Research Article

Infinitely Many Eigenfunctions for Polynomial Problems: Exact Results

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Let $F(x, y) = a_s(x)y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)$ be a real-valued polynomial function in which the degree s of y in $F(x, y)$ is greater than or equal to 1. For any polynomial $y(x)$, we assume that $T: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ is a nonlinear operator with $T(y(x)) = F(x, y(x))$. In this paper, we will find an eigenfunction $y(x) \in \mathbb{R}[x]$ to satisfy the following equation: $F(x, y(x)) = ay(x)$ for some eigenvalue $a \in \mathbb{R}$ and we call the problem $F(x, y(x)) = ay(x)$ a fixed point like problem. If the number of all eigenfunctions in $F(x, y(x)) = ay(x)$ is infinitely many, we prove that (i) any coefficients of $F(x, y)$, $a_s(x), a_{s-1}(x), \dots, a_0(x)$, are all constants in $\mathbb{R}$ and (ii) $y(x)$ is an eigenfunction in $F(x, y(x)) = ay(x)$ if and only if $y(x) \in \mathbb{R}$.

1. Introduction and Preliminaries

Lenstra [1] investigated that

$$F(x, y(x)) = 0 \quad (1)$$

in which $F(x, y)$ is a polynomial function over the algebraic rational number field $\mathbb{Q}(\alpha)$ (where α is an algebraic number). He found a polynomial $y = y(x) \in \mathbb{Q}(\alpha)[x]$ satisfying the polynomial equation

$$F(x, y(x)) = x. \quad (2)$$

Further, Tung [2] extended (2) to solve polynomial solutions (near solutions) $y(x) \in \mathbb{K}[x]$ ($\mathbb{K}$ is a field) for the following equation:

$$F(x, y(x)) = ax^m, \quad (3)$$

where $a \in \mathbb{K}$ is a constant depending on the polynomial solution $y(x)$ and $m \in \mathbb{N}$ a given nonnegative integer.

Moreover, Lai and Chen [3–5] extended (3) to solve $y(x) \in \mathbb{R}[x]$ satisfying the polynomial equation as the form

$$F(x, y(x)) = ap^m(x), \quad x \in \mathbb{R}, \quad (4)$$

where $a \in \mathbb{R}$, $m \in \mathbb{N}$, $p(\cdot)$ is an irreducible polynomial in $x \in \mathbb{R}$, and the polynomial function $F(x, y): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is written by

$$F(x, y) = \sum_{i=0}^s a_i(x) y^i \quad \text{with } s \geq 1, \quad (5)$$

where s denotes the degree $\deg_y F$ of y in $F(x, y)$.

Recently, Chen and Lai [6, 7] research a quasicoincidence problem in which an arbitrary nonzero polynomial function $f(x) \in \mathbb{R}[x]$ is given as follows:

$$F(x, y(x)) = af(x), \quad (6)$$

where a is a constant.

Definition 1 (Chen and Lai, [6]). A polynomial function $y = y(x)$ satisfying (6) is called a quasicoincidence solution corresponding to some real number a . This number a is called a quasicoincidence value corresponding to the polynomial solutions $y = y(x)$.

In this paper, we define a fixed point like problem in which the $y(x) \in \mathbb{R}[x]$ is replaced by the arbitrary polynomial

$f(x) \in \mathbb{R}[x]$ throughout this paper. Then we restate (6) as the following equation:

$$F(x, y(x)) = ay(x). \quad (7)$$

It is a new developed fixed point like problem. We call the polynomial (7) as a *fixed point like* problem. The number of all eigenfunctions in (7) may be infinitely many, or finitely many, or not solvable.

Since there may exist many eigenfunctions corresponding to the eigenvalue a , for convenience, we use the following notations to represent different situations:

- (1) $\mathbf{E}_{\text{function}}$, the set of all eigenfunctions “ $y(x)$ ” satisfying (7);
- (2) $\mathbf{E}_{\text{value}}$, the set of all eigenvalues “ a ” satisfying (7);
- (3) $\mathbf{E}_{\text{function}}(a)$, the set of all eigenfunctions “ $y(x)$ ” corresponding to a fixed eigenvalue “ a ”.

