Research Article

On Distance (r, k) -Fibonacci Numbers and Their Combinatorial and Graph Interpretations

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Received 7 May 2015; Accepted 2 September 2015

Academic Editor: Ali R. Ashrafi

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We introduce three new two-parameter generalizations of Fibonacci numbers. These generalizations are closely related to k -distance Fibonacci numbers introduced recently. We give combinatorial and graph interpretations of distance (r, k) -Fibonacci numbers. We also study some properties of these numbers.

1. Introduction

In general we use the standard terminology of the combinatorics and graph theory; see [1]. The well-known Fibonacci sequence $\{F_n\}$ is defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$. The Fibonacci numbers have been generalized in many ways, some by preserving the initial conditions and others by preserving the recurrence relation. For example, in [2] k -Fibonacci numbers were introduced and defined recurrently for any integer $k \geq 1$ by $F(k, n) = kF(k, n-1) + F(k, n-2)$ for $n \geq 2$ with $F(k, 0) = 0$, $F(k, 1) = 1$. In [3] the following generalization of the Fibonacci numbers was defined: $x_n = 2^r x_{n-1} + x_{n-2}$ for an integer $r \geq 0$ such that $4^{r-1} + 1 \neq 0$ and $n \geq 2$ with $x_0 = 0$ and $x_1 = 1$. Other interesting generalizations of Fibonacci numbers are presented in [4, 5]. In the literature there are different kinds of distance generalizations of F_n . They have many graph interpretations closely related to the concept of k -independent sets. We recall some of such generalizations:

- (1) Reference [6]. Consider $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k+1$ with $F(k, n) = n+1$ for $n \leq k$.
- (2) References [4, 7, 8]. Consider Fibonacci p -numbers $F_p(n) = F_p(n-1) + F_p(n-p-1)$ for any given p ($p =$

$1, 2, 3, \dots$) and $n > p+1$ with $F_p(0) = 0$ and $F_p(n) = 1$ for $1 \leq n \leq p+1$.

- (3) Reference [9]. Consider $Fd^{(1)}(k, n) = Fd^{(1)}(k, n-k+1) + Fd^{(1)}(k, n-k)$ for $n \geq k$ with $Fd^{(1)}(k, n) = 1$ for $n \leq k-1$.
- (4) Reference [9]. Consider $Fd^{(2)}(k, n) = Fd^{(2)}(k, n-k+1) + Fd^{(2)}(k, n-k)$ for $n \geq k$ with $Fd^{(2)}(k, n) = 0$ for $n = 0, \dots, k-2$, $Fd^{(2)}(k, k-1) = 1$, $Fd^{(2)}(1, 1) = 1$, $Fd^{(2)}(2, 2) = 2$, for $k \geq 3$ $Fd^{(2)}(k, k) = 1$.
- (5) Reference [9]. Consider $Fd^{(3)}(k, n) = Fd^{(3)}(k, n-k+1) + Fd^{(3)}(k, n-k)$ for $n \geq 2k-1$ with $Fd^{(3)}(k, n) = 1$ for $n = 0, \dots, k-1$, $Fd^{(3)}(2, 2) = 2$, for $k \geq 3$ $Fd^{(3)}(k, k) = Fd^{(3)}(k, 2k-2) = 3$, for $k+1 \leq n \leq 2k-1$ $Fd^{(3)}(k, n) = 4$.
- (6) Reference [10]. Consider $F_2^{(1)}(k, n) = F_2^{(1)}(k, n-2) + F_2^{(1)}(k, n-k)$ for $n \geq k+1$ with

$$F_2^{(1)}(k, n) = \begin{cases} 1 & \text{if } n \leq k-1 \text{ or } n = k = 1, \\ 2 & \text{if } n = k \geq 2. \end{cases} \quad (1)$$

TABLE 1: Distance (r, k) -Fibonacci numbers $F_r^{(I)}(k, n)$ of the first kind.

