Research Article

A Crank-Nicolson Scheme for the Dirichlet-to-Neumann Semigroup

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The aim of this work is to study a semidiscrete Crank-Nicolson type scheme in order to approximate numerically the Dirichlet-to-Neumann semigroup. We construct an approximating family of operators for the Dirichlet-to-Neumann semigroup, which satisfies the assumptions of Chernoff's product formula, and consequently the Crank-Nicolson scheme converges to the exact solution. Finally, we write a P_1 finite element scheme for the problem, and we illustrate this convergence by means of a FreeFem++ implementation.

1. Introduction

Let Ω be a bounded smooth domain $\Omega \subset \mathbb{R}^n$ and let $\gamma(x) = [\gamma_{i,j}(x)]_{i,j=1}^n$ be a real-valued matrix function, which is known as the electrical conductivity matrix. The matrix $\gamma(x)$ is symmetric and smooth; that is, $\gamma_{ij} = \gamma_{ji} \in C^\infty(\overline{\Omega})$. Its positive eigenvalues are uniformly bounded; that is, there exists $0 < c_1 < c_2$ such that, for all $(x, \xi) \in \Omega \times \mathbb{R}^n$, one has $c_1 \|\xi\|^2 \leq \xi^T \gamma(x) \xi \leq c_2 \|\xi\|^2$, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$.

In this paper, we denote by $X = L^2(\Omega)$, with boundary space $\partial X = L^2(\partial\Omega)$. It is well known that, for any $f \in H^{1/2}(\partial\Omega)$, the following Dirichlet problem,

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0, & \text{in } \Omega, \\ u &= f, & \text{on } \partial\Omega, \end{aligned} \quad (1)$$

has a unique solution $u = L_\gamma f$ in $H^1(\Omega)$, which is called the γ -harmonic lifting of f ; see [1] for more details.

Define the action of the Dirichlet-to-Neumann operator on f as the outward normal derivative of u on the boundary; that is,

$$\Lambda_\gamma(f) := (\nu \cdot \gamma \nabla u)|_{\partial\Omega}, \quad (2)$$

where ν is the outer normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

This operator is defined on its domain:

$$D(\Lambda_\gamma) := \{f \in H^{1/2}(\partial\Omega); \Lambda_\gamma f \in L^2(\partial\Omega)\}. \quad (3)$$

The Dirichlet-to-Neumann semigroup is the trace of the following boundary value problem:

$$(\text{BVP}) \begin{cases} \operatorname{div}(\gamma \nabla u(t, \cdot)) = 0, & \text{in } \Omega, \\ \partial_t u + \nu \cdot \gamma \nabla u = 0, & \text{on } \partial\Omega, \\ u(0, \cdot) = f, & \text{on } \partial\Omega. \end{cases} \quad (4)$$

In [1], Lax has introduced a semigroup in ∂X , where Ω is the unit ball $B(0, 1)$ and γ is the identity matrix. Let v be the solution of the problem:

$$\begin{aligned} \Delta v &= 0, & \text{in } \Omega, \\ Lv &= f, & \text{on } \partial\Omega, \end{aligned} \quad (5)$$

where L is the trace operator on $\partial\Omega$. The Lax semigroup is defined by

$$S(t) = LT(t)L_0, \quad (6)$$

where L_0 is the lifting operator and $T(t)v(x) := v(e^{-t}x)$.

Therefore, if $f \in \partial X$, then, for any $w \in \partial\Omega$,

$$(S(t)f)(w) = v(e^{-t}w). \quad (7)$$

It has been shown that the Lax semigroup is actually the Dirichlet-to-Neumann semigroup.

In [2], it is shown that Lax's representation cannot be generalized if Ω is not the unit ball. This was a motivation for many authors to use Chernoff's formula in order to approximate the solution.