For each $a \in \mathbb{R}$, the cardinal number of $\mathbf{E}_{\text{function}}(a)$, denoted by $|\mathbf{E}_{\text{function}}(a)|$, satisfies the following condition:

$$|\mathbf{E}_{\text{function}}(a)| \leq \deg_y F(x, y). \quad (8)$$

In Section 2, we derive some properties of eigenfunctions of $F(x, y)$. If (7) has infinitely many eigenfunctions, the concerned properties are described in Section 3.

Throughout the paper, we denote the polynomial function by

$$\begin{aligned} F(x, y) &= a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_1(x) y + a_0(x) \\ &= \sum_{i=0}^s a_i(x) y^i \end{aligned} \quad (9)$$

with $s \geq 1$. Moreover, we may assume that $a_0(x)$ is nonzero. Since $a_0(x) = 0$, problem (7) may become

$$a_s(x) y^s + a_{s-1}(x) y^{s-1} + \cdots + a_1(x) y = ay(x). \quad (10)$$

Moreover, if problem (7) has infinitely many eigenfunctions, dividing $y(x)$ by both sides of the above equation, then there may exist infinitely many nonzero eigenfunctions $y(x)$ satisfying

$$a_s(x) y^{s-1} + a_{s-1}(x) y^{s-2} + \cdots + a_1(x) = a \quad (11)$$

for some $a \in \mathbb{R}$. Therefore, this problem becomes a special case of (3).

2. Some Lemmas and a Former Theorem

Throughout this paper, we consider (7) for the polynomial function (9).

Lemma 2. Let $y(x) \in \mathbf{E}_{\text{function}}$. Then

$$y(x) = dp(x) \quad \text{for some } d \in \mathbb{R}, \quad (12)$$

and this $p(x)$ is divisible $a_0(x)$ and is denoted by $p(x) \mid a_0(x)$.

Proof. Since $y(x) \in \mathbf{E}_{\text{function}}$, we have $F(x, y(x)) = ay(x)$ for some $a \in \mathbb{R}$. This means

$$\begin{aligned} a_s(x) y^s(x) + a_{s-1}(x) y^{s-1}(x) + \cdots + a_1(x) y(x) + a_0(x) \\ = ay(x) \end{aligned} \quad (13)$$

for some $a \in \mathbb{R}$. It leads to

$$\begin{aligned} y(x) (a_s(x) y^{s-1}(x) + a_{s-1}(x) y^{s-2}(x) + \cdots + (a_1(x) - a)) \\ = -a_0(x); \end{aligned} \quad (14)$$

then $y(x)$ is a factor of $a_0(x)$. $\square$

In Lemma 2, any eigenfunction is a factor $p(x)$ of $a_0(x)$. Thus we define a class of this factor as follows.

Notation 1. Let $p(x) \in \mathbb{R}[x]$, and we denote $\Phi(p(x)) = \{\alpha p(x) : \alpha \in \mathbb{R}\}$.

According to Notation 1, it is obvious that for any $p(x)$ in $\mathbb{R}[x]$, we have the cardinal number

$$|\Phi(p(x))| = \infty. \quad (15)$$

For convenience, we explain the relations of $\mathbf{E}_{\text{function}}$ and $\Phi(p(x))$ in the following lemma.

Lemma 3. Consider

$$\mathbf{E}_{\text{function}} = \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \cap \mathbf{E}_{\text{function}}. \quad (16)$$

Proof. For any $y(x) \in \mathbf{E}_{\text{function}}$, by Lemma 2, we have $y(x) \mid a_0(x)$. That is,

$$y(x) \in \Phi(p(x)) \quad (17)$$

for some factor $p(x)$ of $a_0(x)$. It follows that

$$\mathbf{E}_{\text{function}} \subseteq \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \quad (18)$$

and we obtain

$$\mathbf{E}_{\text{function}} = \bigcup_{p(x) \mid a_0(x)} \Phi(p(x)) \cap \mathbf{E}_{\text{function}}. \quad (19)$$

$\square$

We will use the definitions of “the pigeonhole principle,” which concert with Grimaldi [8] and the above relation can be explained as the following lemma.

