Weak convergence conditions for semi-Markov stochastic processes in the diffusion approximation scheme without balance condition are studied in the paper. Theory of Markov and semi-Markov processes is used in security market (Black-Scholes equation, Vasicek model, and their modifications) [1], queuing systems [2, 3], engineering [4], biology [5], climate models [6, 7], and publicity models [8]. But numerous papers were devoted to problems of Markov processes convergence. This approach produces errors of mathematical model because exponential distribution of sojourn time in states is supposed. The supposition enables staying in the state any time with nonzero probability. This is unacceptable in physics systems. That is why the subject of this research is semi-Markov processes. Beside this, other techniques of weak convergence research are used in many papers. In this case authors get different sufficient conditions of the convergence. For example, in [9] authors states his results as solution of some martingale problem. It complicates testing these conditions. In [3] author focuses on the convergence of characteristic functions and claims the convergence of characteristic functions for prelimited processes. In [10] author claims the convergence of the generators of the prelimited processes to the generator of some diffusion process.

In contrast to abovementioned works, only moment's conditions on the semi-Markov process local characteristic are used in this paper. Using the term of compensating operator makes it possible to not use convergence of the generators of prelimited processes.

2. Main Result

Consider the conditions of weak convergence of semi-Markov random processes (SMP) in diffusion approximation scheme. Consideration of these problems can be found in [3, 9-15]. Let us consider SMP \( \eta(t), t \geq 0 \), on the probability space \( (\Omega, F, P) \) [11, 16, 17] in Euclidian space \( \mathbb{R}^d, d \geq 1 \), which is generated by the Markov renewal process (MRP)

\[
(\eta_n, \tau_n), \quad n \geq 0;
\]

that is

\[
\eta(t) = \eta(\nu(t)),
\]

where \( \nu(t) = \max\{n \geq 0 : \tau_n < t\} \) is the counting process.

Denote the sojourn time in states \( \theta_n = \tau_n - \tau_{n-1}, n \geq 0 \). MRP is determined by semi-Markov kernel, which sets conditional probabilities of jump's values, and by distribution of the sojourn time in states:

\[
Q(u, dv, t) = P\{\Delta \eta_{n+1} \in dv, \theta_{n+1} \leq t \mid \eta_n = u\} = \Gamma(u, dv) F_n(t),
\]

where \( \Gamma(u, dv) \) is the generator of the process.
where \( u \in \mathbb{R}^d, dv \in \beta_{\mathbb{R}^d}, t \geq 0, \Delta \eta_{n+1} = \eta_{n+1} - \eta_n, \beta_{\mathbb{R}^d} \) is the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \), and

\[
\Gamma(u, dv) = \mathbb{P} \left\{ \Delta \eta_{n+1} \in dv \mid \eta_n = u \right\}.
\]

(4)

\[ F_u(t) = \mathbb{P} \left\{ \theta_{n+1} \leq t \mid \eta_n = u \right\}. \]

(5)

Note that in this section the important fact will be one when the kernel \( Q(u, dv) \) has decomposition:

\[
Q(u, dv) = \Gamma(u, dv) F_u(t)
\]

(6)

because in general this assumption is not valid.

In this paper suppose that balance condition did not hold; it means that

\[
a(u) = \int_{\mathbb{R}^d} v^T(\mathbb{R}^d) \neq 0.
\]

(7)

We can prove weak convergence of the process \( \eta^\varepsilon \Rightarrow \rho \), where \( \rho \) is the solution of differential equation

\[
\frac{d\rho(t)}{dt} = C(\rho(t)),
\]

(8)

where

\[
C(u) = a(u) b(u),
\]

(9)

(10)

\[
b(u) = (\lambda(u))^{-1},
\]

(11)

\[
\lambda(u) = \int_0^\infty t dF_u(t).
\]

Consider stochastic process:

\[
\xi^\varepsilon(t) = \varepsilon^{-1} \left( \varepsilon^2 \eta(\varepsilon^2 t) - \rho(t) \right).
\]

(12)

According to (11), we get

\[
\tau^\varepsilon_n = \varepsilon^2 \tau_n,
\]

\[
\theta^\varepsilon_n = \varepsilon^2 \theta_n.
\]

(13)

Let us define

\[
\xi^\varepsilon_n = \eta(\tau^\varepsilon_n) - \varepsilon^{-1} \rho(\varepsilon^2 \tau^\varepsilon_n),
\]

(14)

\[
\rho^\varepsilon_n = \rho(\varepsilon^2 \tau_n).
\]

(15)

Consider compensating operator for some process.