$k \setminus n$	0	1	2	3	4	5	6	7	8
1	1	1	$r + 1$	$2r + 1$	$r^2 + 3r + 1$	$r^2 + 5r + 2$	$r^3 + 4r^2 + 6r + 2$	$2r^3 + 9r^2 + 8r + 2$	$r^4 + 6r^3 + 15r^2 + 10r + 2$
2	1	1	$2r$	$2r$	$4r^2$	$4r^2$	$8r^3$	$8r^3$	$16r^4$
3	1	1	r	$r^2 + r$	$2r^2$	$2r^3 + r^2$	$r^4 + 3r^3$	$4r^4 + r^3$	$3r^5 + 4r^4$
4	1	1	r	r	$r^3 + r^2$	$r^3 + r^2$	$2r^4 + r^3$	$2r^4 + r^3$	$r^6 + 3r^5 + r^4$
5	1	1	r	r	r^2	$r^4 + r^2$	$r^4 + r^3$	$2r^5 + r^3$	$2r^5 + r^4$
6	1	1	r	r	r^2	r^2	$r^5 + r^3$	$r^5 + r^3$	$2r^6 + r^4$
7	1	1	r	r	r^2	r^2	r^3	$r^6 + r^3$	$r^6 + r^4$

(7) Reference [11]. Consider $F_2^{(2)}(k, n) = F_2^{(2)}(k, n - 2) + F_2^{(2)}(k, n - k)$ for $n \geq k + 1$ with

$$F_2^{(2)}(k, n) = \begin{cases} 0 & \text{if } n \text{ is odd and } n \leq k - 1, \\ 1 & \text{if } n \text{ is even and } n \leq k - 1, \end{cases}$$

$$F_2^{(2)}(k, k) = \begin{cases} 0 & \text{if } k = 1, \\ 1 & \text{if } k \text{ is odd and } k \geq 3, \\ 2 & \text{if } k \text{ is even.} \end{cases} \quad (2)$$

(8) Reference [11]. Consider $F_2^{(3)}(k, n) = F_2^{(3)}(k, n - 2) + F_2^{(3)}(k, n - k)$ for $n \geq k + 1$ with

$$F_2^{(3)}(k, n) = \begin{cases} 1 & \text{if } n \text{ is even and } n \leq k - 1, \\ 2 & \text{if } n \text{ is odd and } n \leq k - 1, \end{cases}$$

$$F_2^{(3)}(k, k) = \begin{cases} 3 & \text{if } k \text{ is odd and } k \geq 3, \\ 2 & \text{if } k \text{ is even or } k = 1. \end{cases} \quad (3)$$

In this paper we introduce three new two-parameter generalizations of distance Fibonacci numbers. They are closely related with the numbers $F_2^{(j)}(k, n)$, $j = 1, 2, 3$, presented in [10, 11]. We show their combinatorial and graph interpretations and we present some identities for them.

2. Distance (r, k) -Fibonacci Numbers

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define distance (r, k) -Fibonacci numbers of the first kind $F_r^{(I)}(k, n)$ by the recurrence relation

$$F_r^{(I)}(k, n) = rF_r^{(I)}(k, n - 2) + r^{k-1}F_r^{(I)}(k, n - k) \quad (4)$$

for $n \geq k + 1$

with the following initial conditions:

$$F_r^{(I)}(k, 0) = F_r^{(I)}(k, 1) = 1,$$

$$F_r^{(I)}(k, n) = r^{\lfloor n/2 \rfloor} \quad \text{for } n = 2, 3, \dots, k - 2,$$

$$F_r^{(I)}(k, k - 1) = r^{\lfloor (k-1)/2 \rfloor} \quad \text{for } k \geq 3,$$

$$F_r^{(I)}(k, k) = r^{k-1} + r^{\lfloor k/2 \rfloor} \quad \text{for } k \geq 2. \quad (5)$$

For $r = 1$ we get $F_1^{(I)}(k, n) = F_2^{(I)}(k, n)$. These numbers were introduced in [10].

If $r = 1$ and $k = 1$, then $F_1^{(I)}(1, n)$ gives the Fibonacci numbers F_n . For $r = 1$ and $k = 3$ the numbers $F_1^{(I)}(3, n)$ are the well-known Padovan numbers.

Table 1 includes the values of $F_r^{(I)}(k, n)$ for special values of k and n .

