

Research Article

Extremal Trees with respect to Number of \((A, B, 2C)\)-Edge Colourings

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We determine the smallest and the largest number of \((A, B, 2C)\)-edge colourings in trees. We prove that the star is a unique tree that maximizes the number of all of the \((A, B, 2C)\)-edge colourings and that the path is a unique tree that minimizes it.

1. Introduction and Preliminary Results

For a general concept, see [1]. The Fibonacci sequence \(\{F_n\}\) is defined recursively by the second-order recurrence relation \(F_n = F_{n-1} + F_{n-2}\) for \(n \geq 2\) with the initial conditions \(F_1 = F_2 = 1\). A related sequence is the Pell sequence \(\{P_n\}\) defined by \(P_n = 2P_{n-1} + P_{n-2}\) for \(n \geq 2\) with \(P_0 = 0\), \(P_1 = 1\). Table 1 includes first terms of the sequence \(\{P_n\}\). The terms of Fibonacci and Pell sequences are called Fibonacci numbers and Pell numbers, respectively. The numbers of the Fibonacci type play an important role in distinct areas of mathematics and they have many different applications and interpretations. Some of them are closely related to the Hosoya index \(Z(G)\) (defined as a number of all matchings in the graph \(G\), including the empty matching) and the Merrifield-Simmons index \(\sigma(G)\) (defined as a number of all independent sets in \(G\), including the empty set); see [2] and its references. It is well-known that \(\sigma(\mathcal{P}_n) = F_n\) and \(Z(\mathcal{P}_n \circ \mathcal{K}_1) = P_{n-1}\), for \(n \geq 1\), where \(\mathcal{P}_n\) is an \(n\)-vertex path, \(\mathcal{K}_n\) is an \(n\)-vertex complete graph, and \(G \circ H\) denotes the corona of two graphs. The numbers of the Fibonacci type in the graph theory were studied intensively also in [3–15].

Consider a simple, undirected graph \(G\) with the vertex set \(V(G)\) and the edge set \(E(G)\). In [11] we introduced an \((a_1A_1, a_2A_2, \ldots, a_pA_p)\)-edge colouring of a graph \(G\) defined in the following way. Let \(G\) be \(p\)-edge coloured graph with the set of colours \(\{A_1, A_2, \ldots, A_p\}\), where \(p \geq 2\). Moreover, let \(a_1, a_2, \ldots, a_p\) be positive integers. We say that a subgraph of \(G\) is \(M\)-monochromatic if all its edges are coloured alike by colour \(M \in \{A_1, A_2, \ldots, A_p\}\). The graph \(G\) is said to be \((a_1A_1, a_2A_2, \ldots, a_pA_p)\)-edge coloured, if every maximal (with respect to set inclusion) \(A_i\)-monochromatic subgraph of \(G\) can be partitioned into edge-disjoint paths of the length \(a_i\), \(i = 1, 2, \ldots, p\). This type of edge colouring of graph generalizes the edge colouring introduced by Piejko and Włoch in [10] and the edge colouring by monochromatic paths introduced by Trojnar-Spelina and Włoch in [13]. Many interesting results concerning some special kinds of \((a_1A_1, a_2A_2, \ldots, a_pA_p)\)-edge colouring of graphs can be found in [10, 11]. We recall some of them.

Let \(\{A, B, C\}\) be the set of colours. By \((A, B, 2C)\)-edge colouring we denote the 3-edge colouring of graph \(G\), such that every \(C\)-monochromatic subgraph of \(G\) can be partitioned into edge-disjoint paths of the even length. Let \(\delta(G)\) be the number of all \((A, B, 2C)\)-edge colourings of the graph \(G\). The following result was given in [10].

Theorem 1 (see [10]). Let \(n \geq 2\) be an integer. Then \(\delta(\mathcal{P}_n) = P_{n-1}\).

Let \(\{s_n\}\) be the sequence defined by the relation \(s_n = 2s_{n-1} + (n - 1)s_{n-2}\) for \(n \geq 2\) with the initial conditions \(s_0 = 1\), \(s_1 = 2\). We can find a few first terms of \(\{s_n\}\) in Table 1.

