Research Article

Bounds on the Spectral Radius of a Nonnegative Matrix and Its Applications

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We obtain the sharp bounds for the spectral radius of a nonnegative matrix and then obtain some known results or new results by applying these bounds to a graph or a digraph and revise and improve two known results.

1. Introduction

First we recall some basic definitions and notations that will be used in this paper. Let A be an $n \times n$ real matrix and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A . Since A is not symmetric in general, the eigenvalues may be complex numbers. Without loss of generality, we assume that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$, and then the spectral radius of A is defined as $\rho(A) = |\lambda_1|$; that is, it is the largest modulus of the eigenvalues of A . By the Perron-Frobenius theorem, we have the following: (1) $\rho(A)$ is an eigenvalue of A if A is a nonnegative matrix; (2) $\rho(A) = \lambda_1$ is simple if A is a nonnegative irreducible matrix.

Let $G = (V, E)$ ($\vec{G} = (V, E)$) be a graph (digraph) with vertex set $V = V(G)$ ($= V(\vec{G})$) $= \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$ (arc set $E = E(\vec{G})$). A graph G (digraph $\vec{G}$) is simple if it has no loops and multiple edges (arcs). For any pairs of vertices $v_i, v_j \in V$, if there is a (directed) path from v_i to v_j , the graph G (digraph $\vec{G}$) is called (strongly) connected. In this paper, we consider finite, simple graphs and digraphs.

Let G be a graph and $\text{diag}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees of G , where d_i is the degree of vertex v_i .

Let $\vec{G}$ be a digraph; $N_G^-(v_i) = \{v_j \in V(\vec{G}) \mid (v_j, v_i) \in E(\vec{G})\}$ and $N_G^+(v_i) = \{v_j \in V(\vec{G}) \mid (v_i, v_j) \in E(\vec{G})\}$ denote the in-neighbors and out-neighbors of v_i , respectively. Let $d_i^- = |N_G^-(v_i)|$ and $d_i^+ = |N_G^+(v_i)|$ denote the indegree and

outdegree of the vertex v_i in $\vec{G}$, respectively, and $\text{diag}(\vec{G}) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$ be the diagonal matrix of the vertex outdegrees of $\vec{G}$.

Let $A(G) = (a_{ij})$ be the $(0, 1)$ adjacency matrix of G , where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Let $A(\vec{G}) = (a_{ij})$ denote the adjacency matrix of $\vec{G}$, where a_{ij} is equal to the number of arcs (v_i, v_j) .

Then the signless Laplacian matrix of G ($\vec{G}$) is defined as

$$\begin{aligned} Q(G) &= \text{diag}(G) + A(G) \\ (Q(\vec{G}) &= \text{diag}(\vec{G}) + A(\vec{G})). \end{aligned} \quad (2)$$

The spectral radii of $A(G)$ and $Q(G)$ ($A(\vec{G})$ and $Q(\vec{G})$), denoted by $\rho(G)$ and $q(G)$ ($\rho(\vec{G})$ and $q(\vec{G})$), are called the (adjacency) spectral radius of G ($\vec{G}$) and the signless Laplacian spectral radius of G ($\vec{G}$), respectively.

Let $G = (V, E)$ be a connected graph and $\vec{G} = (V, E)$ be a strong connected digraph. For $u, v \in V$, the distance from u to v , denoted by $d_G(u, v)$ ($d_{\vec{G}}(u, v)$), is the length of the shortest (directed) path from u to v in G ($\vec{G}$). For $u \in V$, the transmission of vertex u in G ($\vec{G}$) is the sum of distances from u to all other vertices of G ($\vec{G}$), denoted by $\text{Tr}_G(u)$ ($\text{Tr}_{\vec{G}}(u)$).

The distance matrix of G ($\vec{G}$) is the $n \times n$ matrix $\mathcal{D}(G)$ ($\mathcal{D}(\vec{G})$), where $d_{ij} = d_G(v_i, v_j)$ ($\mathcal{D}(\vec{G}) = (d_{ij})$, where $d_{ij} = d_{\vec{G}}(v_i, v_j)$). In fact, for $1 \leq i \leq n$, the transmission of vertex v_i , $\text{Tr}_G(v_i)$ ($\text{Tr}_{\vec{G}}(v_i)$), is just the i th row sum of $\mathcal{D}(G)$ ($\mathcal{D}(\vec{G})$). For convenience, we also call $\text{Tr}_G(v_i)$ ($\text{Tr}_{\vec{G}}(v_i)$) the distance degree (outdegree) of vertex v_i in G ($\vec{G}$), denoted by D_i (D_i^+); that is, $D_i = \sum_{j=1}^n d_{ij} = \text{Tr}_G(v_i)$ ($D_i^+ = \sum_{j=1}^n d_{ij} = \text{Tr}_{\vec{G}}(v_i)$). Similarly, we define $D_i^- = \sum_{j=1}^n d_{ji}$.

