Research Article

On a System of Equations of a Non-Newtonian Micropolar Fluid

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We investigate a problem for a model of a non-Newtonian micropolar fluid coupled system. The problem has been considered in a bounded, smooth domain of $\mathbb{R}^3$ with Dirichlet boundary conditions. The operator stress tensor is given by $\tau(e(u)) = [(\nu + \nu_0 M(|e(u)|^2))e(u)]$. To prove the existence of weak solutions we use the method of Faedo-Galerkin and compactness arguments. Uniqueness and periodicity of solutions are also considered.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0, T)$ is a time interval. The unsteady flows of incompressible fluids in a boundary domain $\Omega \subset \mathbb{R}^d$, $d > 1$, are described by the system of equations

$$\begin{aligned} \rho \frac{\partial u}{\partial t} - \nabla \cdot \tau(e(u)) + \rho(u \cdot \nabla)u &= -\nabla p + \rho f \quad \text{in } Q_T, \\ \nabla \cdot u &= 0 \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \Sigma_T, \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned} \quad (1)$$

where $u = (u_1, u_2, \dots, u_d)$ is the velocity, p represents the pressure, ρ is a positive constant determining the density of a material, $f = (f_1, f_2, \dots, f_d)$ stands for the given external body forces, $\tau : \mathbb{R}_{\text{sym}}^{d^2} \rightarrow \mathbb{R}_{\text{sym}}^{d^2}$ denotes the extra stress tensor, $e : \mathbb{R}^d \rightarrow \mathbb{R}_{\text{sym}}^{d^2}$ denotes the symmetric part of the velocity gradient; that is,

$$e(u) = \frac{1}{2} [\nabla u + (\nabla u)^T], \quad (2)$$

whose components are defined as in [1] by

$$2e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, 2, \dots, d \quad (3)$$

and $\mathbb{R}_{\text{sym}}^{d^2}$ represents the set of all symmetric $d \times d$ matrices; that is,

$$\mathbb{R}_{\text{sym}}^{d^2} = \{D \in \mathbb{R}^{d^2}; D_{ij} = D_{ji}, i, j = 1, 2, \dots, d\}. \quad (4)$$

Note, for example, that when $\tau(e(u))$ is of the form

$$\tau(e(u)) = \mu_0 (1 + |e(u)|^{p-2}) e(u), \quad (5)$$

with $p = 2$, problem (1) turns into the Navier-Stokes system, which is a model for Newtonian fluids. In the expression (5), $|e(u)|$ denotes the usual Euclidean matrix norm. We observe that (5) can be written in the form

$$\tau(e(u)) = \mu_0 M(|e(u)|^2) e(u), \quad (6)$$

where $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, $M \in C^0(0, \infty)$ is the generalized viscosity function. Fluids constituted by (6) are sometimes named fluids with shear-dependent viscosity. Models belonging to this class of non-Newtonian fluid mechanics are frequently used in several fields of chemistry, glaciology, biology, and geology, as discussed by Malek et al. [2].

The first mathematical investigations of problem (1) was done by Ladyzhenskaya in 1963, where she proposed to study system (1) with (5) and $p = 4$. Combining monotone operator theory and compactness arguments, she proved the existence of weak solution to model (1), if $p \geq 1 + (2d/(d+2))$, and their uniqueness if $p \geq (d+2)/2$. See also Lions [3] for another proof of the same results. More results are known about problem (1) obtained in a series of papers, including those of Malek et al. [2], Malek et al. [4], Frehse and Málek [5], Malek et al. [1], and other mathematicians.

The equations below describe the motion of Newtonian micropolar fluids:

$$\begin{aligned} \frac{\partial u}{\partial t} - (\nu + \nu_r) \Delta u + (u \cdot \nabla) u + \nabla p &= 2\nu_r \nabla \times w + f \quad \text{in } Q_T, \\ \frac{\partial w}{\partial t} - (c_a + c_d) \Delta w + (u \cdot \nabla) w - (c_o + c_d - c_a) \nabla (\nabla \cdot w) \\ &+ \lambda w = 2\nu_r \nabla \times u + g \quad \text{in } Q_T, \\ \nabla \cdot u &= 0 \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \Sigma_T, \\ w &= 0 \quad \text{on } \Sigma_T, \\ u(0) &= u_0 \quad \text{in } \Omega, \\ w(0) &= w_0 \quad \text{in } \Omega, \end{aligned} \quad (7)$$

where $u(x, t), w(x, t) \in \mathbb{R}^3$ and $p(x, t) \in \mathbb{R}$, denoting for $(x, t) \in Q$, respectively, the unknown velocity, the micro-rotational velocity, and the hydrostatic pressure of the fluid and λ is a positive constant. The positive constants ν and ν_r are, respectively, the *Newtonian* and *microrotational viscosity*. The positive constants c_0 , c_a , and c_d are called *coefficients of angular viscosities* and satisfy $c_0 + c_d > c_a$.

The main difference with respect to modeled fluids by the Navier-Stokes is that the rotation of the particles is taken into account. The above approach was introduced by Eringen [6]. The nonlinear coupled system (7) can be used to model the behavior of liquid crystals, polymeric fluids, and blood under some circumstances (see, e.g., [7]). These systems have been mainly analyzed in the book of Lukaszewicz [8].

The problem that we study in this work consists in supposing that in system (7) the fluid is of the type (5). More precisely, we investigate the mixed problem: let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0, T)$ is a time interval. We find that $u, w : Q_T \rightarrow \mathbb{R}^3$ and $p : Q_T \rightarrow \mathbb{R}$ solving the following system of equations:

$$\begin{aligned} u' - \nabla \cdot \tau(e(u)) + (u \cdot \nabla) u + \nabla p &= \nabla \times w + f \quad \text{in } Q_T, \\ w' - \nu_1 \nabla \cdot e(w) + (u \cdot \nabla) w + \lambda_1 w &= \lambda_2 \nabla \times u + g \quad \text{in } Q_T, \\ \nabla \cdot u &= 0 \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \Sigma_T, \end{aligned}$$

$$\begin{aligned} w &= 0 \quad \text{on } \Sigma_T, \\ u(0) &= u_0 \quad \text{in } \Omega, \\ w(0) &= w_0 \quad \text{in } \Omega, \end{aligned} \quad (8)$$

where the extra stress tensor is given by $\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u)$, $e(u)$ as in (2) and (3), ν_0 , ν_1 , λ_1 , and λ_2 are positive constants, $u = (u_1, u_2, u_3)$, and $\nabla \times u$ is given by

$$\nabla \times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right); \quad (9)$$

the same holds for $\nabla \times w$. Let us consider $M : (0, \infty) \rightarrow (0, \infty)$ satisfying the hypothesis

$$M \in C^1(0, \infty), \quad M > M_0 > 0, \quad M' > 0, \quad (10)$$

$$c_1 |e(u)|^2 \leq M(|e(u)|^2) \leq c_2 |e(u)|^2, \quad (11)$$

where M_0 , c_1 , and c_2 are positive constants. We observe that if M is a constant function, then problem (8) reduces to problem (7).

2. Notation and Main Results

In order to solve problem (8) we need some notations about Sobolev spaces. We use standard notation of $L^p(\Omega)$, $W^{m,p}(\Omega)$, and $C^p(\Omega)$ for functions that are defined on Ω and range in $\mathbb{R}$ and the notation of $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$, and $\mathbf{C}^p(\Omega)$ for functions that range in $\mathbb{R}^d$. We also work with the spaces $L^p(I; W^{m,p}(\Omega))$ or $L^p(Q_T)$.

