Research Article

Affine Fullerene C\textsubscript{60} in a GS-Quasigroup

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It will be shown that the affine fullerene C\textsubscript{60}, which is defined as an affine image of buckminsterfullerene C\textsubscript{60}, can be obtained only by means of the golden section. The concept of the affine fullerene C\textsubscript{60} will be constructed in a general GS-quasigroup using the statements about the relationships between affine regular pentagons and affine regular hexagons. The geometrical interpretation of all discovered relations in a general GS-quasigroup will be given in the GS-quasigroup C((1/2)(1 + √5)).

1. Introduction

The fullerenes are closed carbon-cage molecules containing only pentagonal and hexagonal rings. C\textsubscript{60} is the first fullerene that was theoretically conceived and experimentally obtained. The geometrical structure of C\textsubscript{60} is a truncated icosahedron with a carbon atom at the corners of each hexagon and a bond along each edge (Figure 1). The sixty-carbon cluster with the geometry of a truncated icosahedron is named buckminsterfullerene [1, 2].

The affine regular icosahedron is defined as an affine image of a regular icosahedron. Let the affine regular icosahedron be given with the pairs of opposite vertices \(a, a'; b, b'; c, c'; d, d'; e, e'; f, f'\) as in Figure 2. Let us divide each edge of this icosahedron into three equal parts and then omit two lateral parts. On each of the twenty faces of the icosahedron on the sides the three segments in their middle parts are left. Let us connect the adjacent ends of these segments, so that an affine regular hexagon is formed on each side of an icosahedron, and an affine regular pentagon appears in the neighborhood of each vertex of icosahedron (Figure 3).

The obtained polyhedron consists of twelve affine regular pentagons and twenty affine regular hexagons. It is an affine version of buckminsterfullerene C\textsubscript{60} which will be called affine fullerene C\textsubscript{60}. It is presented in Figure 3, from where it is obvious how the labels of vertices of that polyhedron are chosen, starting from the labels of vertices of the affine regular icosahedron.

We will prove that the complete affine fullerene C\textsubscript{60} can be presented only by means of the golden section. The concept of a GS-quasigroup will be used in this consideration.

2. GS-Quasigroup

A quasigroup (\(Q, \cdot\)) is said to be a golden section quasigroup or shortly a GS-quasigroup [3] if it satisfies the (mutually equivalent) identities

\(a (ab \cdot c) \cdot c = b,\) \hspace{1cm} (1)

\(a \cdot (a \cdot bc) c = b,\) \hspace{1cm} (2)

and the identity of idempotence

\(aa = a.\) \hspace{1cm} (3)

GS-quasigroups are medial quasigroups; that is, the identity

\(ab \cdot cd = ac \cdot bd\) \hspace{1cm} (4)

is valid [4].
As a consequence of the identity of mediality, the considered GS-quasigroup \((Q, \cdot)\) satisfies the identities of elasticity and left and right distributivity; that is, we have these identities:

\[
\begin{align*}
ab \cdot a &= a \cdot ba, \\
bc &= ab \cdot ac, \\
ab \cdot c &= ac \cdot bc.
\end{align*}
\]

Further, the identities

\[
\begin{align*}
a(ab \cdot b) &= b, \\
(b \cdot ba) a &= b, \\
a(ab \cdot c) &= b \cdot bc, \\
tictac \cdot ba &= cb \cdot b, \\
a(a \cdot bc) &= b(b \cdot ac), \\
(cb \cdot a) a &= (ca \cdot b) b
\end{align*}
\]

and equivalencies

\[
\begin{align*}
ab = c &\iff a = c \cdot cb, \\
ab = c &\iff b = ac \cdot c
\end{align*}
\]

also hold.