Let $\text{Tr}(G) = \text{diag}(D_1, D_2, \dots, D_n)$ be the diagonal matrix of vertex transmissions of G , and let $\text{Tr}(\vec{G}) = \text{diag}(D_1^+, D_2^+, \dots, D_n^+)$ be the diagonal matrix of vertex transmissions of $\vec{G}$. The distance signless Laplacian matrix of G ($\vec{G}$) is the $n \times n$ matrix defined by Aouchiche and Hansen as [1]

$$\begin{aligned} \mathcal{Q}(G) &= \text{Tr}(G) + \mathcal{D}(G) \\ \mathcal{Q}(\vec{G}) &= \text{Tr}(\vec{G}) + \mathcal{D}(\vec{G}). \end{aligned} \quad (3)$$

The spectral radii of $\mathcal{D}(G)$ and $\mathcal{Q}(G)$ ($\mathcal{D}(\vec{G})$ and $\mathcal{Q}(\vec{G})$), denoted by $\rho^{\mathcal{D}}(G)$ and $q^{\mathcal{D}}(G)$ ($\rho^{\mathcal{D}}(\vec{G})$ and $q^{\mathcal{D}}(\vec{G})$), are called the distance spectral radius of G ($\vec{G}$) and the distance signless Laplacian spectral radius of G ($\vec{G}$), respectively.

Let G be a connected graph. The reciprocal distance matrix (also called the Harary matrix) $R(G) = (r_{ij})$ of G is the $n \times n$ matrix, where $(r_{ij}) = 1/d_{ij}$ if $i \neq j$ and $r_{ii} = 0$ for $i = 1, \dots, n$. Clearly, the reciprocal distance matrix $R(G)$ is nonnegative and symmetric.

Let G be a graph and $\vec{G}$ be a digraph; we call G ($\vec{G}$) regular if each vertex of G ($\vec{G}$) has the same degree (outdegree). Other definitions, terminology, and notations not in the article can be found in [2–4].

In recent decades, there are many results on the bounds of the spectral radius of a nonnegative matrix and the various spectral radii of a graph or a digraph, including the spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance signless Laplacian spectral radius, and the spectral radius of the reciprocal distance matrix; see [5–16] and so on.

In this paper, we obtain the sharp bounds for the spectral radius of a nonnegative (irreducible) matrix in Section 2 and then obtain some known results or new results by applying these bounds to a graph in Section 3 or a digraph in Section 4; we revise and improve two known results.

2. Main Results

In this section, we will obtain the sharp bounds for the spectral radius of a nonnegative (irreducible) matrix and revise and improve the result of Theorem 2.9 in [9]. The techniques used in this section are motivated by [7, 9, 14] and so on.

Lemma 1 (see [2]). *If A is an $n \times n$ nonnegative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \dots, r_n$, then $\min_{1 \leq i \leq n} r_i \leq \lambda(A) \leq \max_{1 \leq i \leq n} r_i$. Moreover, if A is irreducible, then one of the equalities holds if and only if the row sums of A are all equal.*

Theorem 2. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \dots, r_n$, where $r_1 \geq r_2 \geq \dots \geq r_n$, and let S be the smallest diagonal element, T be the smallest nondiagonal element, and $\lambda(A)$ be the spectral radius of A . Take $\phi_1 = r_n$ and for $2 \leq l \leq n$,*

$$\begin{aligned} \phi_l &= \frac{r_n + S - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2}. \end{aligned} \quad (4)$$

Let $\phi_t = \max_{1 \leq l \leq n} \{\phi_l\}$ for some $1 \leq t \leq n$. Then $\lambda(A) \geq \phi_t$. Moreover, if A is irreducible, then

- (1) $\lambda(A) = \phi_1 = r_n$ if and only if $r_1 = r_2 = \dots = r_n$.
- (2) $\lambda(A) = \phi_t > r_n$ with $2 \leq t \leq n$ if and only if A satisfies the following conditions:

- (i) $a_{ii} = S$ for $1 \leq i \leq t-1$;
- (ii) $a_{ij} = T > 0$ for $1 \leq i \leq n, 1 \leq j \neq i \leq t-1$;
- (iii) $r_1 = r_2 = \dots = r_{t-1} > r_t = r_{t+1} = \dots = r_n$.

Proof. If $T = 0$, then $\phi_l = \phi_1 = r_n$ for any $2 \leq l \leq n$ by $r_n \geq S$. Thus by Lemma 1 and $r_1 \geq r_2 \geq \dots \geq r_n$, we have $\lambda(A) \geq r_n = \max_{1 \leq l \leq n} \{\phi_l\} = \phi_1$, and if A is irreducible, $\lambda(A) = \phi_1 = r_n$ if and only if $r_1 = r_2 = \dots = r_n$.