**Definition 1.** The compensating operator \( \Gamma^\varepsilon \) for SMP \( \xi^\varepsilon(t), t \geq 0 \), is defined by the relation

\[
\Gamma^\varepsilon \varphi(u, \varepsilon) = \mathbb{E} \left[ \varphi(\xi^\varepsilon_{n+1}, \tau^\varepsilon_n) - \varphi(\xi^\varepsilon_n, \tau^\varepsilon_n) \mid \xi^\varepsilon_n = u, \tau^\varepsilon_n = t \right]
\]

on the test-functions \( \varphi(u, \varepsilon) \).

In this case there is a weak convergence SMP in the scheme of diffusive approximation without balance conditions.

**Theorem 2.** Let the following conditions be satisfied:

(D1) Uniform integrability (bounded time in states):

\[
\lim_{T \to \infty} \sup_{u \in \mathbb{R}^d} \mathbb{E} \int_T^{\infty} F(t) dt = 0.
\]

(16)

(D2) \( \exists C, C_1 > 0, \) that inequality

\[
\mathbb{E} e^{-\varepsilon^2} \leq 1 - C \varepsilon,
\]

(17)

(D3) Boundary of the second moment of jump's value:

\[
\sup_{u \in \mathbb{R}^d} \mathbb{E} \left[ \int_{\mathbb{R}^d} v^T \Gamma(u, dv) \right] < \infty.
\]

(18)

(D4) Kernel \( \Gamma^\varepsilon(u, dv) \) satisfies the following conditions:

\[
a^\varepsilon(z, \varepsilon) = \int_{\mathbb{R}^d} v^T \Gamma(z + \varepsilon u, dv)
\]

\[
= a(z) + \varepsilon a(z, \varepsilon) + \varepsilon \delta^1_1,
\]

(19)

\[
B^\varepsilon(z, \varepsilon) = \int_{\mathbb{R}^d} v^T \Gamma(z + \varepsilon u, dv) = B(z) + \varepsilon \delta^2_2,
\]

where \( \delta^1_1, \delta^2_2 \to 0, \varepsilon \to 0. \)

(D5) Function \( b(u) \) satisfies the condition

\[
b(\varepsilon u) = b(\varepsilon u) + \varepsilon \delta^1_1, \delta^2_2 \to 0, \varepsilon \to 0.
\]

(D6) Convergence of the initial conditions is as follows:

\[
\xi^0(0) \to \xi(0),
\]

(20)

\[
\sup_{\varepsilon > 0} \mathbb{E} \xi^\varepsilon(0) < \infty,
\]

Then weak convergence takes place in \( D([0, T]), T < \infty \), as \( \varepsilon \downarrow 0 \):

\[
\xi^\varepsilon \Rightarrow \xi^0.
\]

(21)
where $\xi^ε(t)$, $t \geq 0$, is the diffusion process with generator
\[ \Gamma^0_ε \varphi (u) = C_1 (u, \rho (t)) \varphi' (u) \]
\[ + \frac{1}{2} \overline{B} (\rho (t)) \varphi'' (u), \]
\[ C_1 (u, \rho (t)) = b (\rho (t)) a_1 (\rho (t), u), \]
\[ \overline{B} (\rho (t)) = b (\rho (t)) \sigma^2 (\rho (t)), \]
\[ \sigma^2 (u) = B (u) - a^2 (u). \tag{22} \]

Remark 3. Boundary operator depends on the averaged evolution $\rho$; that is why it is advisable to consider weak convergence of two-component evolution $(\xi^ε, \rho^ε)$. But, according to [11, 18–20], we will prove theorem only for the process, which consists of parameter of series $ε$, in other words $\xi^ε$.