By $\langle \cdot, \cdot \rangle$ we will represent the duality pairing between X and X' , with X' being the topological dual of the space X . We Also define the followings spaces:

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega) ; \nabla \cdot \varphi = 0\}. \quad (12)$$

$V_p = V_p(\Omega)$ is the closure of $\mathcal{V}$ in the space $\mathbf{W}^{1,p}(\Omega)$, $p \in (1, \infty)$. In particular, $V = V_2$. The norm of gradient in V_p is given by

$$\|\nabla u\|_p \equiv \left[\int_{\Omega} |\nabla u(x)|^p dx \right]^{1/p}. \quad (13)$$

The inner product and norm in V is given, respectively, by

$$((u, v)) = \sum_{i,j=1}^3 \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_j}{\partial x_i}(x) dx, \quad (14)$$

$$\|u\|^2 = \sum_{i,j=1}^3 \int_{\Omega} \left(\frac{\partial u_i}{\partial x_j}(x) \right)^2 dx.$$

$H = H(\Omega)$ is the closure of $\mathcal{V}$ in the space $\mathbf{L}^2(\Omega)$, with inner product and norm defined, respectively, by

$$\begin{aligned} (u, v) &= \sum_{i=1}^3 \int_{\Omega} u_i(x) v_i(x) dx, \\ |u|^2 &= \sum_{i=1}^3 \int_{\Omega} |u_i(x)|^2 dx. \end{aligned} \quad (15)$$

Remark 1. $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ are Hilbert's spaces. We note that $\mathbf{H}_0^1(\Omega) \xhookrightarrow{c} \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$, where the first embeddings are compact.

We introduce the following bilinear and the trilinear forms, as well as the convention of summation of indices, that is, $\alpha_i \beta_j$ instead of $\sum_{i,j=1}^d \alpha_i \beta_j$:

$$a(u, v) = \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x) dx = ((u, v)), \quad \forall u, v \in V, \quad (16)$$

$$b(u, v, w) = \int_{\Omega} u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x) dx, \quad \forall u, v, w \in V. \quad (17)$$

We note that (see Lions [3])

$$b(u, v, w) = -b(u, w, v), \quad \forall u \in V, \quad \forall v, w \in \mathbf{H}_0^1(\Omega). \quad (18)$$

We also introduce the notations

$$Au = -\Delta u, \quad B_u v = (u \cdot \nabla) v, \quad \forall u, v \in V, \quad (19)$$

$$\mathcal{K}u = -\nabla \cdot M(|e(u)|^2) e(u), \quad \forall u \in V. \quad (20)$$

According to this, we have

$$\langle Au, v \rangle = a(u, v), \quad \forall u, v \in V, \quad (21)$$

$$\langle B_u v, w \rangle = b(u, v, w), \quad \forall u \in V, \quad \forall v, w \in \mathbf{H}_0^1(\Omega), \quad (22)$$

$$\langle \mathcal{K}u, v \rangle = \int_{\Omega} M(|e(u)|^2) e_{ij}(u) e_{ij}(v) dx, \quad \forall u, v \in V. \quad (23)$$

Remark 2. We observe that $M' > 0$ implies for all $u_1, u_2 \in V$ that

$$\begin{aligned} & \langle \mathcal{K}u_1 - \mathcal{K}u_2, u_1 - u_2 \rangle \\ &= \int_{\Omega} [M(|e(u_1)|^2) e_{ij}(u_1) - M(|e(u_2)|^2) e_{ij}(u_2)] \\ & \quad \times [e_{ij}(u_1) - e_{ij}(u_2)] dx \geq 0. \end{aligned} \quad (24)$$

Therefore $\mathcal{K} : V \rightarrow V'$ is a monotonous operator.

Definition 3. Let $u_0 \in H, w_0 \in \mathbf{L}^2(\Omega), f \in L^{4/3}(I, V')$, and $g \in L^2(I; \mathbf{H}^{-1}(\Omega))$. A weak solution to (8) is a pair of functions $\{u, w\}$, such that

$$\begin{aligned} u &\in L^\infty(I; H) \cap L^4(I; V_4) \cap L^4(I; V), \\ w &\in L^\infty(I; \mathbf{L}^2(\Omega)) \cap L^2(I; \mathbf{H}_0^1(\Omega)) \end{aligned} \quad (25)$$

satisfying the following identity:

$$\begin{aligned} & \int_0^T \langle u'(t), \varphi \rangle dt + \nu \int_0^T a(u(t), \varphi) dt \\ & + \int_0^T b(u(t), u(t), \varphi) dt \\ & + \nu_0 \int_0^T \int_{\Omega} M(|e(u(t))|^2) e_{ij}(u(t)) e_{ij}(\varphi) dx dt \\ & = \int_0^T (\nabla \times w(t), \varphi) dt \\ & + \int_0^T (f(t), \varphi) dt, \quad \forall \varphi \in \mathcal{D}(I; \mathcal{V}), \\ & \int_0^T \langle w'(t), \phi \rangle dt + \nu_1 \int_0^T a(w(t), \phi) dt \\ & + \nu_1 \int_0^T (\nabla \cdot w(t), \nabla \cdot \phi) dt \\ & + \int_0^T b(u(t), w(t), \phi) dt + \lambda_1 \int_0^T (w(t), \phi) dt \\ & = \lambda_2 \int_0^T (\nabla \times u(t), \phi) dt \\ & + \int_0^T (g(t), \phi) dt, \quad \forall \phi \in \mathcal{D}(I; \mathcal{D}(\Omega)), \\ & u(0) = u_0, \quad w(0) = w_0. \end{aligned} \quad (26)$$

Lemma 4 (Korn's inequality). *Let $1 < p < \infty$. Then, there exists a constant $K_p = K_p(\Omega)$, such that the inequality*

$$K_p \|v\|_{W^{1,p}(\Omega)} \leq \|e(v)\|_{L^p(\Omega)} \quad (27)$$

is fulfilled for all v satisfying either $v \in W_0^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open and bounded with $\partial\Omega \subset C^1$.

Proof. The proof of this lemma can be found in [1], page 169. $\square$

Lemma 5. *Let $p \geq 2, \tau : \mathbb{R}_{sym}^{d^2} \rightarrow \mathbb{R}_{sym}^{d^2}$, and $\Phi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ and the assumptions below are satisfied for all $B, D \in \mathbb{R}_{sym}^{d^2}$ and $i, j, k, l = 1, \dots, d$:*

$$\begin{aligned} & \partial_{ij} \Phi(|D|^2) = \tau_{ij}(D), \\ & \Phi(0) = \partial_{ij} \Phi(0) = 0, \end{aligned} \quad (28)$$

$$\partial_{ij} \partial_{kl} \Phi(|D|^2) B_{ij} B_{kl} \geq C_1 (1 + |D|)^{p-2} |B|^2,$$

$$|\partial_{ij} \partial_{kl} \Phi(|D|^2)| \leq C_2 (1 + |D|)^{p-2}.$$

Then, there exist positive constants $C_i, i = 3, 4, 5$, such that

$$C_3 (1 + |D|^{p-2}) |D|^2 \leq \Phi(|D|^2) \leq C_4 (1 + |D|)^p, \quad (29)$$

$$(\tau(B) - \tau(D)) \cdot (B - D) \geq C_5 |B - D|^2. \quad (30)$$

Proof. The proof of this lemma can be found in [4], page 263. $\square$

Lemma 6 (Vitali). *Let Ω be a bounded domain in $\mathbb{R}^n$ and $f^m : \Omega \rightarrow \mathbb{R}$ integrable for every $m \in \mathbb{N}$. Assume that*

- (1) $\lim_{m \rightarrow \infty} f^m(x)$ exists and is finite for almost all $x \in \Omega$;
- (2) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{m \in \mathbb{N}} \int_H |f^m(x)| dx < \varepsilon, \quad \forall H \in \Omega, \quad |H| < \delta; \quad (31)$$

then

$$\lim_{m \rightarrow \infty} \int_{\Omega} f^m(x) dx = \int_{\Omega} \lim_{m \rightarrow \infty} f^m(x) dx. \quad (32)$$

Proof. The proof of this lemma can be found in [9], page 63. $\square$

Lemma 7. *Consider $d \geq 3$ and $s, r \in \mathbb{R}$, with $s > 2$, $r > d$, verifying $(2/s) + (d/r) = 1$. If $u \in L^r(\Omega)$, then*

$$|b(u, v, w)| \leq c \|u\|_{L^r(\Omega)} \|v\| \|w\|^{2/s} \|w\|^{d/r} \quad (33)$$

for all $v, w \in V$, where $c \geq 0$ is a constant independent of u, v , and w .