Now we consider the case $T > 0$.

Firstly, we show $\lambda(A) \geq \phi_l$ for all $2 \leq l \leq n$.

Since A is a nonnegative matrix, then $a_{p,q} \geq T > 0$ for $1 \leq p \neq q \leq n$. Thus

$$\sum_{j=1}^{l-1} a_{ij} \geq \begin{cases} S + (l-2)T, & \text{if } 1 \leq i \leq l-1; \\ (l-1)T, & \text{if } l \leq i \leq n. \end{cases} \quad (5)$$

Let

$$x = \frac{S - r_n + (2l-3)T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2(l-1)T}. \quad (6)$$

It is easy to show that $x > 1$. Take

$$x_j = \begin{cases} x, & \text{if } 1 \leq j \leq l-1, \\ 1, & \text{if } l \leq j \leq n, \end{cases} \quad (7)$$

and let $\mathbf{U} = \text{diag}(x_1, x_2, \dots, x_n)$ be a diagonal matrix of order n . Let $B = \mathbf{U}^{-1}A\mathbf{U}$, and then B and A have the same eigenvalues, and $\lambda(B) = \lambda(A)$.

Now we consider the row sums of B , say, $s_1, s_2, \dots, s_n$.

Case 1 ($1 \leq i \leq l-1$). Consider

$$\begin{aligned} s_i &= \sum_{j=1}^n \frac{x_j}{x_i} a_{ij} = \sum_{j=1}^{l-1} a_{ij} + \frac{1}{x} \sum_{j=l}^n a_{ij} \\ &= \frac{1}{x} \sum_{j=1}^n a_{ij} + \left(1 - \frac{1}{x}\right) \sum_{j=1}^{l-1} a_{ij} = \frac{1}{x} r_i + \left(1 - \frac{1}{x}\right) \sum_{j=1}^{l-1} a_{ij} \\ &\geq \frac{1}{x} r_i + \left(1 - \frac{1}{x}\right) [S + (l-2)T] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x} (r_i - S) + S + \left(1 - \frac{1}{x}\right) (l-2) T \\
&\geq \frac{1}{x} (r_{l-1} - S) + S + \left(1 - \frac{1}{x}\right) (l-2) T,
\end{aligned} \tag{8}$$

with equality if and only if (a) and (b) hold: (a) $a_{ii} = S$ and $a_{ij} = T$ if $1 \leq j \leq l-1$ with $j \neq i$ and (b) $r_i = r_{l-1}$.

Case 2 ($l \leq i \leq n$). Consider

$$\begin{aligned}
s_i &= \sum_{j=1}^n \frac{x_j}{x_i} a_{ij} = x \sum_{j=1}^{l-1} a_{ij} + \sum_{j=l}^n a_{ij} \\
&= \sum_{j=1}^n a_{ij} + (x-1) \sum_{j=1}^{l-1} a_{ij} = r_i + (x-1) \sum_{j=1}^{l-1} a_{ij} \\
&\geq r_i + (x-1) (l-1) T \geq r_n + (x-1) (l-1) T,
\end{aligned} \tag{9}$$

with equality if and only if (c) and (d) hold: (c) $a_{ij} = T$ if $1 \leq j \leq l-1$ and (d) $r_i = r_n$.

Noting that

$$\begin{aligned}
&r_n + (x-1) (l-1) T \\
&= \frac{1}{x} (r_{l-1} - S) + S + \left(1 - \frac{1}{x}\right) (l-2) T \\
&= \frac{S + r_n - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2} \\
&= \phi_l,
\end{aligned} \tag{10}$$

then, by Lemma 1, we have $\lambda(A) = \lambda(B) \geq \min\{s_1, s_2, \dots, s_n\} \geq \phi_l$.

Noting that $\phi_l \geq \phi_1 = r_n$ by $r_n + T \geq S$, thus $\lambda(A) \geq \phi_t$, where $\phi_t = \max_{1 \leq l \leq n} \{\phi_l\}$ for some $1 \leq t \leq n$.

Let A be irreducible; $\phi_t = \max_{1 \leq l \leq n} \{\phi_l\}$ for some $1 \leq t \leq n$.

Case 1 ($\lambda(A) = \phi_1$). For $2 \leq l \leq n$, by $\phi_l \geq \phi_1$ and $T > 0$, we have $\phi_l = \phi_1 \Leftrightarrow r_{l-1} = r_n$. Then

$$\phi_t = \phi_1 \Leftrightarrow \phi_l = \phi_1 \quad \forall 2 \leq l \leq n \Leftrightarrow r_1 = r_2 = \dots = r_n. \tag{11}$$

On the other hand, by Lemma 1 and $r_1 \geq r_2 \geq \dots \geq r_n$, we have

$$\lambda(A) = r_n \Leftrightarrow r_1 = r_2 = \dots = r_n. \tag{12}$$

By (11), (12), and $\phi_1 = r_n$, (1) holds.