The proof of Theorem 2 consists of two steps.

Step 1. Let us solve the problem of the singular perturbation for CO of process $\xi^ε$ as $ε \downarrow 0$.

Consider an evolution equation
\[ \frac{d \rho (t)}{dt} = C (\rho (t)) \tag{23} \]
that corresponds with
\[ C \varphi (v) = C (v) \varphi' (v) \tag{24} \]
and with semigroup
\[ C_ε \varphi (v) = \varphi \left( v + \int_0^t C (\rho (s)) ds \right). \tag{25} \]

By analogues, an evolution equation
\[ \frac{d \rho^ε (t)}{dt} = -ε^{-1} C (\rho^ε (t)) \tag{26} \]
corresponds with operator
\[ C^ε \varphi (v) = -ε C (v) \varphi' (v) \tag{27} \]
with semigroup
\[ C^ε \varphi (v) = \varphi \left( v - ε^{-1} \int_0^t C (\rho (s)) ds \right). \tag{28} \]

Lemma 4. CO of two-component process $(\xi^ε, \rho^ε)$ on test-functions $\varphi \in C (R^{2d})$ is given by
\[ \Gamma^ε \varphi (u, v) = ε^2 b (v + ε u) \int_0^∞ F_{v+εu} (ds) \]
\[ \cdot \left[ C_ε \varphi (u, v) - \varphi (u, v) \right] \]
\[ \cdot \Gamma_ε (v + ε u, dz), \tag{29} \]
where
\[ \Delta_ε \varphi (u, v) = \varphi (u + ε z, v). \tag{30} \]

Proof. By Definition 1, we got a relation for values of jumps and time of renewals:
\[ \Delta_ε^n = ε \Delta^ε_{n+1} - ε^{-1} \Delta_ε^{n+1}, \]
\[ ε^2 \tau_ε^n. \tag{31} \]

Then
\[ \Delta_ε^{n+1} = \int_0^{ε^2 \tau_ε^n} C (\rho (ε^2 \tau_ε^n + h)) dh. \tag{32} \]

So, according to the condition $ρ^ε_n = v$ we get
\[ \varphi (u, ρ^ε_{n+1}) = \varphi (u, ρ^ε_n + Δρ^ε_{n+1}) = C_ε \varphi (u, v). \tag{33} \]

For embedded chain $ξ^ε_n, n \geq 0$, we get
\[ E \left[ \varphi (ξ^ε_{n+1}, v) | ξ^ε_n = u \right] = E \left[ \varphi (ξ^ε_{n+1}, v) | ξ^ε_n = u \right] = E \left[ \varphi (ξ^ε_n + Δ_ε^n, v) | ξ^ε_n = u \right] \]
\[ + Δ_ε^n \int_0^{ε^2 \tau_ε^n} C (\rho (v + h)) dh, v | ξ^ε_n \right] \tag{34} \]
\[ = u \right] = C_ε \varphi (u, v). \]

Then calculate
\[ \Gamma^ε \varphi (u, v) \]
\[ = E \left[ \varphi (ξ^ε_{n+1}, ρ^ε_{n+1}) - \varphi (u, v) | ξ^ε_n = u, ρ^ε_{n+1} = v \right] \]
\[ = ε^2 b (v + ε u) E \left[ \varphi (ξ^ε_{n+1}, ρ^ε_{n+1} + Δρ^ε_{n+1}) - \varphi (u, v) | η_n = u + ε v \right] \tag{35} \]

So, finally we get representation of the compensating operator for two-component evolution $(\xi^ε, ρ^ε)$:
\[ \Gamma^ε \varphi (u, v) = ε^2 b (v + ε u) \int_0^∞ F_{v+εu} (ds) \]
\[ \cdot \left[ C_ε \varphi (u, v) - \varphi (u, v) \right] \]
\[ \cdot \Gamma_ε (v + ε u, dz), \tag{36} \]

as we wanted to show.