Proof. The proof of this lemma can be found in [3], page 84. $\square$

Theorem 8. *If $d \leq 3$, $u_0 \in H$, $w_0 \in L^2(\Omega)$, $f \in L^{4/3}(I; V')$, and $g \in L^2(I; \mathbf{H}^{-1}(\Omega))$, then there exist a weak solution to problem (8).*

Theorem 9. *Under the assumptions of Theorem 8 with $d = 2$, problem (8) has a unique weak solution.*

Theorem 10 (periodic solutions). *Under the assumptions of Theorem 8 there exist a pair of functions (u, w) such that*

$$\begin{aligned} u &\in L^\infty(I; H) \cap L^4(I; V_4) \cap L^4(I; V), \\ w &\in L^\infty(I; L^2(\Omega)) \cap L^2(I; \mathbf{H}_0^1(\Omega)), \\ (u'(t), \varphi) + \nu a(u(t), \varphi) + \nu_0(\mathcal{K}u(t), \varphi) + (B_u u(t), \varphi) \\ &= (\nabla \times w(t), \varphi) + (f(t), \varphi), \\ (w'(t), \phi) + \nu_1 a(w(t), \phi) + \nu_1(\nabla \cdot w(t), \nabla \cdot \phi) \\ &\quad + (B_u w(t), \phi) + \lambda_1(w(t), \phi) \\ &= \lambda_2(\nabla \times u(t), \phi) + (g(t), \phi), \\ u(0) &= u(T), \quad w(0) = w(T), \\ \forall \varphi \in V, \quad \forall \phi \in \mathbf{H}_0^1(\Omega) \text{ in the sense of } D'(0, T). \end{aligned} \quad (34)$$

Theorem 11. *Assuming that $d \leq 3$, $u_0 \in V \cap V_4$, $w_0 \in \mathbf{H}_0^1(\Omega)$, $f \in L^{4/3}(I; V')$, and $g \in L^2(I; \mathbf{H}^{-1}(\Omega))$ there exist a unique weak solution to problem (8) such that*

$$u' \in L^2(I; H),$$

$$u \in L^\infty(I; V_4),$$

$$u \in L^\infty(I; V), \quad (35)$$

$$w \in L^\infty(I; \mathbf{H}_0^1(\Omega)),$$

$$w' \in L^2(I; L^2(\Omega)).$$

3. Proofs of the Results

Proof of Theorem 8. We will show the existence of a weak solution to system (8) employing the Galerkin approximations. For that purpose we consider $(\varphi_\nu)_{\nu \in \mathbb{N}} \subset V$, a basis of eigenvectors of the Stokes operator and $(\phi_\mu)_{\mu \in \mathbb{N}} \subset \mathbf{H}_0^1(\Omega)$ a basis of eigenvectors of Lamé. We represent by $V_m = [\varphi_1, \dots, \varphi_m] \subset V$ the subspace generated by $\{\varphi_1, \dots, \varphi_m\}$ and $W_m = [\phi_1, \dots, \phi_m] \subset \mathbf{H}_0^1(\Omega)$ the subspace generated by $\{\phi_1, \dots, \phi_m\}$. Let us also consider the pair (u_m, w_m) , such that

$$u_m(x, t) = \sum_{r=1}^m g_{rm}(t) \varphi_r(x), \quad (36)$$

$$w_m(x, t) = \sum_{r=1}^m h_{rm}(t) \phi_r(x)$$

are the solution of the approximate problem

$$\begin{aligned} (u'_m(t), \varphi_r) + \nu(Au_m(t), \varphi_r) + \nu_0(\mathcal{K}u_m(t), \varphi_r) \\ + \langle B_{u_m} u_m(t), \varphi_r \rangle = (\nabla \times w_m(t), \varphi_r) + (f(t), \varphi_r), \\ r = 1, \dots, m, \\ (w'_m(t), \phi_r) - \nu_1(\nabla \cdot e(w), \phi_r) + \langle B_{u_m} w_m(t), \phi_r \rangle \\ + \lambda_1(w_m(t), \phi_r) = \lambda_2(\nabla \times u_m(t), \phi_r) + (g(t), \phi_r), \\ r = 1, \dots, m, \end{aligned}$$

$$u_m(0) = u_{0m} \longrightarrow u_0, \quad \text{strongly in } H,$$

$$w_m(0) = w_{0m} \longrightarrow w_0, \quad \text{strongly in } L^2(\Omega). \quad (37)$$

The system of ordinary differential equations (37) has a local solution on an interval $[0, t_m[$, $0 < t_m < T$. The first estimate permits us to extend this solution to the whole interval $[0, T]$.

First Estimate. We sometimes omit the parameter t . Multiplying both sides of (37)₁ by g_{rm} and (37)₂ by h_{rm} , next adding from $r = 1$ to $r = m$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + \nu \|u_m(t)\|^2 \\ & + \nu_0 \int_{\Omega} M(|e(u_m(t))|^2) |e_{ij}(u_m(t))|^2 dx \end{aligned} \quad (38)$$

$$\begin{aligned} & \leq |w_m(t)| \|u_m(t)\| + \|f(t)\|_{V'} \|u_m(t)\|, \\ & \frac{1}{2} \frac{d}{dt} |w_m(t)|^2 + \nu_1 \|w_m(t)\|^2 \\ & + \nu_1 |\nabla \cdot w_m(t)|^2 + \lambda_1 |w_m(t)|^2 \end{aligned} \quad (39)$$

$$\leq \lambda_2 \|u_m(t)\| |w_m(t)| + \|g(t)\|_{H^{-1}(\Omega)} \|w_m(t)\|,$$

because $b(u, u, u) = b(u, w, w) = 0$, for all $u \in V$, for all $w \in \mathbf{H}_0^1(\Omega)$ (see Lions [3]), $|\nabla \times u_m| = |\nabla u_m| = \|u_m\|$ and $(\nabla \times w_m, u_m) = (w_m, \nabla \times u_m)$ (see Łukaszewicz [8]). Now using Young's inequality we obtain from (38) and (39), respectively:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + \nu \|u_m(t)\|^2 \\ & + \frac{\nu_0 c_1 K}{2} \|u_m(t)\|_{V_4}^4 + \nu_2 \|u_m(t)\|^4 \end{aligned} \quad (40)$$

$$\begin{aligned} & \leq \frac{\nu}{4} \|u_m(t)\|^2 + c_\nu |w_m(t)|^2 \\ & + \frac{\nu_2}{2} \|u_m(t)\|^4 + c_{\nu_2} \|f(t)\|_{V'}^{4/3}, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |w_m(t)|^2 + \nu_1 \|w_m(t)\|^2 \\ & \leq \frac{\nu}{4} \|u_m(t)\|^2 + c_\nu |w_m(t)|^2 \end{aligned} \quad (41)$$

$$+ \frac{\nu_1}{2} \|w_m(t)\|^2 + c_{\nu_1} \|g(t)\|_{H^{-1}(\Omega)}^2.$$

From (27) (Korn's inequality) and (11) we can get

$$\begin{aligned} & \frac{\nu_0}{2} \int_{\Omega} M(|e(u_m)|^2) |e_{ij}(u_m)|^2 dx \geq \frac{\nu_0 c_1}{2} \|e(u_m)\|_{L^4(\Omega)}^4 \\ & \geq \frac{\nu_0 c_1 K}{2} \|u_m\|_{V_4}^4, \end{aligned} \quad (42)$$

$$\begin{aligned} & \frac{\nu_0}{2} \int_{\Omega} M(|e(u_m)|^2) |e_{ij}(u_m)|^2 dx \geq \frac{\nu_0 c_1}{2} \|e(u_m)\|_{L^4(\Omega)}^4 \\ & \geq \nu_2 \|u_m\|^4. \end{aligned}$$