Case 2 ($\lambda(A) = \phi_t > \phi_1$ for some $2 \leq t \leq n$). Then $r_{t-1} > r_n$ and $T > 0$ by $\phi_t > \phi_1 = r_n$.

If $\lambda(A) = \phi_t$, then $s_1 = s_2 = \dots = s_n = \phi_t$ by the above arguments and Lemma 1; thus (a) and (b) hold for $1 \leq i \leq t-1$ and (c) and (d) hold for $t \leq i \leq n$. Thus $a_{ii} = S$ for $1 \leq i \leq t-1$, $r_1 = r_2 = \dots = r_{t-1} > r_t = r_{t+1} = \dots = r_n$ and $a_{ij} = T > 0$ for $1 \leq i \leq n$, $1 \leq j \neq i \leq t-1$. Now (i), (ii), and (iii) follow.

Conversely, if (i), (ii), and (iii) hold, it is easy to show that equality holds. $\square$

Corollary 3. Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \dots, r_n$, where $r_1 \geq r_2 \geq \dots \geq r_n$, and let S be the smallest diagonal element, T be the smallest nondiagonal element, and $\lambda(A)$ be the spectral radius of A . Take $\phi_1 = r_n$ and, for $2 \leq l \leq n$,

$$\begin{aligned}
&\phi_l \\
&= \frac{r_n + S - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2}.
\end{aligned} \tag{13}$$

Let $\phi_t = \max_{1 \leq l \leq n} \{\phi_l\}$ for some $1 \leq t \leq n$. Then $\lambda(A) \geq \phi_t$. Moreover, if A is irreducible with $T = 0$ or A is irreducible and symmetric, then

$$\lambda(A) = \phi_t \quad \text{iff } t = 1, \quad r_1 = r_2 = \dots = r_n. \tag{14}$$

Proof. We complete the proof by the following two cases.

Case 1 ($T = 0$). It is obvious by the proof of Theorem 2.

Case 2 (A is symmetric and $T > 0$). By (i) and (ii), A is symmetric and T is the smallest nondiagonal element. We have $r_1 = r_2 = \dots = r_{t-1} = S + (n-1)T < r_t = \dots = r_n$. It is a contradiction by the fact $r_{t-1} \geq r_t$. $\square$

Similar to the proof of Theorem 2 (so we omit the proof of Theorem 4), we can show Theorem 4 which revises and improves the result of Theorem 2.9 in [9].

Theorem 4. Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \dots, r_n$, where $r_1 \geq r_2 \geq \dots \geq r_n$, and let M be the largest diagonal element, N be the largest nondiagonal element, and $\lambda(A)$ be the spectral radius of A . Take $\phi_1 = r_1$ and, for $2 \leq l \leq n$,

$$\begin{aligned}
&\phi_l \\
&= \frac{r_l + M - N + \sqrt{(r_l + N - M)^2 + 4(l-1)(r_1 - r_l)N}}{2}.
\end{aligned} \tag{15}$$

Let $\phi_t = \min_{1 \leq l \leq n} \{\phi_l\}$ for some $1 \leq t \leq n$. Then $\lambda(A) \leq \phi_t$. Moreover, if A is irreducible, then

- (1) $\lambda(A) = \phi_1 = r_1$ if and only if $r_1 = r_2 = \dots = r_n$.
- (2) $\lambda(A) = \phi_t < r_1$ with $2 \leq t \leq n$ if and only if A satisfies the following conditions:

- (i) $a_{ii} = M$ for $1 \leq i \leq t-1$;
- (ii) $a_{ij} = N > 0$ for $1 \leq i \leq n$, $1 \leq j \neq i \leq t-1$;
- (iii) $r_1 = r_2 = \dots = r_{t-1} > r_t = r_{t+1} = \dots = r_n$.

3. Various Spectral Radii of a Graph

Let G be a graph. In Section 1, the (adjacency) matrix $A(G)$, the signless Laplacian matrix $Q(G)$, the distance matrix $\mathcal{D}(G)$ (if G is connected), the distance signless Laplacian matrix $\mathcal{Q}(G)$ (if G is connected), the reciprocal distance matrix $R(G)$ (if G is connected), the (adjacency) spectral radius $\rho(G)$, the signless Laplacian spectral radius $q(G)$, the distance spectral

radius $\rho^{\mathcal{D}}(G)$, the distance signless Laplacian spectral radius $q^{\mathcal{D}}(G)$, and the spectral radius of the reciprocal distance matrix $\lambda(R(G))$ are defined. Now, in this section, we will apply Theorem 2, Corollary 3, and Theorem 4 to $A(G)$, $Q(G)$, $\mathcal{D}(G)$, $\mathcal{Q}(G)$, and $R(G)$ and obtain some new results or known results.