Let \(C\) be the set of points of the Euclidean plane. For any two different points \(a, b\) we define \(ab = c\) if the point \(b\) divides the pair \(a, c\) in the ratio of the golden section. In [3], it is proved that the points \(a, b, c, d\) are the vertices of a parallelogram denoted by \(\text{Par}(a, b, c, d)\), if and only if there are two points \(p\) and \(q\) such that \(pa = qb\) and \(pd = qc\). It is also shown that if the statement \(\text{Par}(a, b, c, d)\) holds, then the equalities \(pa = qb\) and \(pd = qc\) are equivalent. In a general GS-quasigroup, the notation of a parallelogram can

\[\text{Figure 2}\]

\[\text{Figure 3}\]

3. Affine Regular Pentagons and Hexagons in GS-Quasigroups

From now on, let \((Q, \cdot)\) be any GS-quasigroup. The elements of the set \(Q\) are said to be points.

In each medial quasigroup, the concept of a parallelogram can be introduced by means of two auxiliary points. In [5], it is proved that the points \(a, b, c, d\) are the vertices of a parallelogram denoted by \(\text{Par}(a, b, c, d)\), if and only if there are two points \(p\) and \(q\) such that \(pa = qb\) and \(pd = qc\). It is also shown that if the statement \(\text{Par}(a, b, c, d)\) holds, then the equalities \(pa = qb\) and \(pd = qc\) are equivalent. In a general GS-quasigroup, the notation of a parallelogram can
be characterized by the equivalency \( \text{Par}(a, b, c, d) \Leftrightarrow a \cdot b (ca \cdot a) = d \) (Figure 4).

In [3], some properties of the quaternary relation \( \text{Par} \) on the set \( Q \) are proved. We will mention only the property which will be used afterwards.

**Lemma 1.** From \( \text{Par}(a, b, c, d) \) and \( \text{Par}(c, d, e, f) \) there follows \( \text{Par}(a, b, f, e) \).

We will say that \( b \) is the midpoint of the pair of points \( a, c \) and we write \( M(a, b, c) \) if and only if \( \text{Par}(a, b, c) \) holds. The statement \( M(a, b, c) \) holds if and only if \( c = ba \cdot b \) [3].

The concept of the affine regular hexagon [6] in a GS-quasigroup is defined in the following way. We will say that \((a_1, a_2, a_3, a_4, a_5, a_6)\) is an affine regular hexagon with the vertices \( a_1, a_2, a_3, a_4, a_5, a_6 \) and the center \( o \) and we write \( \text{ARP}_{o}(a_1, a_2, a_3, a_4, a_5, a_6) \) if the statements \( \text{Par}(o, a_{i-1}, a_i, a_{i+1}) \) hold \((i = 1, 2, 3, 4, 5, 6)\), where indexes are taken modulo 6 (Figure 5). The following statement can be proved [6].

**Lemma 2.** An affine regular hexagon is uniquely determined by any three consecutive vertices.

The points \( o, a_1, a_2, a_3, a_4 \) determine the figure which will be denoted by the symbol \( \text{HARH}_{o} (a_1, a_2, a_3, a_4) \), “half” of the affine regular hexagon with the center \( o \) (Figure 5).

The following results [6] will be very useful.

**Lemma 3.** Let \( n \in \mathbb{N}, n \geq 3 \). If the statements \( \text{HARH}_{o} (b_1, a_1, a_2, b_2), \text{HARH}_{o} (b_2, a_2, a_3, b_3), \ldots, \text{HARH}_{o} (b_n-1, a_{n-1}, a_n, b_n) \) are valid, then there exists a unique point \( c_n \) so that the statement \( \text{HARH}_{o} (b_n, a_n, a_1, b_1) \) is valid too. (The case for \( n = 5 \) is illustrated in Figure 6.)

Lemma 3 implies the following statement.

**Lemma 4.** Let \( n \in \mathbb{N}, n \geq 3 \). If the statements \( \text{ARP}_{o} (b_1, a_1, a_2, b_2, d_{21}, d_{22}), \text{ARP}_{o} (b_2, a_2, a_3, b_3, d_{32}, d_{33}), \ldots, \text{ARP}_{o} (b_{n-1}, a_{n-1}, a_n, b_n, d_{n-1, n}, d_{n, n-1}) \) are valid, then there exist unique points \( c_{31}, d_{31}, d_{3n} \) so that the statement \( \text{ARP}_{o} (b_n, a_n, a_1, b_1, d_{n, n}, d_{31}) \) is valid, too.