In [3], Emamirad and Shariftabar proved that the Euler Explicit scheme

$$\begin{aligned} \operatorname{div}(\gamma \nabla u^{m+1}) &= 0, \quad \text{in } \Omega, \\ \frac{1}{\Delta t} (u^{m+1} - u^m) + \frac{\partial u^m}{\partial \nu_\gamma} &= 0, \quad \text{on } \partial\Omega, \\ u^0 &= f, \quad \text{on } \partial\Omega \end{aligned} \quad (8)$$

converges to the solution of the (BVP).

Later, Cherif et al. proposed, in [4], the following Implicit Euler Scheme:

$$\begin{aligned} \operatorname{div}(\gamma \nabla u^{m+1}) &= 0, \quad \text{in } \Omega, \\ \frac{1}{\Delta t} (u^{m+1} - u^m) + \frac{\partial u^{m+1}}{\partial \nu_\gamma} &= 0, \quad \text{on } \partial\Omega, \\ u^0 &= f, \quad \text{on } \partial\Omega, \end{aligned} \quad (9)$$

which converges to the solution of the (BVP).

After studying the implicit and explicit schemes, by the same strategy, we are interested in studying the convergence of the Crank-Nicolson type scheme. This paper is divided into 4 sections, as follows.

First, in Section 2, we introduce the Crank-Nicolson type scheme for evolution equations.

In Section 3, we prove the convergence of a Crank-Nicolson type scheme for the (BVP), by constructing an approximating family, and use Chernoff's product formula in order to prove the convergence in $L^2(\partial\Omega)$.

Finally, in Section 4, we present a numerical implementation of the Crank-Nicolson type scheme in the computational framework of the finite element method.

2. The Crank-Nicolson Scheme

In numerical analysis, the Crank-Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is a second-order method in time and it is numerically stable. The method was developed by John Crank and Phyllis Nicolson in the mid-20th century.

Let A be a m -accretive operator on the Hilbert space H defined as $A : D(A) \rightarrow H$.

Consider the evolution problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + Au &= 0, \quad \text{for } 0 < t < T, \\ u(0, x) &= u_0(x). \end{aligned} \quad (10)$$

We approximate the time derivative by using the finite difference method as follows:

$$\begin{aligned} u(n\Delta t) &\approx u^n \\ \frac{\partial u}{\partial t}(n\Delta t) &= \frac{1}{\Delta t} (u(n+1)\Delta t - u(n\Delta t)) \\ &\approx \frac{1}{\Delta t} (u^{n+1} - u^n), \quad \text{for } n = 0, \dots, \left\lfloor \frac{T}{\Delta t} \right\rfloor. \end{aligned} \quad (11)$$

Crank-Nicolson introduced the scheme:

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{2}Au^n + \frac{1}{2}Au^{n+1} &= 0, \\ \text{for } n &= 0, \dots, \left\lfloor \frac{T}{\Delta t} \right\rfloor, \\ u_0 &\text{ given.} \end{aligned} \quad (12)$$

The Chernoff product formula holds not only in the case of semigroup of contractions but also if the semigroup is bounded, so that the Crank-Nicolson scheme for the (BVP) can be written as

$$(CNS) \begin{cases} \operatorname{div}(\gamma \nabla u^{m+1}) + \operatorname{div}(\gamma \nabla u^m) = 0, & \text{in } \Omega, \\ \frac{1}{\Delta t} (u^{m+1} - u^m) + \frac{1}{2} \left(\frac{\partial u^m}{\partial \nu_\gamma} + \frac{\partial u^{m+1}}{\partial \nu_\gamma} \right) = 0, & \text{on } \partial\Omega, \\ u^0 = f, & \text{on } \partial\Omega. \end{cases} \quad (13)$$

3. Convergence of the Crank-Nicolson Type Scheme

In order to show the convergence of the (CNS), let us recall the following Chernoff theorem.