3.1. Adjacency Spectral Radius of a Graph. Let G be a graph. By applying Corollary 3 and Theorem 4 to the (adjacency) matrix $A(G)$ with $S = 0$, $T = 0$, $M = 0$, $N = 1$, and $r_i = d_i$ for any $1 \leq i \leq n$, we have the following.

Corollary 5. Let G be a graph on n vertices with degree sequence $d_1, d_2, \dots, d_n$, where $d_1 \geq d_2 \geq \dots \geq d_n$. Then one has

$$d_n \leq \rho(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_i - 1 + \sqrt{(d_i + 1)^2 + 4(i-1)(d_1 - d_i)}}{2} \right\}. \quad (16)$$

Moreover, if G is connected, then the left equality holds if and only if G is a regular graph, the right equality holds if and only if G is a regular graph, or there exists some t with $2 \leq t \leq n$ such that G is a bidegreed graph with $d_1 = \dots = d_{t-1} = n-1 > d_t = \dots = d_n$.

Remark 6. The left inequality in Corollary 5 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 5 is the result of Theorem 2.2 in [13].

3.2. Signless Laplacian Spectral Radius of a Graph. Let G be a graph. By applying Corollary 3 and Theorem 4 to the signless Laplacian matrix $Q(G)$ with $S = d_n$, $T = 0$, $M = d_1$, $N = 1$, and $r_i = 2d_i$ for any $1 \leq i \leq n$, we have the following.

Corollary 7. Let G be a graph on n vertices with degree sequence $d_1, d_2, \dots, d_n$, where $d_1 \geq d_2 \geq \dots \geq d_n$. Then one has

$$2d_n \leq q(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}. \quad (17)$$

Moreover, if G is connected, then the left equality holds if and only if G is a regular graph, the right equality holds if and only if G is a regular graph, or there exists some t with $2 \leq t \leq n$ such that G is a bidegreed graph in which $d_1 = \dots = d_{t-1} = n-1 > d_t = \dots = d_n$.

Remark 8. The left inequality in Corollary 7 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 7 is the result of Theorem 3.2 in [15].

3.3. Distance Spectral Radius of a Graph. Let G be a connected graph and d be the diameter of G . Then the distance matrix $\mathcal{D}(G) = (d_{ij})$ is nonnegative and symmetric. By applying Corollary 3 and Theorem 4 to the distance matrix $\mathcal{D}(G)$ with $S = 0$, $T = 1$, $M = 0$, $N = d$, and $r_i = D_i$ for any $1 \leq i \leq n$, we note that $d_{21} = \dots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 9. Let G be a connected graph on n vertices and d be the diameter of G , with distance degree sequence $D_1, D_2, \dots, D_n$ such that $D_1 \geq D_2 \geq \dots \geq D_n$. Let

$$f(i) = \frac{D_n - 1 + \sqrt{(D_n + 1)^2 + 4(i-1)(D_{i-1} - D_n)}}{2}. \quad (18)$$

Then one has

$$\max_{2 \leq i \leq n} \{D_n, f(i)\} \leq \rho^{\mathcal{D}}(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{D_i - d + \sqrt{(D_i + d)^2 + 4d(i-1)(D_1 - D_i)}}{2} \right\}. \quad (19)$$

Moreover, one of the equalities holds if and only if $D_1 = D_2 = \dots = D_n$.

Remark 10. The right inequality in Corollary 9 is the result of Corollary 1.8 in [6].

By applying Theorem 2 and Corollary 3 to the distance matrix $\mathcal{D}(G)$ with $S = 0$, $T = 1$, and $r_i = D_i$ for $i = 1, 2, \dots, n$, we have the following.

Corollary 11 (see [16, Theorem 2]). Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \dots, D_n$ such that $D_1 \geq D_2 \geq D_{i-1} > D_i \geq \dots \geq D_n$ for some $2 \leq i \leq n$. Then

$$\rho^{\mathcal{D}}(G) > \frac{D_n - 1 + \sqrt{(D_n + 1)^2 + 4(i-1)(D_{i-1} - D_n)}}{2}. \quad (20)$$