Adding inequalities (40) and (41) and integrating from 0 to t , with $0 \leq t \leq T$, we conclude

$$\begin{aligned} & (|u_m(t)|^2 + |w_m(t)|^2) + \nu_0 c_1 K \int_0^t \|u_m(s)\|_{V_4}^4 ds \\ & + \nu_2 \int_0^t \|u_m(s)\|^4 ds + \nu_1 \int_0^t \|w_m(s)\|^2 ds \end{aligned} \quad (43)$$

$$\leq C + C \int_0^t (|u_m(s)|^2 + |w_m(s)|^2) ds.$$

By using Gronwall's inequality, we can write

$$|u_m(t)|^2 + |w_m(t)|^2 \leq C. \quad (44)$$

Therefore, it follows from (43) that

$$(u_m) \text{ is bounded in } L^\infty(I; H), \quad (45)$$

$$(u_m) \text{ is bounded in } L^4(I; V_4), \quad (46)$$

$$(u_m) \text{ is bounded in } L^4(I; V), \quad (47)$$

$$(w_m) \text{ is bounded in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (48)$$

$$(w_m) \text{ is bounded in } L^2(I; \mathbf{H}_0^1(\Omega)). \quad (49)$$

Second Estimate. We consider $P_m : V \rightarrow V_m$ as the orthogonal projections from V to V_m :

$$P_m u = \sum_{j=1}^m (u, \varphi_j) \varphi_j, \quad \forall u(t) \in V. \quad (50)$$

We also consider the adjoint operator to P_m which is $P_m^* : V' \rightarrow V'$. We note that $P_m^* u'_m = u'_m$. By the choice of the special basis (φ_j) , we obtain

$$\|P_m\|_{\mathcal{L}(V, V)} \leq 1, \quad \|P_m^*\|_{\mathcal{L}(V', V')} \leq 1. \quad (51)$$

It follows from (37)₁, (21), (22), and (23) that

$$u'_m = -\nu P_m^* A u_m - \nu_0 P_m^* \mathcal{K} u_m - P_m^* B u_m + P_m^* \nabla \times w_m + P_m^* f. \quad (52)$$

We have $|\langle A u_m, v \rangle| \leq \|u_m\| \|v\|$, for all $u_m(t), v(t) \in V$. Therefore (47) implies

$$(A u_m) \text{ is bounded in } L^4(I; V') \hookrightarrow L^{4/3}(I; V'). \quad (53)$$

Let $u_m(t), v(t) \in V$. From (20), Hölder's inequality, and (11) we take

$$\begin{aligned} |\langle \mathcal{K} u_m, v \rangle| & \leq \left| \langle M(|e(u_m)|^2) e(u_m), \nabla v \rangle \right| \leq c |e(u_m)|^3 \|v\| \\ & \leq c |\nabla u_m|^3 \|v\| \leq \|u_m\|^3 \|v\|. \end{aligned} \quad (54)$$

Therefore, from (47), we obtain

$$(\mathcal{K} u_m) \text{ is bounded in } L^{4/3}(I; V'). \quad (55)$$

From $d \leq 3$ we derive $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Using (17) and Hölder's inequality we conclude

$$|\langle B u_m u_m, v \rangle| \leq \|u_m\|_{L^4(\Omega)} \|u_m\| \|v\|_{L^4(\Omega)} \leq c \|u_m\|^2 \|v\| \quad (56)$$

for all $u_m(t), v(t) \in V$. Therefore, from (47)

$$(B u_m u_m) \text{ is bounded in } L^2(I; V') \hookrightarrow L^{4/3}(I; V'). \quad (57)$$

On the other hand, let $w_m(t) \in \mathbf{H}_0^1(\Omega)$, and $v(t) \in V$

$$|\langle \nabla \times w_m, v \rangle| = |\langle w_m, \nabla \times v \rangle| \leq |w_m| \|v\| \leq c \|w_m\| \|v\| \quad (58)$$

(see Lukaszewicz [8], pp. 116). It follows from (49) that

$$(\nabla \times w_m) \text{ is bounded in } L^2(I; V') \hookrightarrow L^{4/3}(I; V'). \quad (59)$$

It follows from (53)–(59), (51), and hypothesis about f that

$$(u'_m) \text{ is bounded in } L^{4/3}(I; V'). \quad (60)$$

Analogously let $R_m : \mathbf{H}_0^1(\Omega) \rightarrow W_m$ be the orthogonal projections

$$R_m w = \sum_{j=1}^m (w, \phi_j) \phi_j, \quad \forall w \in \mathbf{H}_0^1(\Omega). \quad (61)$$

We also consider the adjoint operator to R_m , which is $R_m^* : \mathbf{H}^{-1}(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$. We have $R_m^* w'_m = w'_m$ and by the choice of the special basis (ϕ_j) , we can get

$$\|R_m\|_{\mathcal{L}(H_0^1(\Omega), H_0^1(\Omega))} \leq 1, \quad \|R_m^*\|_{\mathcal{L}(H^{-1}(\Omega), H^{-1}(\Omega))} \leq 1. \quad (62)$$

From (37)₂, (21), (22), and (23)

$$\begin{aligned} w'_m = & -\gamma_1 R_m^* \nabla \cdot e(w_m) - R_m^* B w_m - \lambda_1 R_m^* w_m \\ & + \lambda_2 R_m^* \nabla \times u_m + R_m^* g. \end{aligned} \quad (63)$$

We note that $|\langle \nabla \cdot e(w_m), v \rangle| = |\langle \nabla w_m, \nabla v \rangle| \leq \|w_m\| \|v\|$, for all $w_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Thus, (49) implies

$$(\nabla \cdot e(w_m)) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)). \quad (64)$$

Analogously and by using the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$(w_m) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)). \quad (65)$$

On the other hand $|\langle \nabla \times u_m, v \rangle| \leq \|u_m\| \|v\|$, for all $u_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Now, by using (47), we have

$$(\nabla \times u_m) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)). \quad (66)$$

Finally assuming that $d = 2$, it follows from (18), (22), and Hölder's inequality that

$$\begin{aligned} |\langle B_{u_m} w_m, v \rangle| &= |b(u_m, w_m, v)| = |b(u_m, v, w_m)| \\ &\leq c \|u_m\|_{L^4(\Omega)} \|v\| \|w_m\|_{L^4(\Omega)} \end{aligned} \quad (67)$$

for all $u_m(t), w_m(t), v(t) \in \mathbf{H}_0^1(\Omega)$. Therefore, $\|B w_m\|_{H^{-1}(\Omega)}^2 \leq c \|u_m\| \|u_m\| \|w_m\| \|w_m\|$.

Because if $d = 2$, then, $\|u\|_{L^4(\Omega)}^2 \leq c \|u\| \|u\|$ (see Lions [3]). Now using Young's inequality we get

$$\|B_{u_m} w_m\|_{H^{-1}(\Omega)}^2 \leq c |u_m|^2 \|u_m\|^2 + c |w_m|^2 \|w_m\|^2. \quad (68)$$

Therefore, (47)–(49) and (45) permit us to obtain

$$(B_{u_m} w_m) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)), \quad d = 2. \quad (69)$$

Analogously and assuming that $d = 3$ we obtain from Lemma 7

$$\|B_{u_m} w_m\|_{H^{-1}(\Omega)} \leq \|u_m\|_{L^6(\Omega)} |w_m|^{1/2} \|w_m\|^{1/2}. \quad (70)$$

$H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, because $d = 3$. Thus,

$$\begin{aligned} \|B_{u_m} w_m\|_{H^{-1}(\Omega)}^2 &\leq c \|u_m\|^2 |w_m| \|w_m\| \\ &\leq c \|u_m\|^4 + c |w_m|^2 \|w_m\|^2. \end{aligned} \quad (71)$$

Therefore, (47)–(49) imply that

$$(B_{u_m} w_m) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)), \quad d = 3. \quad (72)$$