Theorem 1 (Chernoff's product formula). *Let X be a Banach space, real or complex, and let $\{V(t); t \geq 0\}$ be a family of contractions on X with $V(0) = I$.*

Assume that there exists $A : D(A) \subseteq X \rightarrow X$ which generates a C_0 -semigroup of contractions $\{S(t); t \geq 0\}$ and that for all $x \in D(A)$ there exists

$$\lim_{h \rightarrow 0} \frac{1}{h} (V(h) - I)x = Ax. \quad (14)$$

Then, for each x in X , one has $\lim_{n \rightarrow \infty} V^n(t/n)x = S(t)x$.

Remark 2. Note that Chernoff's product formula holds not only in the case of semigroup of contractions but also if the semigroup is bounded (see [5]).

This theorem was used to prove the convergence of both implicit and explicit schemes.

Now, in order to prove the convergence of the Crank-Nicolson scheme, we will show that the operator $Z(t)$, defined below, satisfies the 3 conditions of Chernoff's theorem.

For any $x \in \partial\Omega$, we have

$$\begin{aligned}\frac{\partial u^{m+1}}{\partial \nu_\gamma} &= \frac{1}{\Delta x} \left[u^{m+1}(x) - u^{m+1}(x - \Delta x \gamma(x) \nu) \right], \\ \frac{\partial u^m}{\partial \nu_\gamma} &= \frac{1}{\Delta x} \left[u^m(x) - u^m(x - \Delta x \gamma(x) \nu) \right].\end{aligned}\quad (15)$$

Using in (CNS) the approximation given in (15), we get

$$\begin{aligned}\left[1 + \frac{1}{2} \frac{\Delta t}{\Delta x} \right] u^{m+1}(x) - \left[1 - \frac{1}{2} \frac{\Delta t}{\Delta x} \right] u^m(x) \\ - \frac{1}{2} \frac{\Delta t}{\Delta x} u^{m+1}(x - \Delta x \gamma(x) \nu) \\ - \frac{1}{2} \frac{\Delta t}{\Delta x} u^{m+1}(x - \Delta x \gamma(\nu)) = 0.\end{aligned}\quad (16)$$

Taking $\alpha = \Delta t / \Delta x$, we get

$$\begin{aligned}\left[1 + \frac{\alpha}{2} \right] u^{m+1}(x) - \frac{\alpha}{2} u^{m+1}(x - \Delta x \gamma(x) \nu) \\ = \left[1 - \frac{\alpha}{2} \right] u^m(x) + \frac{\alpha}{2} u^{m+1}(x - \Delta x \gamma(x) \nu)\end{aligned}\quad (17)$$

which can be written as

$$W(\Delta t) u^{m+1} = V(\Delta t) u^m, \quad (18)$$

where

$$\begin{aligned}W(\Delta t) f &= \left[1 + \frac{\alpha}{2} \right] u - \frac{\alpha}{2} u(x - \alpha^{-1} t \gamma(x) \nu), \\ V(\Delta t) f &= \left[1 - \frac{\alpha}{2} \right] u + \frac{\alpha}{2} u(x - \alpha^{-1} t \gamma(x) \nu).\end{aligned}\quad (19)$$

Now, we define the family $Q(t)$ as the inverse of $W(t)$ in the following sense:

$$W(\Delta t) Q(\Delta t) f(x) = f(x), \quad (20)$$

and we show that $Z(t) = Q(\Delta t) V(\Delta t)$ satisfies all the assumptions of Chernoff's product formula.

(i) $Z(0)f = f$. For any $x \in \partial\Omega$, we have

$$\begin{aligned}W(0) f(x) &= \left(1 + \frac{\alpha}{2} \right) u(x) - \frac{\alpha}{2} u(x) = u(x) \\ &= f(x)\end{aligned}\quad (21)$$

so, $f(x) = W(0)[Q(0)f(x)] = Q(0)f(x)$.

Moreover,

$$\begin{aligned}V(0) f(x) &= \left(1 - \frac{\alpha}{2} \right) u(x) + \frac{\alpha}{2} u(x) = u(x) \\ &= f(x)\end{aligned}\quad (22)$$

so,

$$V(0) f(x) = f(x). \quad (23)$$

Using (6) and (7), we get

$$W(0) f(x) = V(0) f(x). \quad (24)$$

Therefore, $f(x) = Q(0)V(0)f(x) = Z(0)f(x)$.