3.4. Distance Signless Laplacian Spectral Radius of a Graph. Let G be a connected graph and d be the diameter of G . Then the distance matrix $\mathcal{Q}(G)$ is nonnegative and symmetric. By applying Corollary 3 and Theorem 4 to the distance matrix $\mathcal{Q}(G)$ with $S = D_n$, $T = 1$, $M = D_1$, $N = d$, and $r_i = 2D_i$ for $i = 1, 2, \dots, n$, we note that $d_{21} = \dots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 12. Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \dots, D_n$ such that $D_1 \geq D_2 \geq \dots \geq D_n$ and d be the diameter of G . Let

$$\begin{aligned} f(i) &= \frac{3D_n - 1 + \sqrt{(D_n + 1)^2 + 8(i-1)(D_{i-1} - D_n)}}{2}, \\ g(i) &= \frac{D_1 + 2D_i - d + \sqrt{(2D_i - D_1 + d)^2 + 8d(i-1)(D_1 - D_i)}}{2}. \end{aligned} \quad (21)$$

Then one has

$$\max_{2 \leq i \leq n} \{2D_n, f(i)\} \leq q^{\mathcal{D}}(G) \leq \min_{1 \leq i \leq n} \{g(i)\}. \quad (22)$$

Moreover, one of the equalities holds if and only if $D_1 = D_2 = \dots = D_n$.

Remark 13. The right inequality in Corollary 12 is the result of Theorem 3.8 in [9].

By applying Theorem 2 and Corollary 3 to the distance matrix $\mathcal{Q}(G)$ with $S = D_n$, $T = 1$, and $r_i = 2D_i$ for $i = 1, 2, \dots, n$, we have the following.

Corollary 14. Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \dots, D_n$ such that $D_1 \geq D_2 \geq \dots \geq D_n$ for some $2 \leq i \leq n$. Then $q^{\mathcal{D}}(G) > f(i)$.

3.5. Spectral Radius of the Reciprocal Distance Matrix. By applying Corollary 3 and Theorem 4 to the reciprocal distance matrix $R(G)$ with $S = 0$, $T = 1/d$, $M = 0$, $N = 1$, and $r_i = R_i$ for $i = 1, \dots, n$, we have the following.

Corollary 15. Let G be a connected graph on n vertices, d be the diameter of G , $R_i = \sum_{j=1}^n r_{ij}$, and the row sum sequence be $R_1, R_2, \dots, R_n$ of $R(G)$ satisfying $R_1 \geq R_2 \geq \dots \geq R_n$. Let

$$\begin{aligned} f(i) &= \frac{R_n - 1/d + \sqrt{(R_n + 1/d)^2 + (4/d)(i-1)(R_{i-1} - R_n)}}{2}, \\ g(i) &= \frac{R_i - 1 + \sqrt{(R_i + 1)^2 + 4(i-1)(R_1 - R_i)}}{2}. \end{aligned} \quad (23)$$

Then

$$\max_{2 \leq i \leq n} \{R_n, f(i)\} \leq \lambda(R(G)) \leq \min_{1 \leq i \leq n} \{g(i)\}. \quad (24)$$

Moreover, the left equality holds if and only if $R_1 = R_2 = \dots = R_n$, and the right equality holds if and only if either

$R_1 = R_2 = \dots = R_n$ or there exists some t with $2 \leq t \leq n$ such that G is a graph with $t-1$ vertices of degree $n-1$ and the remaining $n-t+1$ vertices have equal degree less than $n-1$.

Remark 16. The right inequality in Corollary 15 is the result (i) of Theorem 4 in [16].

4. Various Spectral Radii of a Digraph

Let $\vec{G}$ be a strong connected digraph. In Section 1, the adjacency matrix $A(\vec{G})$, the signless Laplacian matrix $Q(\vec{G})$, the distance matrix $\mathcal{D}(\vec{G})$ (if $\vec{G}$ is connected), the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$ (if $\vec{G}$ is connected), the adjacency spectral radius $\rho(\vec{G})$, the signless Laplacian spectral radius $q(\vec{G})$, the distance spectral radius $\rho^{\mathcal{D}}(\vec{G})$, and the distance signless Laplacian spectral radius $q^{\mathcal{D}}(\vec{G})$ are defined. Now, in this section, we will apply Theorem 2, Corollary 3, and Theorem 4 to $A(\vec{G})$, $Q(\vec{G})$, $\mathcal{D}(\vec{G})$, and $\mathcal{Q}(\vec{G})$, obtain some new results or known results, and revise and improve the result of Theorem 2.5 in [11].

4.1. Adjacency Spectral Radius of a Digraph. Let $\vec{G}$ be a digraph. By applying Corollary 3 and Theorem 4 to the (adjacency) matrix $A(\vec{G})$ with $S = 0$, $T = 0$, $M = 0$, $N = 1$, and $r_i = d_i^+$ for $i = 1, \dots, n$, we have the following.