It follows from (64)–(72) and hypothesis about g that

$$(w'_m) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)). \quad (73)$$

We note that (45)–(49), (60), (73), and the Aubin-Lions lemma imply that there exist subsequences of (u_m) and (w_m) , still denoted by (u_m) and (w_m) , such that

$$u_m \rightarrow u \text{ strongly in } L^2(I; H), \quad \text{a.e. in } Q_T, \quad (74)$$

$$u_m \rightarrow u \text{ weak star in } L^\infty(I; H), \quad (75)$$

$$u_m \rightarrow u \text{ weakly in } L^4(I; V), \quad (76)$$

$$u'_m \rightarrow u' \text{ weakly in } L^{4/3}(I; V'), \quad (77)$$

$$w_m \rightarrow w \text{ strongly in } L^2(I; L^2(\Omega)), \quad \text{a.e. in } Q_T, \quad (78)$$

$$w_m \rightarrow w \text{ weak star in } L^\infty(I; L^2(\Omega)), \quad (79)$$

$$w_m \rightarrow w \text{ weakly in } L^2(I; \mathbf{H}_0^1(\Omega)), \quad (80)$$

$$w'_m \rightarrow w' \text{ weakly in } L^2(I; H^{-1}(\Omega)), \quad (81)$$

$$\mathcal{K} u_m \rightarrow \chi \text{ weakly in } L^{4/3}(I; V'). \quad (82)$$

We note that (46) and (60) imply that $u \in C^0(I; \mathbf{H})$. Similarly, (49) and (73) imply that $w \in C^0(I; L^2(\Omega))$. Thus, it does make sense to consider $u(0) = u_0$ and $w(0) = w_0$.

In order to prove that

$$\begin{aligned} \int_0^T b(u_m, u_m, \varphi) dt &\rightarrow \int_0^T b(u, u, \varphi) dt, \quad \forall \varphi \in \mathcal{D}(I; \mathcal{V}), \\ \int_0^T b(u_m, w_m, \phi) dt &\rightarrow \int_0^T b(u, w, \phi) dt, \quad \forall \phi \in \mathcal{D}(I; \mathcal{D}(\Omega)), \end{aligned} \quad (83)$$

we use (74) and (78) (see [1], pp. 210). Now, we note that

$$\int_{Q_T} e_{ij}(w_m) e_{ij}(\phi) dx dt \rightarrow \int_{Q_T} e_{ij}(w) e_{ij}(\phi) dx dt, \quad (84)$$

or equivalently

$$\int_0^T \langle \nabla \cdot e(w_m), \phi \rangle dt \longrightarrow \int_0^T \langle \nabla \cdot e(w), \phi \rangle dt \quad (85)$$

results from (80). The other terms of (37)₂ are obtained in the usual manner. In order to prove that

$$\begin{aligned} & \int_{Q_T} M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) dx dt \\ & \longrightarrow \int_{Q_T} M(|e(u)|^2) e_{ij}(u) e_{ij}(\varphi) dx dt, \end{aligned} \quad (86)$$

we use the fact $\nabla u_m \rightarrow \nabla u$ a.e. in Q_T , (see [5] pp. 565-566). Therefore,

$$|\nabla u_m|^2 \longrightarrow |\nabla u|^2 \quad \text{a.e. in } Q_T; \quad (87)$$

that is,

$$|e(u_m)|^2 \longrightarrow |e(u)|^2 \quad \text{a.e. in } Q_T. \quad (88)$$

Since $M \in C^1(0, \infty)$ we obtain from (88)

$$M(|e(u_m)|^2) \longrightarrow M(|e(u)|^2) \quad \text{a.e. in } Q_T. \quad (89)$$

Thus,

$$M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) \longrightarrow M(|e(u)|^2) e_{ij}(u) e_{ij}(\varphi) \quad (90)$$

a.e. in Q_T and for all $\varphi \in \mathcal{D}(I; \mathcal{D}(\Omega))$. Using (46) and (11) we obtain

$$\begin{aligned} & \int_{Q_T} M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) dx dt \\ & \leq C \int_{Q_T} |e(u_m)|^3 |e_{ij}(\varphi)| dx dt \\ & \leq C \int_{Q_T} |e(u_m)|^3 dx dt \leq C \int_{Q_T} |\nabla u_m|^3 dx dt \leq C. \end{aligned} \quad (91)$$

It follows that

$$M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) \in L^1(Q_T). \quad (92)$$

Moreover, if $H \subset Q_T$ is a measurable set, we have from (11), (46), and Hölder's inequality that

$$\begin{aligned} & \int_H M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) dx dt \\ & \leq c \int_H |e(u_m)|^3 |e(\varphi)| dx dt \\ & \leq c \left(\int_{Q_T} |e(u_m)|^4 dx dt \right)^{3/4} \left(\int_H |e(\varphi)|^4 dx dt \right)^{1/4} \\ & \leq c \left(\int_{Q_T} |\nabla u_m|^4 dx dt \right)^{3/4} |H|^{1/4} \leq c |H|^{1/4}. \end{aligned} \quad (93)$$

Therefore,

$$\sup_{m \in \mathbb{N}} \int_H M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) dx dt \leq c |H|^{1/4}. \quad (94)$$

Assuming that $|H|$ is sufficiently small, we obtain

$$\sup_{m \in \mathbb{N}} \int_H M(|e(u_m)|^2) e_{ij}(u_m) e_{ij}(\varphi) dx dt \leq \varepsilon, \quad (95)$$

for all $\varepsilon \in \mathbb{R}$. Now using (90), (92), (95), and Vitali's lemma we can derive (86). Therefore, we can write $\chi = \mathcal{K}u$ in $L^{4/3}(I; \mathbf{H}^{-1}(\Omega))$. The convergences (74)–(82) and (85) and (86) allow us to pass the limit on system (37), with φ_r and q_r being fixed to obtain

$$\begin{aligned} u' + \nu A u + \nu_0 \mathcal{K}u + B_u u &= \nabla \times w + f \quad \text{in } L^{4/3}(I; V'), \\ w' + \nu_1 \nabla \cdot e(w) + B_u w + \lambda_1 w \\ &= \lambda_2 \nabla \times u + g \quad \text{in } L^2(I; \mathbf{H}^{-1}(\Omega)). \end{aligned} \quad (96)$$

This concludes the proof of Theorem 8. $\square$

Proof of Theorem 9. Let (u_1, w_1) and (u_2, w_2) be weak solutions to Problem (8). Then,

$$\begin{aligned} u_1, u_2 &\in L^\infty(I; H) \cap L^4(I; V) \cap L^4(I; V_4), \\ w_1, w_2 &\in L^\infty(I; \mathbf{L}^2(\Omega)) \cap L^\infty(I; \mathbf{H}_0^1(\Omega)). \end{aligned} \quad (97)$$

Consider $\tilde{u} = u_1 - u_2$ and $\tilde{w} = w_1 - w_2$. Then, $(\tilde{u}, \tilde{w})$ satisfies

$$\begin{aligned} \tilde{u}' + \nu A \tilde{u} + \nu_0 (\mathcal{K}u_1 - \mathcal{K}u_2) + (B_{u_1} u_1 - B_{u_2} u_2) &= \nabla \times \tilde{w}, \\ \tilde{w}' + \nu_1 A \tilde{w} + \nu_1 \nabla \cdot \tilde{w} + (B_{u_1} w_1 - B_{u_2} w_2) \\ &+ \lambda_1 \tilde{w} = \lambda_2 \nabla \times \tilde{u}, \\ \tilde{u}(0) &= \tilde{w}(0) = 0, \end{aligned} \quad (98)$$

where the first equality has been considered in $L^{4/3}(I; V')$ and the second in $L^2(I; \mathbf{H}^{-1}(\Omega))$. We take the duality in (98)₁ and (98)₂ with $\tilde{u}$ and $\tilde{w}$, respectively, to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\tilde{u}|^2 + \nu \|\tilde{u}\|^2 + \nu_0 \langle \mathcal{K}u_1 - \mathcal{K}u_2, \tilde{u} \rangle \\ & + \langle B_{u_1} u_1 - B_{u_2} u_2, \tilde{u} \rangle = \langle \nabla \times \tilde{w}, \tilde{u} \rangle, \\ & \frac{1}{2} \frac{d}{dt} |\tilde{w}|^2 + \nu_1 \|\tilde{w}\|^2 + \nu_1 |\nabla \cdot \tilde{w}|^2 \\ & + \langle B_{u_1} w_1 - B_{u_2} w_2, \tilde{w} \rangle + \lambda_1 |\tilde{w}|^2 = \lambda_2 \langle \nabla \times \tilde{u}, \tilde{w} \rangle, \\ & \tilde{u}(0) = \tilde{w}(0) = 0. \end{aligned} \quad (99)$$

