_G + 10\lambda_H - s_1^+ = 8 \\ & 3\lambda_A + 3\lambda_B + 2\lambda_D + 4\lambda_E + \lambda_F \\ & + \lambda_G + 1.5\lambda_H - s_2^+ = 1 \\ & \lambda_A + \lambda_B + \lambda_D + \lambda_E + \lambda_F + \lambda_G + \lambda_H + s^- = 1 \\ & \lambda_A, \lambda_B, \lambda_D, \lambda_E, \lambda_F, \lambda_G, \lambda_H, s_1^+, s_2^+, s^- \geq 0, \\ & \gamma_C^* = 1 + \frac{1}{2}\left(\frac{2}{8}\right) = 1.125. \end{aligned} \tag{13}$$

The projection is on strong frontier.

In summery,

$$\begin{aligned} \gamma_C^* &= 1.125 \\ \gamma_D^* &= 1.25 \\ \gamma_E^* &= 1.5. \end{aligned} \tag{14}$$

In the case the projection of omitted DMU_p lied on weak frontier, nonzero slacks appear in optimal solution and the following approach is suggested.

4. Modified Nonradial Supper Efficiency Model

First solve the following model:

$$\begin{aligned}
\gamma_p^* = \text{Min} \quad & \frac{1+(1/m)\sum_{i=1}^{m}\left(s_i^+/x_{ip}\right)}{1-(1/s)\sum_{r=1}^{s}\left(s_r^-/y_{rp}\right)} \\
\text{s.t.} \quad & \sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}-s_i^+=x_{ip} \quad i=1,\ldots,m \\
& \sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}+s_r^-=y_{rp} \quad r=1,\ldots,s \\
& \lambda_j\geq 0 \quad j=1,\ldots,n, j\neq p \\
& s_i^+\geq 0 \quad i=1,\ldots,m \\
& s_r^-\geq 0 \quad r=1,\ldots,s.
\end{aligned} \tag{15}$$

The model, (15), can be linearized by the suggested method in [7] and solve by the simplex method.

Suppose that

$$\left(\widehat{x},\widehat{y}\right)=\left(x_p+s^{+*},y_p-s^{-*}\right) \tag{16}$$

is the projection of DMU_p on the frontier of $\overline{PPS}$ and now solve the following model

$$\begin{aligned}
W_p^* = \text{Max} \quad & \frac{(1/m)\sum_{i=1}^{m}\left(\widehat{s}_i^-/\widehat{x}_i\right)}{(1/s)\sum_{r=1}^{s}\left(\widehat{s}_r^+/\widehat{y}_r\right)} \\
\text{s.t.} \quad & \sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}+\widehat{s}_i^-=\widehat{x}_i \quad i=1,\ldots,m \\
& \sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}-\widehat{s}_r^+=\widehat{y}_r \quad r=1,\ldots,s \\
& \widehat{s}_r^+\geq 0 \quad r=1,\ldots,s \\
& \widehat{s}_i^-\geq 0 \quad i=1,\ldots,m, \\
& \lambda_j\geq 0 \quad j=1,\ldots,n,\ j\neq p.
\end{aligned} \tag{17}$$

Set

$$\mu_p^*=\gamma_p^*-W_p^*\ \left(\text{supper efficiency of DMU}_p\right). \tag{18}$$

It is evident that if $(\widehat{x},\widehat{y})$ is strongly efficient, then $W_p^*=0$ and μ_p^* is supper efficiency score of DMU_p.