The points \( a, b, c, d \) successively are said to be the vertices of the golden section trapezoid [7] denoted by \( \text{GST}(a, b, c, d) \) if the identity \( a \cdot b = d \cdot c \) holds (Figure 7). It can be proved that the following equivalency \( \text{GST}(a, b, c, d) \Leftrightarrow c = a(db \cdot b) \) holds. The following statement is also valid.

**Lemma 5.** Any of the three statements \( \text{GST}(a, b, c, d), \text{GST}(b, e, f, c), \text{GST}(d, e, a, b) \) is equivalent (Figure 8); \( ae = df \) is a consequence of the two remaining statements.

In [8], it is proved that any two of the five statements

\[
\begin{align*}
\text{GST}(a, b, c, d), & \quad \text{GST}(b, e, f, c), & \quad \text{GST}(d, e, a, b) \\
\text{GST}(d, e, a, b), & \quad \text{GST}(e, a, b, c)
\end{align*}
\]

(16)

imply the remaining statements.

The points \( a, b, c, d, e \) successively are said to be the vertices of the affine regular pentagon [8] denoted by \( \text{ARP}(a, b, c, d, e) \) if any two (and then all five) of the five statements (16) are valid (Figure 7).

**Lemma 6.** An affine regular pentagon is uniquely determined by any three of its vertices.

Now, we are going to study the relationships between the previously defined geometrical concepts in a general GS-quasigroup.

**Lemma 7.** If the statements \( \text{Par}(d_1, o, b_2, a_1), \text{Par}(d_2, o, b_1, a_2) \) are valid, then

(i) the statements \( d_1b_1 = d_2b_2 \) and \( \text{GST}(a_1, a_2, d_2) \) are equivalent (Figure 8);