(ii) $Z'(0)f = -\Lambda_\gamma f$. The derivative of $W(t)$ with respect to t is

$$W'(t) f = -\frac{\alpha}{2} (-\alpha^{-1} \gamma \nu) \cdot \nabla u (v - \alpha^{-1} t \gamma \nu). \quad (25)$$

At point $t = 0$, we have $W'(0)f = 1/2 \Lambda_\gamma f$.

Similarly, the derivative of $V(t)$ with respect to t is

$$V'(t) f = \frac{\alpha}{2} (-\alpha^{-1} \gamma \nu) \cdot \nabla u (v - \alpha^{-1} t \gamma \nu). \quad (26)$$

At point $t = 0$, we have $V'(0)f = -(1/2) \Lambda_\gamma f$.

But $W(t)Q(t)f = f$, so $W'(t)Q(0)f + W(0)Q'(0)f = 0$, and then $Q'(0)f = -(1/2) \Lambda_\gamma f$.

Now, the derivative of $Z(t)$ with respect to t is

$$Z'(t) f = Q'(t) V(t) f + Q(t) V'(t) f. \quad (27)$$

At $t = 0$, $Z'(0)f = Q'(0)V(0)f + Q(0)V'(0)f = -\Lambda_\gamma f$.

Therefore,

$$Z'(0) f = \Lambda_\gamma f. \quad (28)$$

(iii) $Z(t)$ Is a Contraction. In fact,

$$\begin{aligned}\|W(t) f\|_{L^2(\partial\Omega)}^2 \\ = \int_{\partial\Omega} \left| u(x) + \frac{\alpha}{2} (u(x) - u(x - \alpha^{-1} t \gamma \nu)) \right|^2 d\sigma\end{aligned}\quad (29)$$

but $(u(x) - u(x - \alpha^{-1} t \gamma \nu)) / \alpha^{-1}$ is an upper bound of the normal derivative $\partial u / \partial \nu_\gamma$, so that this term is positive according to Hopf's lemma, and, consequently,

$$\begin{aligned}\int_{\partial\Omega} \left| u(x) + \frac{\alpha}{2} (u(x) - u(x - \alpha^{-1} t \gamma \nu)) \right|^2 d\sigma \\ \geq \int_{\partial\Omega} |u(x)|^2 d\sigma = \|f\|_{L^2(\partial\Omega)}^2\end{aligned}\quad (30)$$

which implies that

$$\|W(t) f\|_{L^2(\partial\Omega)} \geq \|f\|_{L^2(\partial\Omega)}. \quad (31)$$

Now

$$\|f\|_{L^2(\partial\Omega)} = \|W(t) Q(t) f\|_{L^2(\partial\Omega)} \geq \|Q(t) f\|_{L^2(\partial\Omega)}. \quad (32)$$

Thus, $Q(t)$ is of contraction.

Similarly $V(t)$ is also of contraction.

Finally,

$$\begin{aligned}\|Z(t) f\|_{L^2(\partial\Omega)} &= \|Q(t) V(t) f\|_{L^2(\partial\Omega)} \\ &\leq \|Q(t) f\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)}.\end{aligned}\quad (33)$$

Therefore, $Z(t)$ is of contraction.

So, we have proved that $Z(t)$ satisfies all the assumptions of Chernoff's product formula, and, consequently, the Crank-Nicholson scheme converges to its exact solution.