Lemma 4 is proved. \qed

Consider asymptotic behavior of CO, from Lemma 4 as $ε \downarrow 0$.

Lemma 5. On test-functions $φ(u, v) \in C^∞ (R^{2d})$ CO of the process $(\xi^ε, ρ^ε)$ has asymptotic representation
\[ \Gamma^ε \varphi (u, v) = \Gamma^0 \varphi (u, v) + R^ε \varphi (u, v), \tag{37} \]
where $\Gamma^0$ is given by the following relation:

$$\Gamma^0 \varphi (u, v) = C_1 (v, u) \varphi'_u (u, v) + \frac{1}{2} B (v) \varphi''_u (u, v) + C (v) \varphi'_v (u, v)$$

and for the negligible term,

$$\limsup_{\varepsilon \downarrow 0} \frac{\sup_{u,v \in \mathbb{R}^d} |R^\varepsilon \varphi (u, v)|}{\varepsilon} = 0$$

is true.

**Proof.** Let us use an algebraic identity

$$(abc - 1) = (a - 1) + (b - 1) + (c - 1)$$

for test-functions $\varphi \in C_c^\infty (\mathbb{R}^d)$, we get the following relation:

$$\varphi (u, v) \Gamma^0 \varphi (u, v) = \int_0^\infty F_{v+eu} (ds)$$

According to Lemma 4 and algebraic identity (40) we get

$$\Gamma^* \varphi (u, v) = e^{-2s} B (v + eu)$$

Then for the term $I_1$, by semigroups properties and condition (D5), we get the following relation:

$$I_1 = e^{-2s} b (v + eu) \int_0^\infty F_{v+eu} (ds)$$

and for the negligible term, $\limsup_{\varepsilon \downarrow 0} \frac{\sup_{u,v \in \mathbb{R}^d} |R^\varepsilon \varphi (u, v)|}{\varepsilon} = 0$.

From factorization $[C_1 e^\varepsilon \varphi'_u (u, v) + C_2 e^\varepsilon \varphi''_u (u, v)]$, according to (40), terms $C_1 e^\varepsilon \varphi'_u (u, v) + C_2 e^\varepsilon \varphi''_u (u, v)$ were considered. $\Gamma^0$ is built by these terms. It is easy to check that the sum of the rest of the terms is $o(1)$ as $\varepsilon \downarrow 0$, if conditions (D3)–(D5) hold.

Using representation for semigroups $C_1 e^\varepsilon \varphi'_u (u, v)$, and $\Delta_{xz}$, it is easy to show that negligible term is $o(1)$ as $\varepsilon \downarrow 0$.

Lemma 5 is proved.

**Remark 6.** Conditions of relative compactness can also be found in [22, 23].

**Lemma 7.** There is an inequality

$$||^{*} \varphi (u, v) || \leq C_{\varphi}$$

for test-functions $\varphi \in C_c^\infty (\mathbb{R}^d)$, where constant $C_{\varphi}$ depends only on function $\varphi$.

For function $\varphi_0 (u, v) = \sqrt{1 + u^2 + v^2}$ there is a bound

$$||^{*} \varphi_0 (u, v) || \leq C \varphi_0 (u, v), \quad |u| \leq I,$n such that $||^{*} \varphi_0 (u, v) || \leq C \varphi_0 (u, v)$, independent from $\varepsilon$.

**Proof.** Let us use the result of Lemma 5:

$$||^{*} \varphi (u, v) || \leq ||^{*} \varphi (u, v) || + ||R^\varepsilon \varphi (u, v) || \leq \sup_{u,v \in \mathbb{R}^d} |b (v)|$$

$$+ \int_0^\infty \frac{1}{2} \left( b (v) B (v) \right) F_v (ds) C^2 (v) + 2 C (v) a (v)$$

According to the definition of test-function $\varphi$ we have $\sup_{u,v \in \mathbb{R}^d} \max |\varphi|, |\varphi'_u|, |\varphi'_v|, |\varphi''_u|, |\varphi''_v| \leq K < \infty$. To prove the lemma, condition (D3) for a boundary of the first and the second moments, condition (D4), and condition (D2) remain to be used, from which it follows that $|b (u)| < c < \infty$.