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define the distance (r, k) -Fibonacci numbers of the second kind $F_r^{(II)}(k, n)$ by the following recurrence relation:

$$F_r^{(II)}(k, n) = rF_r^{(II)}(k, n - 2) + r^{k-1}F_r^{(II)}(k, n - k) \quad (6)$$

for $n \geq k + 1$

with initial conditions

$$F_r^{(II)}(k, n) = \begin{cases} r^{\lfloor n/2 \rfloor} & \text{for even } n, \\ 0 & \text{for odd } n, \end{cases}$$

for $n = 0, 1, \dots, k - 1$ (7)

$$F_r^{(II)}(k, k) = \begin{cases} 0 & \text{for } k = 1, \\ r^{k-1} & \text{for odd } k, \ k \geq 3, \\ r^{k-1} + r^{k/2} & \text{for even } k. \end{cases}$$

For $r = 1$ we have then $F_1^{(II)}(k, n) = F_2^{(2)}(k, n)$; see [10]. Moreover, for $r = 1$ and $k = 1$, $n \geq 2$ $F_1^{(II)}(k, n) = F_{n-2}$.

In Table 2 a few first words of the distance (r, k) -Fibonacci numbers of the second kind $F_r^{(II)}(k, n)$ for special values of k and n are presented.

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define distance (r, k) -Fibonacci numbers of the third kind $F_r^{(III)}(k, n)$ by the following recurrence relation:

$$F_r^{(III)}(k, n) = rF_r^{(III)}(k, n - 2) + r^{k-1}F_r^{(III)}(k, n - k) \quad (8)$$

for $n \geq k + 1$

TABLE 2: Distance (r, k) -Fibonacci numbers $F_r^{(II)}(k, n)$ of the second kind.

$k \setminus n$	0	1	2	3	4	5	6	7	8
1	1	0	r	r	$r^2 + r$	$2r^2 + r$	$r^3 + 3r^2 + r$	$3r^3 + 4r^2 + r$	$r^4 + 6r^3 + 5r^2 + r$
2	1	0	$2r$	0	$4r^2$	0	$8r^3$	0	$16r^4$
3	1	0	r	r^2	r^2	$2r^3$	$r^4 + r^3$	$3r^4$	$3r^5 + r^4$
4	1	0	r	0	$r^3 + r^2$	0	$2r^4 + r^3$	0	$r^6 + 3r^5 + r^4$
5	1	0	r	0	r^2	r^4	r^3	$2r^5$	r^4
6	1	0	r	0	r^2	0	$r^5 + r^3$	0	$2r^6 + r^4$
7	1	0	r	0	r^2	0	r^3	r^6	r^4

with initial conditions

$$F_r^{(III)}(1, 1) = 2,$$

$$F_r^{(III)}(k, n) = \begin{cases} r^{n/2} & \text{for even } n, \\ 2r^{\lfloor n/2 \rfloor} & \text{for odd } n, \end{cases} \quad (9)$$

for $n = 0, 1, \dots, k-1$

$$F_r^{(III)}(k, k) = \begin{cases} r^{k-1} + r^{k/2} & \text{for even } k, \\ r^{k-1} + 2r^{\lfloor k/2 \rfloor} & \text{for odd } k \geq 3. \end{cases}$$

For $r = 1$ we get $F_1^{(III)}(k, n) = F_2^{(3)}(k, n)$. These numbers were introduced in [11]. For $r = 1$, $k = 1$, and $n \geq 0$ we have $F_1^{(III)}(1, n) = F_{n+1}$. Moreover, for $r = 1$, $k = 4$, and $n \geq 1$ $F_1^{(III)}(4, 2n) = F_n$.

Table 3 includes a few initial words of distance $F_r^{(III)}(k, n)$ for special values of k and n .

By the definition of distance (r, k) -Fibonacci numbers of three kinds we get for $k \geq 1$ and $n \geq 0$ the following relations:

$$F_r^{(III)}(k, n) = 2F_r^{(I)}(k, n) \quad \text{for even } k \text{ and odd } n,$$

$$F_r^{(I)}(k, n) = F_r^{(II)}(k, n) = F_r^{(III)}(k, n) \quad (10)$$

for even k and even n ,

$$F_r^{(II)}(k, n) = 0 \quad \text{for even } k \text{ and odd } n.$$

3. Combinatorial and Graph Interpretations of Distance (r, k) -Fibonacci Numbers

In this section we present some combinatorial and graph interpretations of distance (r, k) -Fibonacci numbers. The classical Fibonacci numbers have many combinatorial interpretations. One of them is the interpretation related to set decomposition. We recall it. Let $X = \{1, 2, \dots, n\}$, $n \geq 1$, and $\mathcal{Y}^* = \{Y_t^* : t \in T\}$ be a family of disjoint subsets of X such that

- (1) $|Y_t^*| \in \{1, 2\}$,
- (2) if $|Y_t^*| = 2$ then Y_t^* contains two consecutive integers,
- (3) $X \setminus \bigcup_{t \in T} Y_t^* = \emptyset$.