5. Modified Input Oriented Nonradial Supper Efficiency Model

First the following model is solved:

$$\begin{aligned}
\xi_p^* = \text{Min} \quad & \left(1+\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^+}{x_{ip}}\right) \\
\text{s.t.} \quad & \sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}-s_i^+=x_{ip} \quad i=1,\ldots,m \\
& \sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}-s_r^{+\prime}=y_{rp} \quad r=1,\ldots,s \\
& s_i^+\geq 0 \quad i=1,\ldots,m \\
& s_r^{+\prime}\geq 0 \quad r=1,\ldots,s \\
& \lambda_j\geq 0 \quad j=1,\ldots,n, j\neq p.
\end{aligned} \tag{19}$$

Let

$$\left(\widetilde{x},\widetilde{y}\right)=\left(x_p+s^{+*},y_p+s^{+*\prime}\right). \tag{20}$$

Now solve the following problem.

$$\begin{aligned}
W_p^* = \text{Max} \quad & \frac{(1/m)\sum_{i=1}^{m}\left(t_i^-/\widetilde{x}_i\right)}{(1/s)\sum_{r=1}^{s}\left(t_r^+/\widetilde{y}_r\right)} \\
\text{s.t.} \quad & \sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}+t_i^-=\widetilde{x}_i \quad i=1,\ldots,m \\
& \sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}-t_r^+=\widetilde{y}_r \quad r=1,\ldots,s \\
& t_i^-\geq 0 \quad i=1,\ldots,m \\
& t_r^+\geq 0 \quad r=1,\ldots,s \\
& \lambda_j\geq 0 \quad j=1,\ldots,n, j\neq p.
\end{aligned} \tag{21}$$

Let

$$\mu_p^*=\xi_p^*-W_p^*. \tag{22}$$

6. Modified Output Oriented Nonradial Supper Efficiency Model

Consider the following model:

$$\begin{aligned}
\eta_p^* = \text{Min} \quad & \left(\frac{1}{1-(1/s)\sum_{i=1}^{s}\left(s_r^-/y_{rp}\right)}\right) \\
\text{s.t.} \quad & \sum_{j=1,j\neq p}^{n}\lambda_j x_{ij}+s_i^{-\prime}=x_{ip} \quad i=1,\ldots,m \\
& \sum_{j=1,j\neq p}^{n}\lambda_j y_{rj}+s_r^-=y_rp \quad r=1,\ldots,s \\
& s_i^{-\prime}\geq 0 \quad i=1,\ldots,m \\
& s_r^-\geq 0 \quad r=1,\ldots,s \\
& \lambda_j\geq 0 \quad j=1,\ldots,n, j\neq p.
\end{aligned} \tag{23}$$

Table 3

DMU	C	D	E	F	G
AP	1.14	1.25	2.00	1.00	1.00
SBM	1.125	1.25	1.50	1.00	1.00
Modified SBM	1.125	1.25	1.25	0.90	0.73

Let

$$\mathrm{DMU}_p \text{ projection } = (\overline{x}, \overline{y}) = \left(x_p - s^{-'*}, y_p - s^{-*}\right). \quad (24)$$

Now solve the following problem:

$$\begin{aligned} W_p^* = \text{ Max } & \frac{(1/m)\sum_{i=1}^{m}\left(\widehat{s}_i^- / \overline{x}_i\right)}{(1/s)\sum_{r=1}^{s}\left(\widehat{s}_r^+ / \overline{y}_r\right)} \\ \text{s.t. } & \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} + s_i^- = \overline{x}_i \quad i = 1, \ldots, m \\ & \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} - s_r^+ = \overline{y}_r \quad r = 1, \ldots, s \\ & s_i^- \geq 0 \quad i = 1, \ldots, m \\ & s_r^+ \geq 0 \quad r = 1, \ldots, s \\ & \lambda_j \geq 0 \quad j = 1, \ldots, n, j \neq p. \end{aligned} \quad (25)$$

Let

$$\rho_p^* = \eta_p^* - W_p^*. \quad (26)$$

7. Conclusion

In this paper, it has been shown that both AP-Model and non-radial supper efficiency model are not able to rank DMUs if the projection of omitted DMU is weak efficient in $\overline{PPS}$. The new method removes this difficulties and the example which is solved by using the new method (modified method) confirms the validity.

The results for comparing the methods are shown in Table 3.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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