Then

$$||^{*} \varphi (u, v) || \leq c K_1 (1 + \varepsilon) < C_{\varphi},$$

where constant $K_1 = K_1 (\varphi)$ depends only on $\varphi$, $C_{\varphi} = 2c K_1$.

To prove condition (46) the properties of the function $\varphi_0$ remain to be remembered; namely

$$\varphi_0'_u \leq 1 \leq \varphi_0,$n such that $\varphi_0'_u \leq 1 \leq \varphi_0$,

$$\varphi_0'_v \leq 1 \leq \varphi_0,$n such that $\varphi_0'_v \leq 1 \leq \varphi_0$.

Lemma 7 is proved.
Lemma 8. \(\xi^\varepsilon, \varepsilon > 0\), is relatively compact family.

Proof. To determine the relative compactness of the family \(\xi^\varepsilon, \varepsilon > 0\), according to Theorem 1.4.6 [21] submartingality of the stochastic process \(\alpha^\varepsilon(t) = \varphi(\xi^\varepsilon(t)) + C_\varphi t\) for nonnegative infinitely differentiable \(\varphi\) and for some constant \(C_\varphi\) and inequality (45) and (46) remains to be shown, where \(\xi^\varepsilon(t) = (\xi^\varepsilon(t), \rho(t))\).

Let us prove that stochastic process \(\alpha^\varepsilon(t)\) is nonnegative submartingale relatively to the stream of \(\sigma\)-algebras \(F^\varepsilon_t = \sigma(\tau^\varepsilon_s, \ s \leq t), \tau^\varepsilon_s(t) = \tau_s(t) + 1: \)

\[
E \left[ \alpha^\varepsilon(t) - \alpha^\varepsilon(s) \mid F^\varepsilon_s \right] = E \left[ \varphi(\xi^\varepsilon(t)) - \varphi(\xi^\varepsilon(s)) \mid F^\varepsilon_s \right] + C_\varphi (t - s)
\]

\[
= E \left[ \int_s^t \Gamma^\varepsilon(\xi^\varepsilon(u)) du \mid F^\varepsilon_s \right] + C_\varphi (t - s)
\]

\[
= E \left[ \int_{\tau^\varepsilon_s(t)}^{\tau^\varepsilon(t)} (\Gamma^\varepsilon(\xi^\varepsilon(u)) + C) du \mid F^\varepsilon_t \right]
\]

\[
+ E \left[ \int_s^{\tau^\varepsilon_s(t)} (\Gamma^\varepsilon(\xi^\varepsilon(u)) + C) du \mid F^\varepsilon_s \right] + C (t - \tau^\varepsilon_s(t) - s + \tau^\varepsilon_s(s)).
\]

Two last terms tend to 0 as \(\varepsilon \downarrow 0\). By Lemma 7

\[
E \left[ \int_{\tau^\varepsilon_s(t)}^{\tau^\varepsilon(t)} (\Gamma^\varepsilon(\xi^\varepsilon(u)) + C) du \mid F^\varepsilon_t \right] \geq 0.
\]

Measurability of the process \(\alpha^\varepsilon\) relatively to the stream \(F^\varepsilon_t\) is obvious. So, \(\alpha^\varepsilon\) is nonnegative submartingale.