(ii) the statements \( d_1b_1 = d_2b_2 \) and \( \text{GST}(a_1, b_2, a_2) \) are equivalent (Figure 8).
Proof. (i) We have the equalities
\[ a_1 = d_1 \cdot o(b_2d_1 \cdot d_1), \quad a_2 = d_2 \cdot o(b_2d_2 \cdot d_2), \]  
and we have to prove the equivalency of the equalities
\[ d_1 \cdot d_1 a_1 = d_2 \cdot d_2 a_2, \quad d_1b_1 = d_2b_2. \]  
However, we get
\[ a_1 = d_1 \cdot o(b_2d_1 \cdot d_1) \quad \text{(6)} \]
\[ \overset{(5)}{=} d_1o \cdot (d_1 \cdot b_2d_1) \cdot d_1 \]
\[ \overset{(11)}{=} d_1o \cdot (d_1 \cdot b_2 \cdot b_2) \quad \text{(4)} \]
\[ \quad \overset{(10)}{=} o \cdot (d_1 \cdot b_2) \cdot ob_2 \]
\[ \quad \overset{(6)}{=} o \cdot (d_1 \cdot b_2) \quad \text{(13)} \]
and thus we get
\[ d_1 \cdot d_1 a_1 \]
\[ = d_1 \cdot d_1 \quad \text{[o \cdot (ob_2 \cdot d_1)]} \quad \text{(1)} \]
\[ \overset{(6)}{=} (d_1 \cdot d_1o) \cdot d_1 \quad \text{[d_1 \cdot (ob_2 \cdot d_1)]} \quad \text{(5),(2)} \]
\[ \overset{(5),(2)}{=} (d_1 \cdot d_1o) \cdot ob_2 \quad \text{[d_1o \cdot (d_1o \cdot b_2)]} \quad \text{(5),(2)} \]
and analogously \[ d_2 \cdot d_2 a_2 = d_2 \cdot o((d_2o \cdot b_2)). \]  
Because of that, it is necessary to prove the equivalency of the equality \[ d_1o \cdot (d_1o \cdot b_2) = d_2o \cdot (d_2o \cdot b_2) \] and the equality \[ b_2 = (d_2 \cdot d_1b_1) \cdot d_1b_1 \] which is, according to (15), equivalent to \[ d_1b_1 = d_2b_2. \]  
As we get
\[ d_2o \cdot (d_2o \cdot b_2) \]
\[ \overset{(8)}{=} d_2o \cdot d_2o \cdot d_2 (d_2b_1 \cdot b_1) \]
\[ \overset{(6)}{=} d_2o \cdot o (d_2b_1 \cdot b_1) \]
\[ \overset{(2),(5)}{=} [d_1 \cdot d_1 (d_2d_1 \cdot d_1)] \quad [o \cdot o (d_2b_1 \cdot b_1)] \]
\[ \overset{(4)}{=} d_1o \cdot [d_1o \cdot (d_2d_1 \cdot d_2b_1) \quad (d_1b_1)] \]
\[ \overset{(6)}{=} d_1o \cdot [d_1o \cdot (d_2d_1 \cdot d_1b_1) \quad (d_1b_1)]. \]
these should be equivalent equalities:
\[ d_1o \cdot (d_1o \cdot b_2) = d_1o \cdot [d_1o \cdot (d_2 \cdot d_1b_1) \quad (d_1b_1)]. \]
\[ b_2 = (d_2 \cdot d_1b_1) \cdot d_1b_1, \]
which is obvious.
(ii) Firstly, let us prove that from GST \( a_1, b_1, b_2, a_2 \) there follows \[ d_1b_1 = d_2b_2. \]  
We have the equalities
\[ d_1 = a_1 \cdot b_2 \quad (oa_1 \cdot a_1), \]
\[ d_2 = a_2 \cdot b_1 \quad (oa_2 \cdot a_2), \]
\[ b_2 = a_1 \quad (a_2b_2 \cdot b_1). \]
Therefore we get
\[ d_1b_1 = \quad [a_1 \cdot b_2 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ = a_1 \quad [a_1 \cdot (a_2b_1 \cdot b_1) \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(6)}{=} [a_1 \cdot a_1 \quad (a_2b_1 \cdot b_1)] \quad [a_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(12)}{=} [a_2b_1 \quad (a_2b_1 \cdot a_1) \quad a_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(7)}{=} [(a_2 \cdot a_2a_1) \quad a_1 \quad b_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(4)}{=} [(a_2 \cdot a_2a_1) \quad a_1 \quad b_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(9)}{=} [a_2b_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(6)}{=} [a_2b_1 \quad (oa_1 \cdot a_1)] \quad b_1 \]
\[ \overset{(7)}{=} (a_2b_1 \quad (a_2 \quad (oa_1 \cdot a_1) \quad b_1), \]
\[ d_2b_2 = [a_2 \cdot b_1 \quad (oa_2 \cdot a_2)] \quad a_1 \quad (a_2b_1 \cdot b_1) \]
\[ \overset{(6)}{=} [a_2b_1 \quad (oa_2 \cdot a_2)] \quad a_1 \quad (a_2b_1 \cdot b_1) \]
so it is necessary to prove the equality
\[ a_2 \left( o a_1 \cdot a_1 \right) \cdot b_1 = a_1 \left( o a_2 \cdot a_2 \right) \left( a_2 b_1 \cdot b_1 \right). \tag{25} \]

Really, we have
\[
\begin{align*}
(9,10) & = (a_1 \cdot a_1 b_1) \cdot \left[ (o \cdot a_2 b_1) b_1 \cdot (a_2 b_1 \cdot b_1) \right] \\
(7) & = [(a_1 \cdot a_1 b_1) \cdot (o \cdot a_2 b_1) (a_2 b_1)] b_1 \\
(4) & = \left( a_1 \left( o a_2 \cdot a_2 \right) \left( a_2 b_1 \cdot b_1 \right) \right) b_1 \\
(7) & = [a_1 \left( o a_2 b_1 \right) \left( a_2 a_2 \right) b_1] \\
(4) & = \left( a_1 \cdot a_1 a_2 \right) \left( o a_2 b_1 \right) b_1 \\
(11) & = (a_1 \cdot a_1 a_2) \left( o a_2 \cdot a_2 \right) b_1 \\
(10) & = [a_2 \left( a_2 a_2 \cdot a_2 \right) \left( o a_2 \cdot a_2 \right)] b_1 \\
(5) & = [(a_2 a_2 a_2) \left( o a_2 \cdot a_2 \right)] b_1 \\
(7) & = [(a_2 a_2 a_2) \left( o a_2 \cdot a_2 \right) a_2 b_1] \\
(4) & = \left[ (o a_1 a_2) \left( o a_2 \cdot a_2 \right) a_2 b_1 \right] \\
(5) & = \left[ (o a_1 a_2) \left( o a_2 \cdot a_2 \right) a_2 b_1 \right] \\
(6) & = \left[ (o a_1 a_2) \left( o a_2 \cdot a_2 \right) a_2 b_1 \right] \\
(5) & = \left[ (o a_1 a_2) \left( o a_2 \cdot a_2 \right) a_2 b_1 \right] \\
(11) & = a_2 \left( o a_1 \cdot a_1 \right) b_1.
\end{align*}
\]

