Research Article

Uniqueness of Solutions to a Nonlinear Elliptic Hessian Equation

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Through an Alexandrov-Fenchel inequality, we establish the general Brunn-Minkowski inequality. Then we obtain the uniqueness of solutions to a nonlinear elliptic Hessian equation on $\mathbb{S}^n$.

1. Introduction

According to a general Brunn-Minkowski inequality, we obtain a proof of the uniqueness of solutions to the following fully nonlinear elliptic Hessian equation:

$$\sigma_k(u_{ij} + u\delta_{ij}) = fu^{p-1} \quad \text{on } \mathbb{S}^n, \quad (1)$$

where u is the support function of convex bodies, u_{ij} are the second-order covariant derivations of u with respect to any orthonormal frame $\{e_1, e_2, \dots, e_n\}$ on $\mathbb{S}^n$, δ_{ij} is the standard Kronecker symbol, $\mathbb{S}^n$ is the unit sphere of n -dimension, f is a positive function defined on $\mathbb{S}^n$, $k \in \{1, 2, \dots, n\}$, $p > 1$, and σ_k is the k th elementary symmetric function defined as follows: for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}. \quad (2)$$

The definition can be extended to any symmetric matrix $W \in \mathbb{R}^{n \times n}$ by $\sigma_k(W) = \sigma_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \lambda_2(W), \dots, \lambda_n(W))$ is the eigenvalue vector of W .

Equation (1) arrives from the geometry of convex bodies. A compact convex subset of Euclidean $(n+1)$ -space $\mathbb{R}^{n+1}$ with nonempty interiors is called a *convex body*. An important concept related to a convex body Q is its support function.

Definition 1. Let M (the boundary of a convex body Q) be a smooth, closed, uniformly convex hypersurface enclosing the

origin in $\mathbb{R}^{n+1}$. Assume that M is parameterized by its inverse Gauss map $X : \mathbb{S}^n \rightarrow M \subset \mathbb{R}^{n+1}$; the *support function* u of M (or Q) is defined by

$$u(x) = \langle x, X(x) \rangle, \quad \forall x \in \mathbb{S}^n, \quad (3)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$.

u is convex after being extended as a function of homogeneous degree 1 in $\mathbb{R}^{n+1}$. Conversely, any continuous convex function u of homogeneous degree 1 determines a convex body as follows:

$$Q = \{y \in \mathbb{R}^{n+1} : y \cdot x \leq u(x), \quad \forall x \in \mathbb{S}^n\}. \quad (4)$$

From some basic concepts to support function, Minkowski sum [see Definition 4], and mixed volumes [see Definition 5], Minkowski developed a set of theories related to convex bodies. If $k = n$ and $p = 1$, (1) is the Monge-Ampère equation corresponding to the classical Minkowski problem

$$\det(u_{ij} + u\delta_{ij}) = f \quad \text{on } \mathbb{S}^n, \quad (5)$$

which has been solved by Nirenberg [1], Pogorelov [2, 3], Cheng and Yau [4], and many others. When $p = 1$, (1) is the classical Christoffel-Minkowski problem:

$$\sigma_k(u_{ij} + u\delta_{ij}) = f \quad \text{on } \mathbb{S}^n. \quad (6)$$

A necessary condition [3] for (6) to have a solution is

$$\int_{\mathbb{S}^n} x_i f(x) ds = 0, \quad \forall i = 1, 2, \dots, n+1, \quad (7)$$

where ds is the standard area form on $\mathbb{S}^n$. Guan et al. [5] obtained that (7) is sufficient for (6) to have an admissible solution [see Definition 6].