Lemma 8 is proved. \(\square\)

According to Lemma 8 \(\xi^\varepsilon, \varepsilon > 0\), is a relatively compact family. To complete the proof of the theorem the family \(\xi^\varepsilon\) that converges to martingale remains to be shown. Consider stochastic processes:

\[
\begin{align*}
\zeta^\varepsilon_t(t) & := \zeta^\varepsilon(\tau^\varepsilon_t(t)), \\
\zeta^\varepsilon_t(t) & := \zeta^\varepsilon(\tau^\varepsilon(t)),
\end{align*}
\]

\(t \geq 0, \) \(\varepsilon > 0\), \( (52) \)

\[
\mu^\varepsilon_t := \varphi(\zeta^\varepsilon(t)) - \int_0^t \Gamma^0(\zeta^\varepsilon(s)) ds.
\]

Then

\[
E \mu^\varepsilon_t = E \left[ \varphi(\zeta^\varepsilon(t)) - \int_0^t \Gamma^0(\zeta^\varepsilon(s)) ds \right]
\]

\[
= E \left[ \varphi(\zeta^\varepsilon(t)) - \varphi(\zeta^\varepsilon(t)) \right]
\]

\[
+ E \left( \varphi(\zeta^\varepsilon(t)) - \int_0^t \Gamma^0(\zeta^\varepsilon(s)) ds \right)
\]

\[
+ E \left( \int_0^{\tau^\varepsilon(t)} \Gamma^0(\zeta^\varepsilon(s)) ds \right)
\]

\[
+ E \left( \int_0^{\tau^\varepsilon(t)} \Gamma^0(\zeta^\varepsilon(s)) - \Gamma^0(\zeta^\varepsilon(s)) ds \right)
\]

\[
\leq 0.
\]

According to Lemma 7 the third term satisfies the relation

\[
E \left( \int_0^{\tau^\varepsilon(t)} \Gamma^0(\zeta^\varepsilon(s)) ds \right) \longrightarrow 0.
\]

By the same way we can prove that the first and the fourth terms tend to 0, because \(\varphi\) is continuous.

The last term tends to 0 by Lemma 5, because

\[
\lim_{\varepsilon \downarrow 0} \Gamma^0(\zeta^\varepsilon(\eta)) = \Gamma^0(\zeta^\varepsilon(\eta))
\]

on test-functions \(\varphi\), which have uniform bounded derivatives of any order.

The second term is equal to

\[
\delta^\varepsilon_t := \varphi(\zeta^\varepsilon_t(t)) - \int_0^t \Gamma^0(\zeta^\varepsilon_t(s)) ds
\]

and has a martingale condition according to Lemma 6.1 [11].

Let us use its martingale condition:

\[
E \Theta^\varepsilon_t = E \varphi(\zeta^\varepsilon_t(0)) = E \varphi(\zeta^\varepsilon(0)).
\]

Finally we have

\[
E \mu^\varepsilon_t = E \varphi(\zeta^\varepsilon_t(0)) + r^\varepsilon,
\]

where \(r^\varepsilon \rightarrow 0\) as \(\varepsilon \downarrow 0\).

Now from theorem’s condition (D6) we get

\[
E \mu_t = E \varphi(\zeta(0));
\]

in another words, \(\zeta\) is martingale.

So, we have checked all conditions of the weak convergence, namely, the compactness of the processes family and the martingality of the limited process. Beside this according to Lemma 5 CO converges to the generator of diffusion process.

Theorem 2 is proved.

3. Numerical Example

Consider semi-Markov process \(\eta(t), t \geq 0\), in \(R^1\). For this process \(\Gamma(u, dv)\) has uniform distribution on \([u-1/2, u+1/2]\)
and \( \theta_{\eta_1} | \eta_n = u \) has Bernoulli distribution with parameters \((4, 1/2)\). Then

\[
a(u) = u \neq 0
\]  

(60)

and average evolution \( \rho(t), t > 0 \), has representation

\[
\rho(t) = \rho_0 e^{t/2},
\]

(61)

where \( \rho_0 \) are initial condition. It is easy to verify the conditions of the theorem. Prelimited processes are shown in Figures 1 and 2 for \( \varepsilon = 10^{-1}, 10^{-2} \).

4. Conclusions

Weak convergence of semi-Markov processes in the diffusive approximation scheme on conditions on the local characteristics of semi-Markov process is studied in this paper.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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