The sequence \(\{s_n\}\) has many distinct interpretations also in graphs. It is worth mentioning that \(s_n\) is the Hosoya index of the corona of the complete graphs \(\mathcal{K}_n\) and \(\mathcal{K}_1\); that is, \(Z(\mathcal{K}_n \circ \mathcal{K}_1) = s_n\) for \(n \geq 1\). For more interpretations see [16, 17].
Table 1: The first terms of \( \{P_n\} \) and \( \{s_n\} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_n )</td>
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<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td>70</td>
<td>169</td>
<td>408</td>
<td>985</td>
<td>2378</td>
<td>5741</td>
</tr>
<tr>
<td>( s_n )</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>43</td>
<td>142</td>
<td>499</td>
<td>1850</td>
<td>7193</td>
<td>12310</td>
<td>538078</td>
<td>2430355</td>
</tr>
</tbody>
</table>

Another interpretation of the sequence \( \{s_n\} \) in graphs, which is closely related to \((A, B, 2C)\)-edge colourings of \( n \)-edge star \( K_{1,n-1} \), was given in [11].

**Theorem 2** (see [11]). Let \( n \) be a positive integer. Then 
\[ \delta(K_{1,n}) = s_n. \]

In [12] Prodinger and Tichy proved that the star is a tree that maximizes the Merrifield-Simmons index, while the path is a tree that minimizes it. In this paper we obtain an analogous result for the number of \((A, B, 2C)\)-edge colourings in trees.

Let \( G \) and \( H \) be given graphs with distinguished vertices \( x \in V(G) \) and \( y \in V(H) \). By \( G^x \) we denote the graph obtained from \( G \) by deleting the vertices \( x \) and \( x \). For \( e \in E(G) \) the notation \( G \setminus \{ x \} \) means the graph obtained from \( G \) by deleting the edge \( e \). We prove the following.

**Theorem 3.** Let \( x \in V(G) \), \( y \in V(H) \) and let \( x_1, x_2, \ldots, x_k \in V(G) \) be neighbours of \( x \) and let \( y_1, y_2, \ldots, y_l \in V(H) \) be neighbours of \( y \), where \( k \) and \( l \) are positive integers. Then
\[ \delta(G^x \oplus H^y) = 2\delta(G)\delta(H) + \delta(H) \sum_{i=1}^k \delta(G \setminus \{ xx_i \}) \]
\[ + \delta(G) \sum_{j=1}^l \delta(H \setminus \{ yy_j \}) \]  
(1)

\[ \delta(G^x \circ H^y) \geq \delta(G) \delta(H) \]
\[ + k \sum_{i=1}^k \delta(G \setminus \{ xx_i \}) \delta(H \setminus \{ xx_i \}) \]  
(2)

Furthermore, if \( l = 1 \) then
\[ \delta(G^x \circ H^y) = \delta(G) \delta(H) \]
\[ + \delta(H \setminus \{ yy_j \}) \sum_{i=1}^k \delta(G \setminus \{ xx_i \}) \]  
(3)

Proof. By \( \delta_A \), \( \delta_B \), and \( \delta_C \) we denote the number of all \((A, B, 2C)\)-edge colourings of the graph \( G^x \oplus H^y \) such that an edge \( xy \) has a colour \( A \), \( B \), or \( C \), respectively. It can be easily seen that \( \delta_A \) and \( \delta_B \) are equal to the number of all \((A, B, 2C)\)-edge colourings of the graph \( G \) multiplied by the number of all \((A, B, 2C)\)-edge colourings of the graph \( H \). Moreover, \( \delta_C \) is equal to the number of all \((A, B, 2C)\)-edge colourings of the graph \( H \) multiplied by the number of all \((A, B, 2C)\)-edge colourings of graphs \( G \setminus \{ xx_i \} \), where \( i = 1, 2, \ldots, k \), plus the number of all \((A, B, 2C)\)-edge colourings of the graph of the graph \( G^x \oplus H^y \) with exactly \( l \) \( C \)-monochromatic paths \( \{e \} \subset E(G^x \oplus H^y) \), such that \( e \in \{e_1, e_2, \ldots, e_k\} \) and \( e' \in \{e_1', e_2', \ldots, e_l'\} \). Observing that \( \zeta(0) = \delta(K)\delta(H) \) and
\[ \zeta(1) = \sum_{i=1}^k \sum_{j=1}^l \delta(G \setminus \{ xx_i \}) \delta(H \setminus \{ yy_j \}) \]  
(5)

It should be noted that \( \delta(G^x \circ H^y) = \sum_{t=0}^r \zeta(t) \) and so \( \delta(G^x \circ H^y) \leq \zeta(0) + \zeta(1) \), which gives inequality (2). Moreover, if \( l = 1 \) then also \( r = 1 \) and we obtain equality (3). This completes the proof.