Now, we are going to prove that \( d_1 b_1 = d_2 b_2 \) implies \( \text{GST}(a_1, b_1, b_2, a_2) \). According to (i), from the hypotheses of (ii), there follows the statement \( \text{GST}(d_1, a_1, a_2, d_2) \) and, from this statement and the equality \( d_1 b_1 = d_2 b_2 \), according to Lemma 5, there follows the statement \( \text{GST}(a_1, b_1, b_2, a_2) \).

**Lemma 8.** If the statements \( \text{HARH}_o(c_1, b_1, b_2, c_2), \text{HARH}_d(c_1, b_1, b_2, c_1), \) and \( \text{HARH}_d(a_2 b_2, c_2, e_2) \) are valid, then the statement \( \text{GST}(a_1, b_1, b_2, a_2) \) and equality \( d_1 b_1 = d_2 b_2 \) are equivalent (Figure 9).

**Proof.** The assumptions of the lemma imply the statements \( \text{Par}(o, b_2, b_1, c_1), \text{Par}(o, b_1, b_2, c_2), \text{Par}(b_1, c_1, d_1, a_1), \) and \( \text{Par}(b_2, c_2, d_2, a_2), \) and then, according to Lemma 1, parallelograms \( \text{Par}(o, b_2, a_1, d_1), \text{Par}(o, b_1, a_2, d_2) \) follow from the first and the third, and the second and the fourth parallelogram, respectively. Owing to these last statements, according to Lemma 7(ii), statements \( \text{GST}(a_1, b_1, b_2, a_2) \) and \( d_1 b_1 = d_2 b_2 \) are equivalent.

**Lemma 9.** With the assumption \( \text{ARH}(a_0, b_0, b_2, c_0, c_2, a_0) \), the statement \( f' a_0 = e' a_0 \) follows from the equalities \( d' b_2 = f' b_0 \), \( d' c_0 = e' c_0 \) (Figure 10).

**Proof.** Supposing that a more precise statement \( \text{ARH}_o(a_0, b_0, b_2, c_0, c_2, a_0) \) is valid, so the statements \( \text{Par}(c_0, c_0, o, b_0) \) and \( \text{Par}(b_0, o, a_0, a_0) \) are valid. From the statements \( \text{Par}(c_0, c_0, o, b_0) \), \( d' c_0 = e' c_0 \) there follows \( d' b_2 = f' b_0 \), which together with \( d' b_2 = f' b_0 \) gives the equality \( f' b_0 = e' a_0 \), and this statement and the statement \( \text{Par}(b_0, o, a_0, a_0) \) imply the equality \( f' b_0 = e' a_0 \).
4. Construction of an Affine Fullerene C\textsubscript{60} in a GS-Quasigroup

In this section, we are going to construct an affine fullerene C\textsubscript{60} in a general GS-quasigroup by means of the previously discovered statements about affine regular pentagons and hexagons in a general GS-quasigroup.

**Theorem 11.** An affine fullerene C\textsubscript{60} can be constructed in each GS-quasigroup.

**Proof.** For the sake of clarity, each step of the proof is precisely presented in figures in the GS-quasigroup C((1/2)(1 + √5)) and each sequence of the proof of the theorem can be followed on the Schlegel diagram (Figure 12).

Let us start with the four given points \( b_1, b_2, c_1, c_2 \). The affine regular hexagons ARH\((a_0, b_1, b_2, c_1, c_2, a_1)\) and ARH\((b_1, b_2, c_1, c_2, a_1, b_2)\) can be constructed according to Lemma 2.