It is well known that the number of all families $\mathcal{Y}^*$ is equal to the classical Fibonacci numbers F_n . We introduce analogous interpretation of distance (r, k) -Fibonacci numbers.

Let $r \geq 1$ and $X = \{1, 2, \dots, n\}$, $n \geq 2$, be the set of n integers. Let $k \geq 3$. Assume that $\mathcal{R}_n$ is a multifamily of two-element subsets of X such that

$$\mathcal{R}_n = \left\{ \underbrace{\{1, 2\}, \{1, 2\}, \dots, \{1, 2\}}_{r\text{-times}}, \underbrace{\{2, 3\}, \dots, \{2, 3\}}_{r\text{-times}}, \dots, \underbrace{\{n-1, n\}, \dots, \{n-1, n\}}_{r\text{-times}} \right\}. \quad (11)$$

For fixed t , $1 \leq t \leq n-k$ by $\mathcal{R}(k, t)$ we denote a subfamily of $\mathcal{R}_n$ such that $\mathcal{R}(k, t) = \{\{t+j, t+j+1\} : j = 0, 1, \dots, k-2, t = 1, 2, \dots, n-k+1\}$. Analogously for fixed t' we define $\mathcal{R}(k, t') = \{\{t, t+1\} : t = 1, 2, \dots, n-1\}$.

Let $\mathcal{R}_{t,t'}^{(j)}$, $j = 1, 2, 3$, be a subfamily of $\mathcal{R}_n$ such that $\mathcal{R}_{t,t'}^{(j)} = \mathcal{R}(k, t) \cup \mathcal{R}(k, t')$ and

- (a) for each $R(k, t'_1), R(k, t'_2) \in \mathcal{R}(k, t')$, $t'_1 \neq t'_2$ holds $|t'_1 - t'_2| \geq 2$, for each $R(k, t_1), R(k, t_2) \in \mathcal{R}(k, t)$, $t_1 \neq t_2$, holds $|t_1 - t_2| \geq k$,
- (b) for each $R_t \in \mathcal{R}_{t,t'}^{(j)}$, holds $|R_t| \in \{2, k\}$ for $t \in T$

and exactly one of the following conditions for $R_t \in \mathcal{R}_{t,t'}^{(j)}$ and $j = \text{I, II, III}$, respectively, is satisfied:

- (c1) $X \setminus \bigcup_{t \in T} R_t = \emptyset$ or $X \setminus \bigcup_{t \in T} R_t = \{n\}$,
- (c2) $X \setminus \bigcup_{t \in T} R_t = \emptyset$,
- (c3) $|X \setminus \bigcup_{t \in T} R_t| \in \{0, 1\}$ and if $p \in X \setminus \bigcup_{t \in T} R_t$ then either $p = 1$ or $p = n$.

Assume that the condition (c1) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(1)}$ we will call a decomposition with repetitions of the set X with the rest at the end.

Assume that the condition (c2) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(2)}$ we will call a perfect decomposition with repetitions of the set X .

Assume that the condition (c3) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(3)}$ we will call a decomposition with repetitions of the set X with the rest at the end or at the beginning.

Theorem 1. Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Then the number of all decompositions with repetitions of the set X with the rest at the end is equal to the number $F_r^{(I)}(k, n)$.

TABLE 3: Distance (r, k) -Fibonacci numbers $F_r^{(III)}(k, n)$ of the third kind.

$k \setminus n$	0	1	2	3	4	5	6	7	8
1	1	2	$r+2$	$3r+2$	r^2+5r+2	$4r^2+7r+2$	r^3+9r^2+9r+2	$5r^3+9r^2+16r+4$	$r^4+14r^3+18r^2+18r+4$
2	1	2	$2r$	$4r$	$4r^2$	$8r^2$	$8r^3$	$16r^3$	$16r^4$
3	1	2	r	r^2+2r	$3r^2$	$2r^3+2r^2$	r^4+5r^3	$5r^4+2r^3$	$3r^5+7r^4$
4	1	2	r	$2r$	r^3+r^2	$2r^3+2r^2$	$2r^4+r^3$	$4r^4+2r^3$	$r^6+3r^5+r^4$
5	1	2	r	$2r$	r^2	r^4+2r^2	$2r^4+r^3$	$2r^5+2r^3$	$4r^5+r^4$
6	1	2	r	$2r$	r^2	$2r^2$	r^5+r^3	$2r^5+2r^3$	$2r^6+r^4$
7	1	2	r	$2r$	r^2	$2r^2$	r^3	r^6+2r^3	$2r^6+2r^4$