Theorem 3. *The operator $Z(t)$ satisfies the Chernoff conditions; hence, the Crank-Nicolson method is convergent.*

4. Numerical Results

In the last section of this paper, we present the resolution of (CNS) in the particular case where γ is the unit matrix and Ω is the open unit disk. The point in dealing with this case is that the exact solution is known to be $v(e^{-t}x, e^{-t}y)$, where v is the solution of the Laplace equation $\Delta v = 0$, with $v = u_0$ on the boundary (u_0 is any regular function). Thus, we can compare this exact solution with the approximate solution obtained via our Crank-Nicolson scheme. For this test, we choose $u_0(x, y) = 0.5(x^2 - y^2) + y + 0.5$. The variational formulation of the (CNS) can be derived by multiplying both sides of the problem by a test function v :

$$\int_{\Omega} \gamma \Delta u^{m+1} v \, dx + \int_{\Omega} \gamma \Delta u^m v \, dx = 0. \quad (34)$$

Using Green's formula, we get

$$\begin{aligned} - \int_{\Omega} \gamma \nabla u^{m+1} \nabla v \, dx - \int_{\partial\Omega} \frac{2}{\Delta t} [u^{m+1} - u^m] v \, d\sigma \\ - \int_{\Omega} \gamma \nabla u^m \nabla v \, dx = 0. \end{aligned} \quad (35)$$

Multiplying by Δt , we get

$$\begin{aligned} \int_{\Omega} \Delta t \gamma \nabla u^{m+1} \nabla v \, dx + 2 \int_{\partial\Omega} u^{m+1} v \, d\sigma \\ = - \int_{\Omega} \Delta t \gamma \nabla u^m \nabla v \, dx + 2 \int_{\partial\Omega} u^m v \, d\sigma \end{aligned} \quad (36)$$

which is of the form

$$a(u^{m+1}, v) = l(v), \quad (37)$$

where the bilinear form

$$\begin{aligned} a(u^{m+1}, v) &= \int_{\Omega} \Delta t \gamma \nabla u^{m+1} \nabla v \, dx + 2 \int_{\partial\Omega} u^{m+1} v \, d\sigma, \\ l(v) &= - \int_{\Omega} \Delta t \gamma \nabla u^m \nabla v \, dx + 2 \int_{\partial\Omega} u^m v \, d\sigma \end{aligned} \quad (38)$$

is the linear form.

Using this variational formulation, the problem can be approximate via P_1 -finite elements implemented in Freefem++.

The actual algorithm is as follows.

Choose an initial u_0 and, for a given Δt , where $t < N\Delta t$, we have the following:

for ($t = 0; t < N\Delta t$) do

(i) $u^{n+1} = Z(u^n)$

(ii) compute u^{n+1} via the variational problem

(iii) $n \leftarrow n + 1$

end do.

As an illustration, let us represent the exact (Figure 1) and the approximate solution (Figure 2), the L^2 -error between the exact and the approximate solutions (Figure 3), and the decrement of this L^2 -error according to the fineness of the meshing (Figure 4).

FIGURE 1: The exact solution.

FIGURE 2: The numerical solution.

FIGURE 3: The difference between the exact and the numerical solutions.

FIGURE 4: The L^2 -error according to the fineness of the mesh.

5. Conclusion

As a conclusion, we have proved the convergence of the (CNS) by showing that the constructed family of operators satisfies the assumptions of Chernoff's product formula and consequently approximated numerically the Dirichlet-to-Neumann semigroup.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] P. D. Lax, *Functional Analysis*, Wiley-Interscience, New York, NY, USA, 2012.
- [2] H. Emamirad and I. Laadnani, "An approximating family for the Dirichlet-to-Neumann semigroup," *Advances in Differential Equations*, vol. 11, no. 3, pp. 241–257, 2006.
- [3] H. Emamirad and M. Sharifitabar, "On explicit representation and approximations of Dirichlet-to-Neumann semigroup," *Semigroup Forum*, vol. 86, no. 1, pp. 192–201, 2013.
- [4] M. A. Cherif, T. El Arwadi, H. Emamirad, and J.-M. Sac-épée, "Dirichlet-to-Neumann semigroup acts as a magnifying glass," *Semigroup Forum*, vol. 88, no. 3, pp. 753–767, 2014.
- [5] K. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.