Owing to Lemma 6, affine regular pentagons ARP\((b_1, b_2, b_3, b_4, b_5)\) and ARP\((c_1, c_2, c_3, c_4, c_5)\) can be obtained. If we apply Lemma 2 again, we can get ARH\((b_1, b_2, c_1, c_2, d_1, d_2)\) and ARH\((c_1, c_2, e_1, e_2, a_1)\).

According to Theorem 10, the existence of these obtained affine regular hexagons and affine regular pentagons around an affine regular hexagon will result in the existence of the affine regular pentagon ARP\((a_0, a_1, a_2, a_3, a_4)\) (Figure 13).

According to Lemma 2, the already obtained points \( a_0, a_1, a_2, a_3, a_4, a_5, a_6 \) uniquely determine ARH\((d_0, d_0, a_0, a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_5, a_5, a_6, a_6)\) and then, because of Lemma 4, the statements ARH\((d_0, d_0, f_1, f_1, d_1, d_1)\), ARH\((f_1, f_1, a_0, a_0, b_1, b_1, b_2, b_2)\), ARH\((b_1, b_1, a_0, a_0, c_1, c_1, c_2, c_2)\), and ARH\((c_1, c_1, a_0, a_0, e_1, e_1, e_2, e_2)\) imply the statement ARH\((f_1, f_1, a_0, a_0, a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_5, a_5, a_6, a_6)\).

In the same way, the statements ARH\((e_0, b_1, b_1, b_2, b_2, f_1, f_1, c_0, c_0, d_1, d_1)\), ARH\((c_0, c_0, b_1, b_1, a_0, a_0, e_0, e_0)\), and ARH\((a_0, a_0, b_1, b_1, f_1, f_1, a_1, a_1)\) imply ARH\((f_0, f_0, b_1, b_1, b_2, b_2, c_0, c_0, d_1, d_1)\) and, analogously, the statements ARH\((e_0, c_0, c_0, e_0, e_0, f_1, f_1)\), ARH\((e_0, c_0, c_0, a_0, a_0, e_0, e_0)\), ARH\((a_0, a_0, c_0, c_0, b_1, b_1)\), and ARH\((b_1, b_1, e_0, e_0, d_1, d_1, b_2, b_2)\) imply ARH\((d_0, d_0, f_1, f_1, d_1, d_1)\).
These two precisely described procedures will also be used later. It will also be denoted which figure is used for the geometrical presentation of the obtained implications in the GS-quasigroup $C((1/2)(1 + \sqrt{5}))$.

Now, according to Theorem 10, the statements ARH $(d_x, b_x, a_x, c_x, e_x, d'_x)$, ARH $(b_t, b_x, a_t, c_t, e_t, d'_t)$, ARH $(c_t, c_t, e_t, f_t, d'_t)$, ARH $(b_t, b_t, a_t, a_t, e_t, d'_t)$, ARH $(e_t, a_t, c_t, b_t, a_t, e_t, d'_t)$, and ARH $(a_t, a_t, e_t, f_t, d'_t)$ imply ARH $(d'_x, b_x, a_x, c_x, d'_t, c'_t)$. The statements ARH $(e'_x, e'_x, e'_x, e'_x, e'_x, e'_x)$ and ARH $(f'_d, f'_d, f'_d, f'_d, f'_d)$ can be obtained similarly (Figure 15).

Thanks to Lemma 4, the statements ARH $(e'_t, e'_t, e'_t, e'_t, e'_t, e'_t)$, ARH $(c'_t, c'_t, d'_t, f'_t, f'_t, e'_t)$, and ARH $(f'_d, f'_d, f'_d, f'_d, f'_d, f'_d)$ imply ARH $(d'_t, c'_t, d'_t, f'_t, f'_t, a'_t)$. We can find the affine regular hexagons ARH $(b'_d, b'_d, b'_d, b'_d, b'_d, b'_d)$ and ARH $(c'_t, c'_t, f'_t, f'_t, d'_t, d'_t, c'_t)$ in the same way (Figure 16).