Proof (induction on n). Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Let $X = \{1, 2, \dots, n\}$. Denote by $d(n)$ the number of all decompositions with repetitions of X with the rest at the end. Let $n = 2$. Then it is easily seen that there are exactly r decompositions of X . Thus we get $d(2) = r = F_r^{(I)}(k, 2)$. Let $n \geq 3$. Assume that equality $d(n) = F_r^{(I)}(k, n)$ holds for an arbitrary n . We will show that $d(n+1) = F_r^{(I)}(k, n+1)$.

Let $d^2(n+1)$ and $d^k(n+1)$ denote the number of all decompositions R with repetitions of the set $X = \{1, 2, \dots, n+1\}$ with the rest at the end such that $\{1, 2\} \in R$ and $\{1, 2, \dots, k\} \in R$, respectively. It is easily seen that

$$d(n+1) = d^2(n+1) + d^k(n+1). \quad (12)$$

Moreover, we get

$$\begin{aligned} d^2(n+1) &= d^k(n-1), \\ d^k(n+1) &= d^k(n+1-k). \end{aligned} \quad (13)$$

By the induction hypothesis and by recurrence (4) we obtain

$$\begin{aligned} d(n+1) &= d(n-1) + d(n+1-k) \\ &= F_r^{(I)}(k, n-1) + F_r^{(I)}(k, n+1-k) \\ &= F_r^{(I)}(k, n+1), \end{aligned} \quad (14)$$

which ends the proof. $\square$

Analogously as Theorem 1 we can prove the following.

Theorem 2. Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Then the number of all perfect decompositions with repetitions of the set X is equal to the number $F_r^{(II)}(k, n)$.

Theorem 3. Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Then the number of all decompositions with repetitions of the set X with the rest at the end or at the beginning is equal to the number $F_r^{(III)}(k, n)$.

Distance (r, k) -Fibonacci numbers of three kinds have a graph interpretation, too. It is connected with k -distance H -matchings in graphs. We recall the definition of a k -distance H -matching. Let G and H be any two graphs, let $k \geq 1$ be an integer, and a k -distance H -matching M of G is a subgraph of G such that all connected components of M are isomorphic

to H and for each two components H_1 and H_2 from M for each $x \in V(H_1)$ and $y \in V(H_2)$ holds $d_G(x, y) \geq k$. In case of $k = 1$ and $H = K_2$ we obtain the definition of matching in classical sense. If M covers the set $V(G)$ (i.e., $V(M) = V(G)$), then we say that M is a perfect matching of G . For $k = 2$ and $H = K_1$ the definition of k -distance H -matchings reduces to the definition of an independent set of a graph G . In the literature the generalization of H -matching of a graph G is considered, too. For a given collection $\mathcal{H} = H_1, H_2, \dots, H_n$ of graphs a $\mathcal{H}$ -matching $\mathcal{M}$ of G is a family of subgraphs of G such that each connected component of $\mathcal{M}$ is isomorphic to some H_i , $1 \leq i \leq n$. Moreover, the empty set is a $\mathcal{H}$ -matching of G , too. If $H_i = H$ for all $i = 1, 2, \dots, n$, then we obtain the definition of H -matching.

Among $\mathcal{H}$ -matchings we consider such $\mathcal{H}$ -matchings, where H_i , $i = 1, 2, \dots, n$, belong to the same class of graphs, namely, 2-vertex or k -vertex paths (P_2 and P_k , resp.), $k \geq 3$.