We note that

$$\begin{aligned} \langle B_{u_1} u_1(t) - B_{u_2} u_2(t), \tilde{u}(t) \rangle &= b(\tilde{u}(t), u_1(t), \tilde{u}(t)), \\ \langle B_{u_1} w_1(t) - B_{u_2} w_2(t), \tilde{w}(t) \rangle &= b(\tilde{u}(t), w_2(t), \tilde{w}(t)). \end{aligned} \quad (100)$$

From the monotonicity of $\mathcal{K}$ we have $\langle \mathcal{K}u_1 - \mathcal{K}u_2, \tilde{u} \rangle \geq 0$. Thus,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\tilde{u}|^2 + \nu \|\tilde{u}\|^2 + b(\tilde{u}(t), u_1(t), \tilde{u}(t)) &\leq \langle \nabla \times \tilde{w}, \tilde{u} \rangle, \\ \frac{1}{2} \frac{d}{dt} |\tilde{w}|^2 + \nu_1 \|\tilde{w}\|^2 + \lambda_1 |\tilde{w}|^2 + b(\tilde{u}(t), w_2(t), \tilde{w}(t)) \\ &\leq \lambda_2 \langle \nabla \times \tilde{u}, \tilde{w} \rangle. \end{aligned} \quad (101)$$

Adding the inequalities above, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\tilde{u}|^2 + |\tilde{w}|^2) + \nu \|\tilde{u}\|^2 + \nu_1 \|\tilde{w}\|^2 \\ \leq |\tilde{w}| \|\tilde{u}\| + \lambda_2 \|\tilde{u}\| |\tilde{w}| \\ + |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \tilde{w})|. \end{aligned} \quad (102)$$

In other words,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\tilde{u}|^2 + |\tilde{w}|^2) + \nu \|\tilde{u}\|^2 + \nu_1 \|\tilde{w}\|^2 \\ \leq \frac{\nu}{4} \|\tilde{u}\|^2 + \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu |\tilde{w}|^2 \\ + |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \tilde{w})|. \end{aligned} \quad (103)$$

Considering $d = 2$, we get $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Moreover (see Lions [3])

$$|u|_{L^4(\Omega)} \leq c |u|^{1/2} \|u\|^{1/2}. \quad (104)$$

Thus, using (17), Hölder's inequality, (104), and Young's inequality we take

$$\begin{aligned} |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \tilde{w})| \\ \leq \|\tilde{u}\|_{L^4(\Omega)}^2 \|u_1\| + \|\tilde{u}\|_{L^4(\Omega)} \|w_2\| \|\tilde{w}\|_{L^4(\Omega)} \\ \leq c \|u_1\| \|\tilde{u}\| \|\tilde{u}\| + \|w_2\| \|\tilde{u}\|^{1/2} \|\tilde{u}\|^{1/2} \|\tilde{w}\|^{1/2} \|\tilde{w}\|^{1/2} \\ \leq \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu \|u_1\|^2 |\tilde{u}|^2 \\ + \sqrt{\frac{\nu}{2}} \|\tilde{u}\| \sqrt{2\nu_1} \|\tilde{w}\| + c \|\tilde{u}\| \|\tilde{w}\| \|w_2\|^2 \\ \leq \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu \|u_1\|^2 |\tilde{u}|^2 + \frac{\nu}{4} \|\tilde{u}\|^2 + \nu_1 \|\tilde{w}\|^2 \\ + c \|w_2\|^2 |\tilde{u}|^2 + c \|w_2\|^2 |\tilde{w}|^2. \end{aligned} \quad (105)$$

It follows from (103) that we can write

$$\frac{1}{2} \frac{d}{dt} (|\tilde{u}|^2 + |\tilde{w}|^2) \leq c (1 + \|u_1\|^2 + \|w_2\|^2) (|\tilde{u}|^2 + |\tilde{w}|^2). \quad (106)$$

Integrating from 0 to t we obtain

$$\begin{aligned} |\tilde{u}(t)|^2 + |\tilde{w}(t)|^2 \\ \leq c \int_0^t (1 + \|u_1(s)\|^2 + \|w_2(s)\|^2) (|\tilde{u}(s)|^2 + |\tilde{w}(s)|^2) ds. \end{aligned} \quad (107)$$

Applying Gronwall's inequality in (107), we deduce by using (47) and (49) that

$$u_1(t) = u_2(t), \quad w_1(t) = w_2(t), \quad \forall t \in [0, T]. \quad (108)$$

Theorem 9 has been proved. $\square$

Proof of Theorem 10. Under the assumptions and notations defined in the proof of the Theorem 8 we know that the system (37) has a solution whatever initial value $(u_m(0), w_m(0)) \in V_m \times W_m$. To prove Theorem 10, we first show that there exist an approximate solution for (37), such that

$$(u_m(0), w_m(0)) = (u_m(T), w_m(T)). \quad (109)$$

For this purpose, let us take $\varphi = u_m$ and $\phi = w_m$ in (37) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_m|^2 + \nu \|u_m\|^2 + \nu_0 \int_\Omega M(|e(u_m)|^2) |e_{ij}(u_m)|^2 dx \\ \leq |w_m| \|u_m\| + \|f\|_{V'} \|u_m\|, \end{aligned} \quad (110)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |w_m|^2 + \nu_1 \int_\Omega |e_{ij}(w_m)|^2 dx + \lambda_1 |w_m|^2 \\ \leq \lambda_2 \|u_m\| |w_m| + |g| |w_m|, \end{aligned} \quad (111)$$

because $b(u, u, u) = 0$ (see Lions [3]), $|\nabla \times u_m| = |\nabla u_m| = \|u_m\|$, and $(\nabla \times w_m, u_m) = (w_m, \nabla \times u_m)$ (see Łukaszewicz [8]). Using (11) and (10) and (27) (Korn's inequality), we obtain from (110) and (111), respectively:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_m|^2 + \nu \|u_m\|^2 + \nu_0 M_0 |e(u_m)|^2 \\ \leq |w_m| \|u_m\| + \|f\|_{V'} \|u_m\|, \end{aligned} \quad (112)$$

$$\frac{1}{2} \frac{d}{dt} |w_m|^2 + \nu_1 K \|w_m\|^2 \leq \lambda_2 \|u_m\| |w_m| + |g| |w_m|. \quad (113)$$

After usual computations, we can derive

$$\begin{aligned} \frac{d}{dt} (|u_m|^2 + |w_m|^2) + \nu \|u_m\|^2 + \nu_1 K \|w_m\|^2 \\ \leq C (|u_m|^2 + |w_m|^2 + \|f\|_{V'}^2 + |g|^2). \end{aligned} \quad (114)$$

Considering the embeddings $V \hookrightarrow H$ and $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, there exists a constant c_2 such that

$$\begin{aligned} \frac{d}{dt} (|u_m|^2 + |w_m|^2) + c_2 |u_m|^2 + c_2 |w_m|^2 \\ \leq C (|u_m|^2 + |w_m|^2 + \|f\|_{V'}^2 + |g|^2). \end{aligned} \quad (115)$$