Lemma 4. Suppose that the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$; there exists a factor $p(x)$ of $a_0(x)$ such that the cardinal number

$$|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}| = \infty. \quad (20)$$

Proof. By Lemma 3, we obtain

$$\begin{aligned} (\infty \Rightarrow) |\mathbf{E}_{\text{function}}| &= \left| \bigcup_{p(x)|a_0(x)} \Phi(p(x)) \cap \mathbf{E}_{\text{function}} \right| \\ &\leq \sum_{p(x)|a_0(x)} |\Phi(p(x)) \cap \mathbf{E}_{\text{function}}|. \end{aligned} \quad (21)$$

Since the number of all factor $p(x)$ of $a_0(x)$ is at most $2^{\deg a_0(x)}$, by pigeonhole's principle, it yields

$$|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}| = \infty \quad (22)$$

for some factor $p(x)$ of $a_0(x)$. $\square$

In order to solve the problem (7), [6, Lemma 3 and Theorem 11] are needed as follows.

Lemma 5 (see [6, Lemma 3]). *Assume that the number of all quasicoincidence solutions (defined in Definition 1) is infinitely many; then, for any two quasicoincidence solutions $y_1(x)$ and $y_2(x)$, we have*

$$y_1(x) - y_2(x) = \lambda g(x) \quad (23)$$

for some constant $\lambda \in \mathbb{R}$ and some factor $g(x)$ of $f(x)$.

Theorem 6 (see [6, Theorem 11]). *Assume that the number of all quasicoincidence solutions (defined in Definition 1) is infinitely many; then*

$$F(x, y) = \sum_{i=0}^s c_i \frac{f(x)}{g^i(x)} (y - y_1(x))^i \quad (24)$$

for some $y_1(x) \in \mathbb{R}[x]$, $g(x)$ is a factor of $f(x)$, and $c_i \in \mathbb{R}$ for $i = 0, 1, \dots, s$.

3. Main Theorems

In this section, we consider $F(x, y) = ay(x)$ for the polynomial function $F(x, y)$ defined in (9).

We investigate the fixed point like problem of simple polynomial functions $F(x, y)$ with $s = 1$ at first. Theorems 7 and 8 describe the necessary and sufficient results of these simple functions.

Theorem 7. *Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x)$ for some $a_1(x), a_0(x) \in \mathbb{R}[x]$. If the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$, then*

- (i) $a_1(x) \in \mathbb{R}$;
- (ii) any eigenfunction $y(x)$ of (7) is of the form

$$y(x) = \lambda a_0(x) \quad (25)$$

for some $\lambda \in \mathbb{R}$.

Proof. Since $|\mathbf{E}_{\text{function}}| = \infty$, by Lemma 4, there exists a factor $p(x)$ of $a_0(x)$ such that

$$|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}| = \infty. \quad (26)$$

There exist two different eigenfunctions $y_1(x), y_2(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}$ with

$$\begin{aligned} y_1(x) &= \alpha_1 p(x), \\ y_2(x) &= \alpha_2 p(x), \end{aligned} \quad (27)$$

for different constants $\alpha_1, \alpha_2 \in \mathbb{R}$. Since $y_1(x), y_2(x) \in \mathbf{E}_{\text{function}}$, we have

$$\begin{aligned} F(x, y_1(x)) &= ay_1(x), \\ F(x, y_2(x)) &= by_2(x), \end{aligned} \quad (28)$$

where $a, b \in \mathbf{E}_{\text{value}}$. It follows that

$$\begin{aligned} F(x, y_1(x)) &= a_1(x)y_1(x) + a_0(x) = ay_1(x), \\ F(x, y_2(x)) &= a_1(x)y_2(x) + a_0(x) = by_2(x). \end{aligned} \quad (29)$$