**2. The Largest Number of \((A, B, 2C)\)-Edge Colourings in Trees**

In this section we will show that, among all trees with the given number of vertices \( n \), the star \( K_{1,n-1} \) maximizes the...
number of \((A, B, 2C)\)-edge colourings. Moreover the star 
\(K_{1, n-1}\) is the unique tree with such property. To prove it we need the
following.

**Theorem 4.** Let \(m \geq 2\) be an integer. Then for a graph \(G\) and 
arbitrary \(x \in V(G)\) one has

\[
\delta \left( G^x \ast K_{1,m} \right) < \delta \left( G^x \ast K_{1,m}^v \right),
\]

where \(v\) is the leaf of \(K_{1,m}\) and \(v_0\) is the center of \(K_{1,m}^v\).

**Proof.** Let \(V(K_{1,m}) = \{v_0, v_1, \ldots, v_m\}\) and 
\(E(K_{1,m}) = \{v_0v_1, v_0v_2, \ldots, v_0v_m\}\). Let \(x \in V(G)\) be the vertex of degree 
\(k \geq 1\) and let \(x_1, x_2, \ldots, x_k \in V(G)\) be neighbours of \(x\). By (3) in 
Theorem 3 we have

\[
\delta \left( G^x \ast K_{1,m} \right) = \delta \left( G \ast \left( K_{1,m} \setminus \{v_0, v_1, \ldots, v_m\} \right) \right),
\]

and by inequality (2) in Theorem 3 we have

\[
\delta \left( G^x \ast K_{1,m}^v \right) \geq \delta \left( G \ast \left( K_{1,m} \setminus \{v_0, v_1, \ldots, v_m\} \right) \right),
\]

and so (8) gives

\[
\delta \left( G^x \ast K_{1,m} \right) < \delta \left( G \ast \left( K_{1,m} \setminus \{v_0, v_1, \ldots, v_m\} \right) \right).
\]

Since \(m \geq 2\), then from (7) and (9) we have \(\delta \left( G^x \ast K_{1,m} \right) < \delta \left( G^x \ast K_{1,m}^v \right)\),
which completes the proof.

**Theorem 5.** Let \(n \geq 2\) be an integer and let \(T_n\) be a tree with \(n\)
vertices. Then

\[
\delta \left( T_n \right) \leq \delta \left( K_{1,n-1} \right).
\]

**Proof (by induction on the number of degree \(k \geq 2\) in the tree \(T_n\)).** Let \(T_n\) be an \(n\)-vertex tree with exactly \(t\) vertices of
degree \(k \geq 2\). If \(t = 1\) then the result is obvious, because \(T_{n-1}\)
is an \(n\)-vertex star \(K_{1,n-1}\). Assume that inequality (10) holds
for \(T_{n,t}\) with arbitrary \(t \geq 1\). We will prove that it holds for
\(T_{n,t+1}\). Note that for each tree \(T_{n,t+1}\) there exists \(x \in V(T_{n,t+1})\),
such that \(T_{n,t+1}\) is isomorphic to \(T_{n-mt+1} \ast K_{1,m-1}^v\), where \(v\)
is the leaf of the star \(K_{1, m-1}\) and \(m \geq 3\). Applying Theorem 4 we have

\[
\delta \left( T_{n,t+1} \right) = \delta \left( T_{n-mt+1} \ast K_{1,m-1}^{v_0} \right),
\]

where \(v_0\) is the center of the star \(K_{1,m-1}^{v_0}\). Note that \(T_{n-mt+1} \ast K_{1,m-1}^{v_0}\) is the \(n\)-vertex tree with \(t\) vertices of the degree \(k \geq 2\). Thus, by the induction hypothesis we have \(\delta(T_{n,t+1}) \leq \delta(K_{1,m-1})\), which completes the proof.