Multiplying by $e^{c_2 t}$ and integrating on $[0, t]$, we obtain

$$\begin{aligned} |u_m(t)|^2 + |w_m(t)|^2 &\leq e^{-c_2 t} (|u_m(0)|^2 + |w_m(0)|^2) \\ &+ C + C \int_0^t (|u_m(s)|^2 + |w_m(s)|^2) ds. \end{aligned} \quad (116)$$

By Gronwall's inequality, we can write

$$|u_m(t)|^2 + |w_m(t)|^2 \leq e^{-c_2 t} (|u_m(0)|^2 + |w_m(0)|^2) + C, \quad (117)$$

for all $t \in [0, T]$. Let $\theta(t) = e^{-c_2 t}$; we have $0 < \theta(t) < 1$. Thus,

$$|u_m(T)|^2 + |w_m(T)|^2 \leq \theta (|u_m(0)|^2 + |w_m(0)|^2) + C, \quad (118)$$

where $\theta = \theta(t)$ is a positive constant, such that $0 < 1 - \theta < 1$. Therefore, $C < C/(1 - \theta)$. Taking $R > 0$, such that $C/(1 - \theta) < R^2$, we obtain $C < (1 - \theta)R^2$. Choosing the initial data $(u_m(0), w_m(0)) \in V_m \times W_m$, such that

$$|u_m(0)|^2 < \frac{R^2}{2}, \quad |w_m(0)|^2 < \frac{R^2}{2}. \quad (119)$$

We obtain from (118)

$$\begin{aligned} |u_m(T)|^2 + |w_m(T)|^2 &\leq \theta (|u_m(0)|^2 + |w_m(0)|^2) \\ &+ C < \theta R^2 + (1 - \theta) R^2 = R^2. \end{aligned} \quad (120)$$

Therefore, $|u_m(0)|^2 + |w_m(0)|^2 < R^2$ implies that $|u_m(T)|^2 + |w_m(T)|^2 < R^2$.

Now we define $\sigma : \mathcal{B}_R(0) \cap (V_m \times W_m) \rightarrow \mathcal{B}_R(0) \cap (V_m \times W_m)$, such that

$$\sigma(u_m(0), w_m(0)) = (u_m(T), w_m(T)), \quad (121)$$

where $\mathcal{B}_R(0) = \{(u, w) \in H \times L^2(\Omega); |u|^2 + |w|^2 < R\}$. We note that σ is a continuous function because the solution of the (37) depends continuously of the initial data. We also note that (118) implies $\sigma(\mathcal{B}_R(0)) \subset \mathcal{B}_R(0)$. Therefore, it follows from Brower fixed-point theorem that σ has a fixed point:

$$(u_{0m}, w_{0m}) \in B_R(0) \subset V_m \times W_m. \quad (122)$$

In other words, $\sigma(u_{0m}, w_{0m}) = (u_{0m}, w_{0m})$. Taking the initial data (u_{0m}, w_{0m}) in (37), that is, $(u_m(0), w_m(0)) = (u_{0m}, w_{0m})$, we obtain

$$(u_m(0), w_m(0)) = (u_m(T), w_m(T)). \quad (123)$$

Therefore, (37) has a periodic solution. Next, we obtain estimates to (37) with the initial data (u_{0m}, w_{0m}) as in the proof of Theorem 8. We obtain

$$u_m \rightarrow u \text{ weakly in } L^4(I; V_4), \quad (124)$$

$$u'_m \rightarrow u' \text{ weakly in } L^{4/3}(I; V'), \quad (125)$$

$$w_m \rightarrow w \text{ weakly in } L^2(I; \mathbf{H}_0^1(\Omega)), \quad (126)$$

$$w'_m \rightarrow w' \text{ weakly in } L^2(I; \mathbf{H}^{-1}(\Omega)), \quad (127)$$

where u is the solution to problem (8) in the sense of Definition 3. The convergences (124) and (125) allow us to derive

$$\int_0^T \frac{d}{dt} (u_m(s), v) \theta(s) ds \rightarrow \int_0^T \frac{d}{dt} (u(s), v) \theta(s) ds, \quad (128)$$

for all $v \in V$ and $\theta \in \mathcal{D}(0, T)$, with $\theta(T) = 0$. In other words,

$$(u_m(0), v) \rightarrow (u(0), v), \quad \forall v \in V. \quad (129)$$

The same argument with $\theta \in \mathcal{D}(0, T)$ and $\theta(0) = 0$ allows us to derive

$$(u_m(T), v) \rightarrow (u(T), v), \quad \forall v \in V. \quad (130)$$

It follows from (129) and (130) that $u(0) = u(T)$. Analogously, from (126) and (127), we obtain

$$\begin{aligned} (w_m(0), \bar{v}) &\rightarrow (w(0), \bar{v}), \quad \forall \bar{v} \in \mathbf{L}^2(\Omega), \\ (w_m(T), \bar{v}) &\rightarrow (w(T), \bar{v}), \quad \forall \bar{v} \in \mathbf{L}^2(\Omega). \end{aligned} \quad (131)$$

Therefore, $w(0) = w(T)$. Theorem 10 has been proved. $\square$

Proof of Theorem 11. Let us consider $\tau : \mathbb{R}_{\text{sym}}^{d^2} \rightarrow \mathbb{R}_{\text{sym}}^{d^2}$ and the corresponding potential $\Phi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that

$$\begin{aligned} \tau(D) &= M(|D|^2)D, \quad \forall D \in \mathbb{R}_{\text{sym}}^{d^2}, \\ \Phi(|D|^2) &= \int_0^{|D|^2} M(s) ds, \quad \forall D \in \mathbb{R}_{\text{sym}}^{d^2}. \end{aligned} \quad (132)$$

It is possible to verify (see Malek et al. [1]) that (132) satisfy the assumptions (28) of Lemma 5.

To obtain some estimate for u'_m , we make $\varphi_r = u'_m$ in (37)₁, to obtain applying Schwarz's inequality

$$\begin{aligned} |u'_m|^2 + \nu \int_{\Omega} e_{ij}(u_m) e_{ij}(u'_m) dx \\ + \nu_0 \int_{\Omega} \tau_{ij}(e(u_m)) e_{ij}(u'_m) dx \\ \leq \int_{\Omega} |u_{m_i}| \left| \frac{\partial u_{m_j}}{\partial x_i} \right| |u'_{m_j}| dx \\ + |u'_m| |\nabla \times w_m| + |f| |u'_m|. \end{aligned} \quad (133)$$

Remark 12. We note that (see [4])

$$\int_{\Omega} \tau_{ij}(e(v)) e_{ij}(v') dx = \frac{d}{dt} \int_{\Omega} \Phi(|e(v)|^2) dx. \quad (134)$$

Applying Young's inequality, (49) and (134), we obtain from (141) that

$$\begin{aligned} \frac{1}{2} |u'_m|^2 + \frac{\nu}{2} \frac{d}{dt} |e(u_m)|^2 + \nu_0 \frac{d}{dt} \int_{\Omega} \Phi(|e(u_m)|^2) dx \\ \leq c + c \int_{\Omega} |u_m|^2 |\nabla u_m|^2 dx. \end{aligned} \quad (135)$$

We observe that (46) implies $u_m(t) \in V_4 \hookrightarrow \mathbf{L}^\infty(\Omega)$ because $d \leq 3$. On the other hand, (47) implies $\nabla u_m(t) \in \mathbf{L}^2(\Omega)$. Thus, applying the Hölder's inequality in (135), integrating on $(0, t)$, with $t \leq T$, and after applying Korn's inequality, we obtain

$$\begin{aligned} & \int_0^t |u'_m(s)|^2 ds + \nu_1 K \|u_m\|^2 + 2\nu_0 \int_\Omega \Phi(|e(u_m)|^2) dx \\ & \leq c + 2\nu_0 \int_\Omega \Phi(|e(u_m(0))|^2) dx + \nu_1 K \|u_m(0)\|^2 \quad (136) \\ & + c \int_0^t |u_m(s)|_{L^\infty(\Omega)}^2 \|u_m(s)\|^2 ds. \end{aligned}$$