By (27) and (29), we have

$$\begin{aligned} a_1(x)(\alpha_1 p(x)) + a_0(x) &= a(\alpha_1 p(x)), \\ a_1(x)(\alpha_2 p(x)) + a_0(x) &= b(\alpha_2 p(x)). \end{aligned} \quad (30)$$

By (30), we get

$$a_1(x)(\alpha_1 p(x) - \alpha_2 p(x)) = a\alpha_1 p(x) - b\alpha_2 p(x). \quad (31)$$

Since $\alpha_1 \neq \alpha_2$ and $p(x)$ is nonzero, it follows that

$$a_1(x) = \frac{a\alpha_1 - b\alpha_2}{\alpha_1 - \alpha_2} \in \mathbb{R}. \quad (32)$$

For any $y(x) \in \mathbf{E}_{\text{function}}$, we have

$$F(x, y(x)) = a_1(x)y(x) + a_0(x) = ay(x). \quad (33)$$

By (32), we let $a_1(x) = \widetilde{a_1} \in \mathbb{R}$, (33) becomes

$$\widetilde{a_1}y(x) + a_0(x) = ay(x), \quad (34)$$

and it follows that

$$a_0(x) = (a - \widetilde{a_1})y(x). \quad (35)$$

Owing to $a_0(x) \neq 0$, then we obtain

$$y(x) = \widetilde{\lambda} a_0(x), \quad (36)$$

where $\widetilde{\lambda} = 1/a - \widetilde{a_1}$. $\square$

The following theorem is the sufficient conditions of Theorem 7.

Theorem 8. *Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x)$ for some $a_1(x), a_0(x) \in \mathbb{R}[x]$. If*

- (i) $a_1(x) \in \mathbb{R}$,
(ii) any eigenfunction $y(x)$ of (7) is of the form

$$y(x) = \lambda a_0(x), \quad (37)$$

for some $\lambda \in \mathbb{R}$,

then $|\mathbf{E}_{\text{function}}| = \infty$.

Proof. By (i), we let $a_1(x) = a_1 \in \mathbb{R}$, then $F(x, y(x)) = ay(x)$ for some $a \in \mathbb{R}$. This implies

$$a_1 y(x) + a_0(x) = ay(x) \quad (38)$$

and then $y(x) = (1/(a-a_1))a_0(x)$ is an eigenfunction of (7) for any constant $a \neq a_1$. It follows that

$$\infty = \left| \left\{ \frac{1}{a-a_1} a_0(x) : a \in \mathbb{R} - \{a_1\} \right\} \right| \leq |\mathbf{E}_{\text{function}}|; \quad (39)$$

then $|\mathbf{E}_{\text{function}}| = \infty$. $\square$

In Theorems 7 and 8, problem (7) with $\deg_y F = 1$ is introduced. In the following theorems, we deal with (7) with $\deg_y F \geq 2$ when the number of all eigenfunctions is infinitely many.

Theorem 9. Suppose that the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$ and $\deg_y F(x, y) \geq 2$. Then the polynomial $F(x, y)$ can be represented as

$$F(x, y) = \sum_{i=0}^s \beta_i y^i \quad (40)$$

for some constants $\beta_i \in \mathbb{R}$.

Proof. Since $|\mathbf{E}_{\text{function}}| = \infty$, by Lemma 4, there exists a factor $p(x)$ of $a_0(x)$ satisfying

$$|\Phi(p(x)) \cap \mathbf{E}_{\text{function}}| = \infty. \quad (41)$$

Let $y_1(x)$ be an eigenfunction in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}$ such that

$$F(x, y_1(x)) = a_1 y_1(x) \quad (42)$$

for some eigenvalue $a_1 \in \mathbb{R}$. By Remainder Theorem, we get

$$F(x, y) = (y - y_1(x)) F_1(x, y) + a_1 y_1(x), \quad (43)$$

where $F_1(x, y)$ is the quotient and $a_1 y_1(x)$ is the remainder.