Firey [6] generalized the Minkowski sum to p -sum [see Definition 4] from $p = 1$ to $p \geq 1$ in 1962. Later, Lutwak [7] extended the classical surface area measure to the p -sum cases. Also in [7], Lutwak first introduced the general Minkowski problem, which is called L_p -Minkowski problem thereafter. In the smooth category, L_p -Minkowski problem is equivalent to considering the following Monge-Ampère equation:

$$\det(u_{ij} + u\delta_{ij}) = fu^{p-1} \quad \text{on } \mathbb{S}^n. \quad (8)$$

The uniqueness of L_p -Minkowski problem for $p > 1$ and $p \neq n+1$ (the uniqueness holds up to a dilation if $p = n+1$) has been solved in [7]. However, the uniqueness for $p < 1$ is difficult and still open. In [8], Jian et al. obtained that, for any $-n-1 < p < 0$, there exists a positive function $f \in C^\infty(\mathbb{S}^n)$ to guarantee that (8) has two different solutions, which means that we need more conditions to consider the uniqueness.

When considering cases $1 \leq k < n$, attention is paid to the generalized Christoffel-Minkowski problem. In the smooth category, we need to study the k -Hessian equation (1).

For (1), Hu et al. [9] got the existence and uniqueness of solutions to (1) when $1 \leq k < n$ and $p > k+1$ under appropriate conditions. However, the uniqueness of (1) when $p < 1$ has not been solved well. In this paper, we study the uniqueness of (1) for $p > 1$.

Our main result is the following.

Theorem 2. Suppose u is a positive admissible solution of

$$\sigma_k(u_{ij} + u\delta_{ij}) = fu^{p_0} \quad \text{on } \mathbb{S}^n, \quad (9)$$

where $1 \leq k < n$, $k \in \mathbb{Z}$, $p_0 \in \mathbb{R}^+ \setminus \{k\}$, and f is a positive function defined on the unit sphere $\mathbb{S}^n$ and then the uniqueness holds. If $p_0 = k$, the uniqueness holds up to a dilation, which means that if u solves (9), then $\{au : \forall a \in \mathbb{R}^+\}$ are the whole solutions of (9).

Remark 3. Here, we rewrite (1) by (9), where $p_0 = p - 1$.

The organization of this paper is as follows. In Section 2, we show some basic concepts and lemmas which have been obtained by Guan et al. in [10]. In Section 3, we prove two useful propositions according to the methods in [11]. In the last section, we prove the main theorem.

2. Preliminaries

Definition 4. Given two convex bodies Q_1 and Q_2 in $\mathbb{R}^{n+1}$ with respective support functions u_1, u_2 , and $\lambda, \mu \geq 0$ ($\lambda + \mu > 0$), the Minkowski sum $\lambda Q_1 + \mu Q_2 \subset \mathbb{R}^{n+1}$ is defined by the convex body whose support function is $\lambda u_1 + \mu u_2$.

For $p \geq 1$, let Q_1 and Q_2 be two convex bodies containing the origin in $\mathbb{R}^{n+1}$ in their interiors, and $\lambda, \mu \geq 0$ ($\lambda + \mu > 0$). The convex body $\lambda \circ Q_1 +_p \mu \circ Q_2$, whose support function is given by $(\lambda u_1^p + \mu u_2^p)^{1/p}$, is called Firey's p -sum of Q_1 and Q_2 , where “ $+_p$ ” means the p -summation and “ $\circ$ ” means Firey's multiplication.

Definition 5. Let $Q_1, Q_2, \dots, Q_r$ be convex bodies in $\mathbb{R}^{n+1}$ and the volume of their Minkowski sum

$$Q = \lambda_1 Q_1 + \lambda_2 Q_2 + \dots + \lambda_r Q_r, \quad \lambda_i \geq 0, \quad (10)$$

is an $(n+1)$ th degree homogeneous polynomial of the family $\lambda_1, \lambda_2, \dots, \lambda_r$. Specially, the volume of Q is

$$\begin{aligned} \text{Vol}(Q) &= \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2 + \dots + \lambda_r Q_r) \\ &= \sum_{i_1, i_2, \dots, i_{n+1}=1}^r \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_{n+1}} V(Q_{i_1}, Q_{i_2}, \dots, Q_{i_{n+1}}), \end{aligned} \quad (11)$$

where the functions V are symmetric. Then $V(Q_1, Q_2, \dots, Q_{n+1})$ is called the Minkowski mixed volume of $Q_1, Q_2, \dots, Q_{n+1}$.