Corollary 17. Let $\vec{G}$ be a digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \dots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \dots \geq d_n^+$. Then one has

$$\begin{aligned} d_n^+ &\leq \rho(\vec{G}) \\ &\leq \min_{1 \leq i \leq n} \left\{ \frac{d_i^+ - 1 + \sqrt{(d_i^+ + 1)^2 + 4(i-1)(d_1^+ - d_i^+)}}{2} \right\}. \end{aligned} \quad (25)$$

Moreover, if $\vec{G}$ is a strong connected digraph, then the left equality holds if and only if $\vec{G}$ is a regular digraph, the right equality holds if and only if $\vec{G}$ is a regular digraph, or there exists some t with $2 \leq t \leq n$ such that $\vec{G}$ is a bidegreed digraph with $d_1^+ = \dots = d_{t-1}^+ > d_t^+ = \dots = d_n^+$ and the indegrees $d_1^- = \dots = d_{t-1}^- = n-1$.

4.2. Signless Laplacian Spectral Radius of a Digraph. Let $\vec{G}$ be a digraph. By applying Corollary 3 and Theorem 4 to the signless Laplacian matrix $Q(\vec{G})$ with $S = d_n^+$, $T = 0$, $M = d_1^+$, $N = 1$, and $r_i = 2d_i^+$ for $i = 1, \dots, n$, we have the following.

Corollary 18. Let $\vec{G}$ be a digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \dots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \dots \geq d_n^+$. Then one has

$$2d_n^+ \leq q(\vec{G}) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2} \right\}. \quad (26)$$

Moreover, if $\vec{G}$ is a strong connected digraph, then the left equality holds if and only if $\vec{G}$ is a regular digraph, the right equality holds if and only if $\vec{G}$ is a regular digraph, or there exists some t with $2 \leq t \leq n$ such that $\vec{G}$ is a bidegreed digraph with $d_1^+ = \dots = d_{t-1}^+ > d_t^+ = \dots = d_n^+$ and the indegrees $d_1^- = \dots = d_{t-1}^- = n-1$.

Remark 19. The left inequality in Corollary 18 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 18 revises and improves Proposition 20.

Proposition 20 (see [11, Theorem 2.5]). Let $\vec{G}$ be a strong connected digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \dots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \dots \geq d_n^+$. Then one has

$$q(\vec{G}) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2} \right\}. \quad (27)$$

Moreover, if $i = 1$, the equality holds if and only if $\vec{G}$ is a regular digraph. If $2 \leq i \leq n$, the equality holds if and only if $\vec{G}$ is a regular digraph or a bidegreed digraph in which $d_1^+ = d_2^+ = \dots = d_{i-1}^+ = n-1$ and $d_i^+ = \dots = d_n^+ = \delta^+$.

The following example shows that the result of Proposition 20 is incorrect.

Example 21. Let $n \geq 5$ and D_1 is shown in Figure 1. For D_1 , the outdegree sequence is $3 = d_1^+ > d_2^+ = d_3^+ = \dots = d_n^+ = 2$ and the indegree $d_1^- = n-1$. We have $q(D_1) = 3 + \sqrt{3}$ by direct computation. It is clear that

$$q(D_1) = 3 + \sqrt{3} = \min_{1 \leq i \leq n} \left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2} \right\}. \quad (28)$$

4.3. Distance Spectral Radius of a Digraph. Let $\vec{G}$ be a strong connected digraph and d be the diameter of $\vec{G}$. By applying Theorems 2 and 4 to the distance matrix $\mathcal{D}(\vec{G})$ with $S = 0, T = 1, M = 0, N = d$, and $r_i = D_i^+$ for $i = 1, \dots, n$, we note that $d_{21} = \dots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 22. Let $\vec{G}$ be a strong connected digraph on n vertices with distance outdegree sequence $D_1^+, D_2^+, \dots, D_n^+$ such that $D_1^+ \geq D_2^+ \geq \dots \geq D_n^+$, and let d be the diameter of $\vec{G}$. Let

$$\begin{aligned} f(i) &= \frac{D_n^+ - 1 + \sqrt{(D_n^+ + 1)^2 + 4(i-1)(D_{i-1}^+ - D_n^+)}}{2}, \\ g(i) &= \frac{D_i^+ - d + \sqrt{(D_i^+ + d)^2 + 4d(i-1)(D_1^+ - D_i^+)}}{2}. \end{aligned} \quad (29)$$

Then one has

$$\max_{2 \leq i \leq n} \{D_n^+, f(i)\} \leq \rho^{\mathcal{D}}(\vec{G}) \leq \min_{1 \leq i \leq n} \{g(i)\}. \quad (30)$$

Moreover, the left equality holds if and only if $D_1^+ = \dots = D_n^+$ or there exists some t with $2 \leq t \leq n$ such that $D_1^+ = \dots = D_{t-1}^+ > D_t^+ = \dots = D_n^+$ and $D_1^- = \dots = D_{t-1}^- = n-1$ and the right equality holds if and only if $D_1^+ = \dots = D_n^+$.