**Remark 6.** From Theorems 4 and 5 we can see that the star 
\(K_{1, n-1}\) is a unique graph which maximizes the number of 
\((A, B, 2C)\)-edge colourings in trees of given order \(n\).

3. The Smallest Number of \((A, B, 2C)\)-Edge Colourings in Trees

Now we show that, among all trees with the given number of 
vertices \(n\), the path \(P_n\) minimizes the number of all 
\((A, B, 2C)\)-edge colourings and that it is the unique tree with
such property. To prove it we need some initial results. First
we prove the following property of the Pell numbers.

**Theorem 7.** Let \(q \geq 2\) and \(m_i \geq 1\) be integers for \(i = 1, 2, \ldots, q\). Then

\[
\sum_{i=1}^{q} P_{m_i} > P_{m-1},
\]

where \(m = m_1 + m_2 + \cdots + m_q + 1\).

**Proof (by induction on \(q\)).** For \(q = 2\) we have

\[
\sum_{i=1}^{2} P_{m_i} = P_{m_1} + P_{m_2} > P_{m_1} > P_{m_1}.
\]

We can check the above inequality using the well-known 
identity for the Pell numbers

\[
P_n P_{n+1} + P_{n-1} P_m = P_{m+1}.
\]

Assume that inequality (12) holds for an arbitrary \(q \geq 2\). We show that it holds for \(q + 1\); namely,

\[
\sum_{i=1}^{q+1} P_{m+q+1} > P_{m+q+1}.
\]

Using (14) and the induction hypothesis we obtain

\[
\sum_{i=1}^{q+1} P_{m+q+1} = \sum_{i=1}^{q} P_{m+q+1} - m_i P_{m_i} + \sum_{i=1}^{q} P_{m_i} P_{m_i+1}
\]

\[
= \sum_{i=1}^{q} \left[ P_{m_i+1} - P_{m_i} P_{m_i+1} \right] P_{m_i} + P_{m_i} P_{m_i+1}
\]

\[
= P_{m_i} P_{m_i+1} + P_{m_i} P_{m_i+1} + \sum_{i=1}^{q} P_{m_i} P_{m_i+1}
\]

\[
> P_{m_i} P_{m_i+1} + P_{m_i} P_{m_i+1} + \sum_{i=1}^{q} P_{m_i} P_{m_i+1}
\]

\[
= P_{m+q+1}.
\]

which ends the proof.
Let $m_1, m_2, \ldots, m_q$, $q \geq 2$, be positive integers. By $\delta_{m_{i}}$, where $i = 1, 2, \ldots, q$, we denote the subdivision of a star $\mathcal{S}_{i}$ in such a way that in the resulting graph an $i$th edge of $\mathcal{S}_{i}$ is replaced by a path of length $m_i$. For $m_1 = m_2 = \cdots = m_q = 1$ the graph $\mathcal{S}_{1}$ is a star $\mathcal{S}_{q}$ and for $q = 2$ the graph $\mathcal{S}_{2}$ is a path $\mathcal{P}_{m_{1}+1}$. For $q = 3$ the star subdivision $\mathcal{S}_{m_{1}}$ is called a tripod. In [11] we proved that

$$\delta \left( \mathcal{S}_{m_{1}} \right) = P_{m_{1}+1} \cdot \mathcal{P}_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$$

for all positive integers $m_1, m_2, m_3$. We will use the following notations:

$$V \left( \mathcal{S}_{m_{1}} \right) = \left\{ y_0 \cup \left\{ y_1, y_2, \ldots, y_{m_{1}} \right\} \right\}$$

$$U \left\{ y_1, y_2, \ldots, y_{m_{1}} \right\} \cup \cdots \cup \left\{ y_{1}, y_2, \ldots, y_{m_{1}} \right\}$$

$$\sum_{i=1}^{q} \left( \delta_{m_{i}} \right) = \delta \left( \mathcal{S}_{m_{1}} \right) \cdot \mathcal{P}_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$$

Thus by (20), (22), and Theorem 1 we have

$$\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$$

By the induction hypothesis we have $\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m}$ and so (23) gives

$$\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$$

Using identity (14) we have $\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$, which completes the proof.