If we apply Theorem 10, we will discover three new affine regular pentagons (Figure 17). The statement ARH $(d'_t, d'_t, d'_t, d'_t, d'_t)$ follows from ARH $(d'_f, d'_f, d'_f, d'_f, d'_f, a'_f, a'_f, a'_f, a'_f)$, ARH $(f'_d, f'_d, b'_f, b'_f, b'_f, b'_f, a'_f, a'_f)$, ARH $(f'_d, f'_d, f'_d, f'_d, f'_d, f'_d, a'_f, a'_f, a'_f)$, ARH $(a'_f, a'_f, a'_f, a'_f, a'_f)$, ARH $(f'_d, f'_d, c'_f, c'_f, d'_f, d'_f)$, and ARH $(a'_d, a'_d, a'_d, d'_d, d'_d, a'_d, a'_d)$, and similarly we get ARH $(e'_t, e'_t, e'_t, e'_t, e'_t, e'_t)$ and ARH $(f'_d, f'_d, f'_d, f'_d, f'_d, f'_d)$.

Now, we are in a position to use Lemma 4 whose application gives the statements about three new affine regular hexagons (Figure 18). The statements ARH $(b'_d, b'_d, b'_d, b'_d, b'_d, b'_d)$, ARH $(c'_t, c'_t, c'_t, c'_t, c'_t, c'_t)$, ARH $(a'_d, a'_d, a'_d, a'_d, a'_d, a'_d)$, ARH $(f'_d, f'_d, f'_d, f'_d, f'_d, f'_d)$, and ARH $(a'_d, a'_d, a'_d, a'_d, a'_d, a'_d)$ imply ARH $(c'_t, c'_t, c'_t, c'_t, c'_t, c'_t)$.
the affine regular hexagons ARH\((a'_l, e_{a_l}, e_{c_l}, c'_l, c'_d, c'_e)\) and ARH\((b'_l, f_{b_l}, f_{a_l}, a'_d, a'_e, b'_e)\).

If we apply Theorem 10, we will obtain the three new affine regular pentagons ARP\((a'_l, a'_l, a'_d, a'_e, a'_e)\), ARP\((b'_l, b'_l, b'_d, b'_d, b'_e)\), and ARP\((c'_l, c'_l, c'_d, c'_d, b'_e)\) (Figure 19).

Finally, the application of Lemma 3 will allow us to close the complete structure. We have to prove that the points \(b'_l, a'_l, a'_d, c'_l, c'_d, b'_e\) are the vertices of an affine regular hexagon.

By applying Lemma 3 we get that the statements HARP\((e_{a_l}, a'_l, d_{a_l}, e_{a_l})\), HARP\((e_{a_l}, a'_l, d_{a_l}, d_{a_l})\), HARP\((d_{a_l}, a'_l, d_{a_l}, f_{a_l})\), and HARP\((f_{a_l}, d_{a_l}, d_{a_l}, b'_e)\) imply HARP\((b'_l, a'_l, a'_d, c'_l, c'_d, b'_e)\).

Analogously, we have that HARP\((a'_l, b'_l, b'_l, f_{b_l})\), HARP\((f_{b_l}, b'_l, b'_l, d_{b_l})\), and HARP\((b'_l, b'_l, b'_l, a'_l)\), and HARP\((a'_l, c'_l, c'_d, d_{c_l})\), HARP\((c_l, c'_l, c'_d, f_{c_l})\), HARP\((f_{c_l}, c'_l, c'_d, e_{c_l})\), and HARP\((e_{c_l}, c'_l, c'_d, b'_e)\) imply HARP\((a'_l, c'_l, c'_d, b'_e)\).

These obtained three halves of affine regular hexagons HARP\((b'_l, a'_l, a'_d, c'_l)\), HARP\((c'_l, b'_l, b'_e, a'_l)\), and HARP\((a'_l, c'_l, c'_d, b'_e)\) determine the affine regular hexagon ARH\((b'_l, a'_l, a'_d, c'_l, c'_d, b'_e)\) (Figure 19). This completes the proof of the theorem.

Thus, we have proved that any affine fullerene \(C_{60}\) can be obtained only by applying the golden section.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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