From the above identity and considering any eigenfunction $y(x)$ in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$ with $F(x, y(x)) = ay(x)$, we substitute (43) by taking $y = y(x)$ above and it becomes

$$\begin{aligned} (ay(x) =) F(x, y(x)) \\ = (y(x) - y_1(x)) F_1(x, y(x)) + a_1 y_1(x). \end{aligned} \quad (44)$$

Since $y_1(x), y(x) \in \Phi(p(x))$, we have

$$y_1(x) = \lambda_1 p(x), \quad (45)$$

$$y(x) = \lambda p(x) \quad (46)$$

for some different constants λ_1 and λ . Substituting (45) and (46) into (44), it becomes

$$a\lambda p(x) = (\lambda p(x) - \lambda_1 p(x)) F_1(x, y(x)) + a_1 \lambda_1 p(x) \quad (47)$$

and it leads to

$$F_1(x, y(x)) = \frac{a\lambda - a_1 \lambda_1}{\lambda - \lambda_1} \in \mathbb{R} \quad (48)$$

for any eigenfunction $y(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$.

By (48), there exist infinitely many quasicoincidence solutions in $\Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$ to satisfy

$$F_1(x, y) = af(x) \quad (49)$$

with $f(x) = 1$. This problem is a quasicoincidence problem; then by Theorem 6, we have

$$F_1(x, y) = \sum_{i=0}^{s-1} c_i \frac{f(x)}{g^i(x)} (y - y_1(x))^i. \quad (50)$$

Moreover, since $f(x) = 1$ and $f(x)/g^i(x) \in \mathbb{R}[x]$ for any $i = 0, 1, 2, \dots, s-1$, it implies that $g(x) \in \mathbb{R}$ and by Lemma 5, any $y_2(x), y_3(x) \in \Phi(p(x)) \cap \mathbf{E}_{\text{function}}/\{y_1(x)\}$, we have

$$y_2(x) - y_3(x) = dg(x) = d' \in \mathbb{R}. \quad (51)$$

By definitions of $\Phi(p(x))$, $y_2(x)$, and $y_3(x)$ can also be represented as

$$\begin{aligned} y_2(x) &= \lambda_2 p(x), \\ y_3(x) &= \lambda_3 p(x) \end{aligned} \quad (52)$$

for some $\lambda_2, \lambda_3 \in \mathbb{R}$. By (51), it follows that

$$y_2(x) - y_3(x) = (\lambda_2 - \lambda_3) p(x) \in \mathbb{R}. \quad (53)$$

Moreover, by (53), this implies that $p(x) \in \mathbb{R}$ and by (45), $y_1(x) \in \mathbb{R}$, say, $y_1(x) = b_1$ and (50) implies that

$$F_1(x, y) = \sum_{i=0}^{s-1} c_i (y - b_1)^i = \sum_{i=0}^{s-1} d_i y^i \quad (54)$$

for some $d_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, s-1$.

By (54), (43) implies that

$$\begin{aligned} F(x, y) &= (y - y_1(x)) F_1(x, y) + a_1 y_1(x) \\ &= (y - b_1) \sum_{i=0}^{s-1} d_i y^i + a_1 b_1 = \sum_{i=0}^s \beta_i y^i. \end{aligned} \quad (55)$$

$\square$

Conversely, if $F(x, y)$ can be expressed as in Theorem 9, then the cardinal number $|\mathbf{E}_{\text{function}}| = \infty$; this problem becomes the sufficient conditions of Theorem 9.