**Theorem 9.** Let $m_1, m_2, \ldots, m_q$, $q \geq 2$, be positive integers and let $m = m_1 + m_2 + \cdots + m_q + 1$. Then for a graph $G$ and arbitrary $x \in V(G)$ one has

$$\delta \left( G^{x} + \mathcal{S}_{m_{1}} \right) > \alpha \left( G^{x} + \mathcal{P}_{m} \right)$$

where $y_0$ is the center of $\mathcal{S}_{m_{1}}$, and $v$ is the leaf of the path $\mathcal{P}_{m}$.

**Proof.** Let $x \in V(G)$ be the vertex of degree $k \geq 1$ and let $x_1, x_2, \ldots, x_k \in V(G)$ be neighbours of $x$. Note that $G^{x} + \mathcal{S}_{m_{1}}$ is isomorphic to $G^{x} + \mathcal{S}_{m_{1}}$, where $y_0$ is the center of $\mathcal{S}_{m_{1}}$, and $v$ is the leaf of $\mathcal{P}_{m-1}$, and $m_1$ is a positive integer. Thus, applying (3) of Theorem 3 we obtain

$$\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m_{1}+1} + \cdots + \mathcal{P}_{m_{q}+1}$$

Moreover equality (1) of Theorem 3 gives

$$\delta \left( G^{x} + \mathcal{P}_{m} \right) = 2 \delta \left( G^{x} + \mathcal{P}_{m} \right) + \cdots + \mathcal{P}_{m-1}$$

By (27), Theorem 1, and the definition of $(P_n)$ we obtain

$$\delta \left( G^{x} + \mathcal{P}_{m} \right) = \delta \left( G^{x} + \mathcal{P}_{m} \right) + \delta \left( \mathcal{P}_{m} \right) \sum_{i=1}^{k} \delta \left( G \setminus \left\{ x_i \right\} \right)$$

From Theorem 8 we have $\delta \left( \mathcal{S}_{m_{1}} \right) > P_{m_{1}}$ and

$$\delta \left( \mathcal{S}_{m_{1}} \right) \cdot \mathcal{P}_{m_{1}} + \cdots + \mathcal{P}_{m_{q}+1}$$

which ends the proof.

**Theorem 10.** Let $n \geq 2$ be an integer and let $T_n$ be a tree with $n$ vertices. Then

$$\delta \left( T_n \right) \geq \delta \left( \mathcal{P}_{n} \right)$$
Proof (by induction on the number of vertices of degree $k \geq 3$ in the tree $T_n$). Let $T_{n,t}$ be the $n$-vertex tree with exactly $t$ vertices of degree $k \geq 3$. If $t = 0$ then the result is obvious, because $T_{n,0}$ is an $n$-vertex path $P_n$. Assume that inequality (29) holds for $T_{n,t}$ with arbitrary $t \geq 0$. We will prove that it holds for $T_{n,t+1}$. Note that for each tree $T_{n,t+1}$ there exists $x \in V(T_{n,t+1})$, such that $T_{n,t+1}$ is isomorphic to $T_n^x - m_1 + \delta_{m_1,m_2,\ldots,m_q}$, where $y_0$ is the center of $\delta_{m_1,m_2,\ldots,m_q}$, $m = m_1 + m_2 + \cdots + m_q + 1$, and $q \geq 2$. Applying Theorem 9 we have

$$\delta(T_{n,t+1}) = \delta(T_n^x + \delta_{y_0}) > \delta(T_n^x + \delta_{y_0}),$$

where $v$ is the leaf of the path $P_m$. Note that $T_n^x - m_1 + \delta_{y_0}$ is the $n$-vertex tree with $t$ vertices of degree $k \geq 3$. Thus by the induction hypothesis we have $\delta(T_{n,t+1}) \geq \delta(T_n^x)$ and the proof is complete.

Remark II. From Theorems 9 and 10 we can see that the path $P_m$ is a unique graph which minimizes the number of $(A, B, 2C)$-edge colourings in trees of given order $n$.

From previous theorems we have also the following.

Corollary 12. If $T_n$ is a tree with the number of vertices $n \geq 2$, then

$$P_{n-1} \leq \delta(T_n) \leq s_{n-1}.$$  \hspace{1cm} (31)

Moreover, if $T_n$ is different from $P_n$ and $K_{n-1}$, then

$$P_n < \delta(T_n) < s_{n-1}.$$  \hspace{1cm} (32)

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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