Definition 6. For $k \in \{1, 2, \dots, n\}$, let Γ_k be the convex cone in $\mathbb{R}^n$ which is determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \dots, \sigma_k(\lambda) > 0\}. \quad (12)$$

A function $u \in C^2(\mathbb{S}^n)$ is called k -convex if

$$W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\} \in \Gamma_k, \quad \forall x \in \mathbb{S}^n, \quad (13)$$

and u is called an admissible solution to (1) if u is k -convex and satisfies (1).

Definition 7. Let $A_1, A_2, \dots, A_m$ be symmetric real $k \times k$ matrices, $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$; the determinant of $\lambda_1 A_1 + \dots + \lambda_m A_m$ is a homogeneous polynomial of degree k in $\lambda_1, \lambda_2, \dots, \lambda_m$. Namely,

$$\begin{aligned} \det(\lambda_1 A_1 + \dots + \lambda_m A_m) \\ = \sum_{i_1, \dots, i_k=1}^m \lambda_{i_1} \dots \lambda_{i_k} D_k(A_{i_1}, \dots, A_{i_k}). \end{aligned} \quad (14)$$

In fact, the coefficient $\lambda_{i_1} \dots \lambda_{i_k}$ depends only on $A_{i_1}, \dots, A_{i_k}$; then they are uniquely determined. $D_k(A_1, \dots, A_k)$ is called the mixed discriminant of $A_1, \dots, A_k$.

For later applications, we collect some results here which have been proved in [10].

Lemma 8. Let $u_1, u_2, \dots, u_{n+1}$ be the support function of convex bodies $Q_1, Q_2, \dots, Q_{n+1}$, respectively. Denoting Minkowski mixed volume $V(Q_1, Q_2, \dots, Q_{n+1})$ by $V(u_1, u_2, \dots, u_{n+1})$ and

$$W_m = \{(u_m)_{ij} + u_m \delta_{ij}\}, \quad m = 1, 2, \dots, n+1, \quad (15)$$

then

$$\begin{aligned} V(u_1, u_2, \dots, u_{n+1}) \\ = \int_{\mathbb{S}^n} u_1 D_n(W_2, W_3, \dots, W_{n+1}) ds, \end{aligned} \quad (16)$$

where $D_n(W_2, W_3, \dots, W_{n+1})$ is the mixed discriminant [see Definition 7] of $W_2, W_3, \dots, W_{n+1}$.

Remark 9. For all $1 \leq k \leq n$, setting $u_{k+2} = \dots = u_{n+1} = 1$, then

$$\begin{aligned} V(u_1, \dots, u_{k+1}, 1, \dots, 1) &:= V_{k+1}(u_1, u_2, \dots, u_{k+1}) \\ &= \int_{\mathbb{S}^n} u_1 D_k(W_2, W_3, \dots, W_{k+1}) ds, \end{aligned} \quad (17)$$

where $D_k(W_2, W_3, \dots, W_{k+1})$ is the mixed discriminant of $W_2, W_3, \dots, W_{k+1}$. Furthermore, if $u_1 = u_2 = \dots = u_{n+1} = u$, denote $V(u_1, u_2, \dots, u_{n+1}) := V(u)$ and $V_{k+1}(u_1, u_2, \dots, u_{k+1}) := V_{k+1}(u)$; then

$$\begin{aligned} V(u) &= \int_{\mathbb{S}^n} u \det(u_{ij} + u\delta_{ij}) ds, \\ V_{k+1}(u) &= \int_{\mathbb{S}^n} u \sigma_k(u_{ij} + u\delta_{ij}) ds. \end{aligned} \quad (18)$$

Lemma 10. V is a symmetric multilinear form on $(C^2(\mathbb{S}^n))^{n+1}$.