Theorem 10. Assume that

$$F(x, y) = \sum_{i=0}^s \beta_i y^i \quad (56)$$

for some $\beta_i \in \mathbb{R}$ for $i = 0, 1, \dots, s$; then

$$|\mathbf{E}_{\text{function}}| = \infty. \quad (57)$$

Proof. For any $y(x) = c \in \mathbb{R}$,

$$\begin{aligned} F(x, y(x)) &= \sum_{i=0}^s c_i y^i(x) \\ &= \sum_{i=0}^s c_i c^i \text{ (this is a constant)} = ac, \end{aligned} \quad (58)$$

for some $a = \sum_{i=0}^s c_i c^i / c \in \mathbb{R}$. Then $\mathbb{R} \subseteq \mathbf{E}_{\text{function}}$ and then $|\mathbf{E}_{\text{function}}| = \infty$. $\square$

In fact, if $|\mathbf{E}_{\text{function}}| = \infty$, then $\mathbf{E}_{\text{function}} = \mathbb{R}$ and we prove it as follows.

Theorem 11. If $|\mathbf{E}_{\text{function}}| = \infty$, we have

$$\mathbf{E}_{\text{function}} = \mathbb{R}. \quad (59)$$

Proof. Since $|\mathbf{E}_{\text{function}}| = \infty$, by the proof of Theorem 10, we have

$$\mathbb{R} \subseteq \mathbf{E}_{\text{function}}. \quad (60)$$

Conversely, considering any $y(x) \in \mathbf{E}_{\text{function}}$, we have

$$F(x, y(x)) = \sum_{i=0}^s \beta_i y^i(x) = ay(x) \quad (61)$$

for some eigenvalue $a \in \mathbb{R}$. By Lemma 2, we have $y(x) \mid \beta_0$; this implies $y(x) \in \mathbb{R}$ and then $\mathbf{E}_{\text{function}} \subseteq \mathbb{R}$. This proof is completed. $\square$

From Theorems 9 and 11, we can easily obtain the following two corollaries.

Corollary 12. Let $F(x, y) = \sum_{i=0}^s a_i(x) y^i \in \mathbb{R}[x, y]$, $s \geq 2$, with $a_j(x) \notin \mathbb{R}$ for some j , then $|\mathbf{E}_{\text{function}}| < \infty$.

Proof. This result can be immediately obtained from Theorem 9. $\square$

Corollary 13. If there exists an eigenfunction $y(x) \in \mathbf{E}_{\text{function}}$ with $y(x) \notin \mathbb{R}$, then $|\mathbf{E}_{\text{function}}| < \infty$.

Proof. This result can be immediately obtained from Theorem 11. $\square$

From Theorems 7, 8, and 10 and Corollary 12, we provide some examples of fixed point like problem (7) for some $a \in \mathbb{R}$, which have infinitely many eigenfunctions and do not have infinitely many eigenfunctions as follows.

Example 14. In the following examples, we by the form of $F(x, y)$, we can decide whether the number of all eigenfunctions in (7) is infinitely many or not.

- (1) If $(x, y) = xy + 1$, there do not exist infinitely many eigenfunctions (Theorem 7).
- (2) If $F(x, y) = xy + x$, there do not exist infinitely many eigenfunctions (Theorem 7).
- (3) If $F(x, y) = y + x$, there exist infinitely many eigenfunctions and

$$\mathbf{E}_{\text{function}} = \{\lambda x : \lambda \in \mathbb{R}\} \text{ (Theorem 8)}. \quad (62)$$

- (4) If $F(x, y) = -y^2 + 7y + 1$, there exist infinitely many eigenfunctions and

$$\mathbf{E}_{\text{function}} = \mathbb{R} \text{ (Theorem 10)}. \quad (63)$$

- (5) If $F(x, y) = xy^2 + xy + 1$, there do not exist infinitely many eigenfunctions (Corollary 12).
- (6) If $F(x, y) = \sum_{i=0}^s c_i y^i + x$, $s \geq 2$, for any constants $c_i \in \mathbb{R}$, there do not exist infinitely many eigenfunctions (Corollary 12).

We would like to provide one open problem as follows.

Further Development. For a real-valued polynomial function $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, if $|\mathbf{E}_{\text{function}}| < \infty$, can we find a co-NP complete algorithm to solve all eigenfunctions $y = y(x)$ satisfying (7)?

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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