4.4. Distance Signless Laplacian Spectral Radius of a Digraph. Let $\vec{G}$ be a strong connected digraph and d be the diameter of $\vec{G}$. By applying Theorems 2 and 4 to the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$ with $S = D_n^+, T = 1, M = D_1^+, N = d$, and $r_i = 2D_i^+$ for $i = 1, \dots, n$, we note two facts: the first fact is that (i) and (iii) of (2) in Theorem 2 cannot hold at the same time by $a_{ii} = D_i^+ = \sum_{1 \leq j \leq n} d_{ij}$ and $r_i = 2D_i^+$, and the second fact is that $d_{21} = \dots = d_{n1} = d$ implies a contradiction. Then we have the following.

FIGURE 1: The digraphs D_1 .

Corollary 23. Let $\vec{G}$ be a strong connected digraph on n vertices with distance outdegree sequence $D_1^+, D_2^+, \dots, D_n^+$ such that $D_1^+ \geq D_2^+ \geq \dots \geq D_n^+$, and let d be the diameter of $\vec{G}$. Let

$$\begin{aligned} f(i) &= \frac{3D_n^+ - 1 + \sqrt{(D_n^+ + 1)^2 + 8(i-1)(D_{i-1}^+ - D_n^+)}}{2}, \\ g(i) &= \frac{D_1^+ + 2D_i^+ - d + \sqrt{(2D_i^+ - D_1^+ + d)^2 + 8d(i-1)(D_1^+ - D_i^+)}}{2}. \end{aligned} \quad (31)$$

Then one has

$$\max_{2 \leq i \leq n} \{D_n^+, f(i)\} \leq q^{\mathcal{D}}(\vec{G}) \leq \min_{1 \leq i \leq n} \{g(i)\}. \quad (32)$$

Moreover, one of the equalities holds if and only if $D_1^+ = \dots = D_n^+$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] M. Aouchiche and P. Hansen, "Two Laplacians for the distance matrix of a graph," *Linear Algebra and Its Applications*, vol. 439, no. 1, pp. 21–33, 2013.
- [2] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1979.
- [3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, UK, 1976.
- [4] D. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, UK, 2009.
- [5] S. B. Bozkurt and D. Bozkurt, "On the signless Laplacian spectral radius of digraphs," *Ars Combinatoria*, vol. 108, pp. 193–200, 2013.
- [6] Y. Y. Chen, H. Q. Lin, and J. L. Shu, "Sharp upper bounds on the distance spectral radius of a graph," *Linear Algebra and Its Applications*, vol. 439, no. 9, pp. 2659–2666, 2013.
- [7] X. Duan and B. Zhou, "Sharp bounds on the spectral radius of a nonnegative matrix," *Linear Algebra and Its Applications*, vol. 439, no. 10, pp. 2961–2970, 2013.
- [8] C.-X. He, Y. Liu, and Z.-H. Zhao, "Some new sharp bounds on the distance spectral radius of graph," *MATCH. Communications in Mathematical and in Computer Chemistry*, vol. 63, no. 3, pp. 783–788, 2010.
- [9] W. X. Hong and L. H. You, "Further results on the spectral radius of matrices and graphs," *Applied Mathematics and Computation*, vol. 239, pp. 326–332, 2014.
- [10] W. Hong and L. You, "Spectral radius and signless Laplacian spectral radius of strongly connected digraphs," *Linear Algebra and Its Applications*, vol. 457, pp. 93–113, 2014.
- [11] W. W. Lang and L. G. Wang, "Sharp bounds for the signless Laplacian spectral radius of digraphs," *Applied Mathematics and Computation*, vol. 238, pp. 43–49, 2014.
- [12] A. D. Maden, K. C. Das, and A. S. Çevik, "Sharp upper bounds on the spectral radius of the signless Laplacian matrix of a graph," *Applied Mathematics and Computation*, vol. 219, no. 10, pp. 5025–5032, 2013.
- [13] J. L. Shu and Y. R. Wu, "Sharp upper bounds on the spectral radius of graphs," *Linear Algebra and Its Applications*, vol. 377, pp. 241–248, 2004.
- [14] L. H. You, Y. J. Shu, and P. Z. Yuan, "Sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix and its applications," *Linear and Multilinear Algebra*, 2016.
- [15] L. H. You, Y. J. Shu, and X. D. Zhang, "A sharp upper bound for the spectral radius of a nonnegative matrix and applications," *Czechoslovak Mathematical Journal*, vol. 66, no. 3, pp. 701–715, 2016.
- [16] B. Zhou and N. Trinajstić, "Further results on the largest eigenvalues of the distance matrix and some distance-based matrices of connected (molecular) graphs," *Internet Electronic Journal of Molecular Design*, vol. 6, no. 12, pp. 375–384, 2007.