Observing that (46) implies $\|u_m(t)\|_{L^\infty(\Omega)}^2 \in L^2(0, T)$ and also $\|u_m(t)\|^2 \in L^2(0, T)$, because of (47), so we can obtain by using Hölder's inequality in (136)

$$\begin{aligned} & \int_0^t |u'_m(s)|^2 ds + \nu_1 K \|u_m\|^2 \\ & + 2\nu_0 \int_\Omega \Phi(|e(u_m)|^2) dx \leq c \quad (137) \end{aligned}$$

because (29) and $u_0 \in V \cap V_4$. Now we note that (29) and Korn's inequality imply

$$\int_\Omega \Phi(|e(u_m)|^2) dx \geq C_3 \|e(u_m)\|_{L^4(\Omega)}^4 \geq C_3 K \|u_m\|_{V_4}^4. \quad (138)$$

It follows from (137) that

$$\int_0^t |u'_m(s)|^2 ds + \nu_1 K \|u_m\|^2 + C_3 K \|u_m\|_{V_4}^4 \leq c. \quad (139)$$

Inequality (139) permits us to obtain the following estimates:

$$\begin{aligned} & (u'_m) \text{ is bounded in } L^2(I; H), \\ & (u_m) \text{ is bounded in } L^\infty(I; V), \quad (140) \\ & (u_m) \text{ is bounded in } L^\infty(I; V_4). \end{aligned}$$

Now we will obtain some estimate for w'_m . For that purpose, we make $\phi_r = w'_m$ in (37)₂ to obtain

$$\begin{aligned} & |w'_m|^2 + \nu_1 \int_\Omega e_{ij}(w_m) e_{ij}(w'_m) dx + \frac{\lambda_1}{2} \frac{d}{dt} |w_m|^2 \\ & \leq \int_\Omega |u_{m_i}| \left| \frac{\partial w_{m_j}}{\partial x_i} \right| |w'_{m_j}| dx + \lambda_2 \|u_m\| |w'_m| + |g| |w'_m|. \quad (141) \end{aligned}$$

Applying Young's inequality and using (47) we derive

$$\begin{aligned} & \frac{1}{2} |w'_m|^2 + \frac{\nu_1}{2} \frac{d}{dt} |e(w_m)|^2 + \frac{\lambda_1}{2} \frac{d}{dt} |w_m|^2 \\ & \leq c + c \int_\Omega |u_m|^2 |\nabla w_m|^2 dx. \quad (142) \end{aligned}$$

We observe that (46) implies $u_m(t) \in V_4 \hookrightarrow \mathbf{L}^\infty(\Omega)$ because $d \leq 3$. On the other hand, (49) implies $\nabla w_m(t) \in \mathbf{L}^2(\Omega)$. Thus, applying Hölder's inequality in (142), integrating on $(0, t)$, with $t \leq T$, and after applying Korn's inequality, we obtain

$$\begin{aligned} & \int_0^t |w'_m(s)|^2 ds + \nu_1 K \|w_m\|^2 \\ & \leq c + c \int_0^t |u_m(s)|_{L^\infty(\Omega)}^2 \|w_m(s)\|^2 ds \quad (143) \end{aligned}$$

because $w_0 \in \mathbf{H}_0^1(\Omega)$. Furthermore (46) implies $\|u_m(t)\|_{L^\infty(\Omega)}^2 \in L^2(0, T)$ and (49) implies $\|u_m(t)\| \in L^1(0, T)$. Thus, by using Gronwall's inequality in (143), we conclude

$$(w'_m) \text{ is bounded in } L^2(I; \mathbf{L}^2(\Omega)), \quad (144)$$

$$(w_m) \text{ is bounded in } L^\infty(I; \mathbf{H}_0^1(\Omega)). \quad (145)$$

In order to prove uniqueness of solution to the case $d = 3$ we observe that by assuming $d = 3$ we have $H_0^1 \hookrightarrow L^6(\Omega)$. From Lemma 7 with $s = 4$, $r = 6$ and Young's inequality we derive

$$\begin{aligned} & |b(\tilde{u}, u_1, \tilde{u})| + |b(\tilde{u}, w_2, \tilde{w})| \\ & \leq c \|\tilde{u}\|_{L^6(\Omega)} \|u_1\| |\tilde{u}|^{1/2} \|\tilde{u}\|^{1/2} \\ & + c \|\tilde{u}\|_{L^6(\Omega)} \|w_2\| |\tilde{w}|^{1/2} \|\tilde{w}\|^{1/2} \\ & \leq c \|u_1\| |\tilde{u}|^{1/2} \|\tilde{u}\|^{3/2} \\ & + c \|w_2\| |\tilde{w}|^{1/2} \|\tilde{u}\| \|\tilde{w}\|^{1/2} \quad (146) \\ & \leq \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu \|u_1\|^4 |\tilde{u}|^2 \\ & + \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu \|w_2\|^2 |\tilde{w}| \|\tilde{w}\| \\ & \leq \frac{\nu}{4} \|\tilde{u}\|^2 + c_\nu \|u_1\|^4 |\tilde{u}|^2 \\ & + \frac{\nu}{4} \|\tilde{u}\|^2 + \nu_1 \|\tilde{w}\|^2 + c_{\nu_1} \|w_2\|^4 |\tilde{w}|^2. \end{aligned}$$

It follows from (103) that

$$\frac{1}{2} \frac{d}{dt} (|\tilde{u}|^2 + |\tilde{w}|^2) \leq c (1 + \|u_1\|^4 + \|w_2\|^4) (|\tilde{u}|^2 + |\tilde{w}|^2). \quad (147)$$

Integrating from 0 to t we obtain

$$\begin{aligned} & |\tilde{u}(t)|^2 + |\tilde{w}(t)|^2 \\ & \leq c \int_0^t (1 + \|u_1(s)\|^4 + \|w_2(s)\|^4) (|\tilde{u}(s)|^2 + |\tilde{w}(s)|^2) ds. \quad (148) \end{aligned}$$

Applying Gronwall's inequality in (148), we deduce by using (47) and (145) that

$$u_1(t) = u_2(t), \quad w_1(t) = w_2(t), \quad \forall t \in [0, T]. \quad (149)$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] J. Malek, J. Necas, M. Rokyta, and M. Ruzicka, *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Chapman & Hall, New York, NY, USA, 1st edition, 1996.
- [2] J. Malek, K. R. Rajagopal, and M. Ruzicka, "Existence and regularity of solution and stability of the rest state for fluids with shear dependent viscosity," *Mathematical Models and Methods in Applied Sciences*, vol. 5, no. 6, pp. 789–812, 1995.
- [3] J. L. Lions, *Quelques Methodes de Resolution Des Problemes Aux Limites Non Lineaires*, Dunod, Paris, France, 1969.
- [4] J. Malek, J. Necas, and M. Ruzicka, "On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$," *Advances in Differential Equations*, vol. 6, no. 3, pp. 257–302, 2001.
- [5] J. Frehse and J. Málek, "Problems due to the no-slip boundary in incompressible fluid dynamics," in *Geometric Analysis and Nonlinear Partial Differential Equations*, pp. 559–571, Springer, Berlin, Germany, 2003.
- [6] A. C. Eringen, "Theory of micropolar fluids," *Journal of Mathematical Fluid Mechanics*, vol. 16, pp. 1–18, 1966.
- [7] C. Calmelet-Eluhu and D. R. Majumdar, "Flow of a micropolar fluid through a circular cylinder subject to longitudinal and torsional oscillations," *Mathematical and Computer Modelling*, vol. 27, no. 8, pp. 69–78, 1998.
- [8] G. Lukaszewicz, *Micropolar Fluids, Theory and Applications*, Modeling and Simulations in Science, Engineering and Technology, Birkhauser, Boston, Mass, USA, 1999.
- [9] H. W. Alt, *Lineare Funktionalanalysis*, Springer, Berlin, Germany, 2nd edition, 1992 (German).