Consider a multipath P_n^r , where $n \geq 2$, $r \geq 1$, $V(P_n^r) = \{x_1, x_2, \dots, x_n\}$, and

$$\begin{aligned} E(P_n^r) &= \left\{ \underbrace{\{x_1, x_2\}, \dots, \{x_1, x_2\}}_{r\text{-times}}, \right. \\ &\quad \underbrace{\{x_2, x_3\}, \dots, \{x_2, x_3\}}_{r\text{-times}}, \dots, \\ &\quad \left. \underbrace{\{x_{n-1}, x_n\}, \dots, \{x_{n-1}, x_n\}}_{r\text{-times}} \right\}. \end{aligned} \quad (15)$$

Let $n \geq 2$, $k \geq 3$, and $r \geq 1$ be integers. In the graph terminology the number $F_r^{(I)}(k, n)$ is equal to the number of special $\{P_2, P_k\}$ -matchings M of the multipath P_n^r such that at most one vertex, namely, x_n , does not belong to a $\{P_2, P_k\}$ -matching of the graph P_n^r . We will call such matchings M a quasi-perfect matching of P_n^r . The number $F_r^{(II)}(k, n)$ is equal to the number of such $\{P_2, P_k\}$ -matchings of P_n^r that both vertex x_1 and vertex x_n belong to some $\{P_2, P_k\}$ -matchings M and M' , respectively, of the graph P_n^r . In other words the number $F_r^{(III)}(k, n)$ is equal to all perfect $\{P_2, P_k\}$ -matchings M of the graph P_n^r .

The number $F_r^{(III)}(k, n)$ is equal to the number of special $\{P_k, P_2\}$ -matchings of the multipath P_n^r such that at most one vertex either vertex x_1 or x_n does not belong to a $\{P_2, P_k\}$ -matching of the graph P_n^r .

Let $\sigma(P_n^r)$ be the number of all perfect $\{P_2, P_k\}$ -matchings M of the graph P_n^r .

Theorem 4. Let $r \geq 1$, $k \geq 3$, and $n \geq 2$ be integers. Then $\sigma(P_n^r) = F_r^{(II)}(k, n)$.

Proof. Consider a multipath P_n^r where vertices from $V(P_n^r) = \{x_1, x_2, \dots, x_n\}$ are numbered in the natural fashion. Let $\sigma_k(n)$ and $\sigma_2(n)$ be the number of perfect $\{P_2, P_k\}$ -matchings M of P_n^r such that $x_n, x_{n-1} \in V(M)$ and $x_n, x_{n-1}, \dots, x_{n-k} \in V(M)$, respectively. It is easily seen that $\sigma_k(n) + \sigma_2(n) = \sigma(P_n^r)$.

Let M be an arbitrary perfect $\{P_2, P_k\}$ -matching of P_n^r , $k \geq 3$. Consider two cases:

$$(1) \{x_{n-1}, x_n\} \in E(P_k), \text{ where } P_k \in M.$$

Then we can choose the edge $\{x_{n-1}, x_n\}$ on r ways. Moreover, $M = M' \cup \{P_k\}$, where M' is an arbitrary $\{P_2, P_k\}$ -matching of the graph $P_n^r \setminus \{x_n, x_{n-1}, \dots, x_{n-k+1}\}$ which is isomorphic to the multipath P_{n-k}^r . Hence $\sigma_k(n) = r^{k-1} \sigma(P_{n-k}^r)$.

$$(2) \{x_{n-1}, x_n\} \in E(P_2), \text{ where } P_2 \in M.$$

Proving analogously as in case (1) we obtain $\sigma_2(n) = r \sigma(P_{n-2}^r)$.

Consequently

$$\sigma(P_n^r) = \sigma_k(n) + \sigma_2(n) = r^{k-1} \sigma(P_{n-k}^r) + r \sigma(P_{n-2}^r). \quad (16)$$

Claim

$$\sigma(P_n^r) = r^{k-1} F_r^{(II)}(k, n-k) + r F_r^{(II)}(k, n-2). \quad (17)$$

Proof. Assume now that the set $X = \{1, 2, \dots, n\}$ corresponds to $V(P_n^r)$ with the numbering in the natural fashion. Let $\mathcal{R}(t, t') = \{R_t : t \in T\} \cup \{R'_t : t' \in T\}$ be a multifamily of X which gives a perfect decomposition of the set X . Then every R_t and R'_t correspond to subgraph $P_{|R_t|}$ and $P_{|R'_t|}$ for $t, t' \in T$, respectively, of P_n^r . By Theorem 2 we get

$$\begin{aligned} \sigma(P_n^r) &= \sigma_k(n) + \sigma_2(n) \\ &= r^{k-1} F_r^{(II)}(k, n-k) + r F_r^{(II)}(k, n-2). \end{aligned} \quad (18)$$

Moreover, by (6) we obtain $\sigma(P_n^r) = F_r^{(II)}(k, n)$, which ends the proof. $\square$

Analogously we can prove combinatorial interpretations of numbers $F_r^{(I)}(k, n)$ and $F_r^{(III)}(k, n)$.

4. Identities for Distance (r, k) -Fibonacci Numbers

In this section we give some identities and some relations between distance (r, k) -Fibonacci numbers of three types.

Theorem 5. For $k \geq 1$, $n \geq 2k-2$, and $j = I, II, III$,

$$\begin{aligned} F_r^{(j)}(k, n) &= r F_r^{(j)}(k, n-2) + r^{k-2} F_r^{(j)}(k, n-k+2) \\ &\quad - r^{2k-3} F_r^{(j)}(k, n-2k+2). \end{aligned} \quad (19)$$

Proof. We give the proof for distance (r, k) -Fibonacci numbers of the first kind. By the definition of numbers $F_r^{(I)}(k, n)$, we have

$$\begin{aligned} r F_r^{(I)}(k, n-2) + r^{k-2} F_r^{(I)}(k, n-k+2) \\ - r^{2k-3} F_r^{(I)}(k, n-2k+2) &= r F_r^{(I)}(k, n-2) \\ + r^{k-2} (r F_r^{(I)}(k, n-k) + r^{k-1} F_r^{(I)}(k, n-2k+2)) \\ - r^{2k-3} F_r^{(I)}(k, n-2k+2) &= r F_r^{(I)}(k, n-2) \\ + r^{k-1} F_2^{(I)}(k, n-k) &= F_r^{(I)}(k, n), \end{aligned} \quad (20)$$

which ends the proof. $\square$

Corollary 6. For $n \geq 2$ $F_n = (1/2)(F_{n-2} + F_{n+1})$.

Proof. For $r = 1$, $j = I$, and $k = 1$ by (19) we obtain

$$F_1^{(I)}(1, n) = F_n = F_{n-2} + F_{n+1} - F_n. \quad (21)$$

Hence

$$F_n = \frac{1}{2} (F_{n-2} + F_{n+1}). \quad (22)$$

$\square$

Theorem 7. For $r \geq 1$, $k \geq 2$, and $n \geq 1$,

$$F_r^{(I)}(k, n) = F_r^{(II)}(k, n) + F_r^{(I)}(k, n-1). \quad (23)$$

Proof (induction on n). For $n = 1$ we have

$$F_r^{(I)}(k, 1) = 1 = F_r^{(II)}(k, 1) + F_r^{(I)}(k, 0). \quad (24)$$

Assume that equality (23) is true for an arbitrary n . We will prove it for $n+1$. By the recurrence (6) and by induction hypothesis we get

$$\begin{aligned} F_r^{(I)}(k, n+1) &= r F_r^{(I)}(k, n-1) + r^{k-1} F_r^{(I)}(k, n+1-k) \\ &= r (F_r^{(II)}(k, n-1) + F_r^{(I)}(k, n-2)) \\ &\quad + r^{k-1} (F_r^{(II)}(k, n+1-k) + F_r^{(I)}(k, n-k)) \\ &= r F_r^{(II)}(k, n-1) + r^{k-1} F_r^{(II)}(k, n+1-k) \\ &\quad + r F_r^{(I)}(k, n-2) + r^{k-1} F_2^{(II)}(k, n-k) \\ &= F_r^{(II)}(k, n+1) + F_r^{(I)}(k, n), \end{aligned} \quad (25)$$

which ends the proof. $\square$

Analogously we can prove the following.

Theorem 8. For $r \geq 1$, $k \geq 3$, and $n \geq 0$,

$$2 F_r^{(I)}(k, n) = F_r^{(II)}(k, n) + F_r^{(III)}(k, n). \quad (26)$$

Theorem 9. For $r \geq 1$, $k \geq 2$, $n \geq 2k$, and $j = I, II, III$,

$$\begin{aligned} F_r^{(j)}(k, n) &= r^2 F_r^{(j)}(k, n-4) + 2 r^k F_r^{(j)}(k, n-k-2) \\ &\quad + r^{2k-2} F_r^{(j)}(k, n-2k). \end{aligned} \quad (27)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author would like to thank the referee for helpful comments and suggestions for improving an earlier version of this paper.

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