Lemma 11. For any function $u \in C^2(\mathbb{S}^n)$, $W = \{u_{ij} + u\delta_{ij}\}$, $1 \leq k < n$, we have the Minkowski type integral formula,

$$\int_{\mathbb{S}^n} u \sigma_k(W) ds = \int_{\mathbb{S}^n} \sigma_{k+1}(W) ds, \quad (19)$$

where ds is the standard area element on $\mathbb{S}^n$.

The following is a form of Alexandrov-Fenchel inequality for positive k -convex functions which comes from [10].

Lemma 12 (Alexandrov-Fenchel inequality). If $u_1, u_2, \dots, u_k$ are k -convex, u_1 is positive, and there exists $l \in \{2, 3, \dots, k\}$ such that $u_l \geq 0$ on $\mathbb{S}^n$, then, for any $v \in C^2(\mathbb{S}^n)$,

$$\begin{aligned} V_{k+1}^2(v, u_1, u_2, \dots, u_k) \\ \geq V_{k+1}(u_1, u_1, u_2, \dots, u_k) V_{k+1}(v, v, u_2, \dots, u_k), \end{aligned} \quad (20)$$

with equality if and only if $v = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \dots, a_{n+1}$.

3. Two Important Propositions

Now we prove two important propositions. The methods we use are from [11].

Proposition 13. Suppose $u_0, u_1 > 0$ are k -convex; then

$$\begin{aligned} V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) \\ \geq (1-t)V_{k+1}^{1/(k+1)}(u_0) + tV_{k+1}^{1/(k+1)}(u_1), \end{aligned} \quad (21)$$

$\forall t \in [0, 1],$

with equality if and only if $u_0 = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \dots, a_{n+1}$.

Proof. We only need to prove that

$$F(t) = V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) \quad (22)$$

is concave on $[0, 1]$. Setting $u_t = (1-t)u_0 + tu_1$, $t \in [0, 1]$, we have

$$F(t) = V_{k+1}^{1/(k+1)}\left(\overbrace{u_t, \dots, u_t}^{k+1}\right). \quad (23)$$

By the symmetric multilinear property of V , it is obvious that

$$\begin{aligned} F'(t) &= V_{k+1}^{1/(k+1)-1}\left(\overbrace{u_t, \dots, u_t}^{k+1}\right) V_{k+1}\left(-u_0 \right. \\ &\quad \left. + u_1, \overbrace{u_t, \dots, u_t}^k\right), \end{aligned} \quad (24)$$

$$\begin{aligned} F''(t) &= kV_{k+1}^{1/(k+1)-2}\left(\overbrace{u_t, \dots, u_t}^{k+1}\right) \\ &\quad \cdot \left[V_{k+1}\left(\overbrace{u_t, \dots, u_t}^{k+1}\right) \right. \\ &\quad \cdot V_{k+1}\left(-u_0 + u_1, -u_0 + u_1, \overbrace{u_t, \dots, u_t}^{k-1}\right) \\ &\quad \left. - V_{k+1}^2\left(-u_0 + u_1, \overbrace{u_t, \dots, u_t}^k\right) \right] \leq 0, \end{aligned} \quad (25)$$

where the last inequality uses (20); thus F is a concave function on $[0, 1]$. The equality condition is checked easily. $\square$

Proposition 14 (general Brunn-Minkowski inequality). Suppose $u_0, u_1 > 0$ are k -convex, then

$$\begin{aligned} \int_{\mathbb{S}^n} u_1 \sigma_k((u_0)_{ij} + u_0 \delta_{ij}) ds \\ \geq V_{k+1}^{1/(k+1)}(u_1) V_{k+1}^{1-1/(k+1)}(u_0), \end{aligned} \quad (26)$$

with equality if and only if $u_0 = au_1 + \sum_{i=1}^{n+1} a_i x_i$ for some constants $a, a_1, \dots, a_{n+1}$.

Proof. Setting

$$\begin{aligned} F(t) &= V_{k+1}^{1/(k+1)}((1-t)u_0 + tu_1) \\ &\quad - (1-t)V_{k+1}^{1/(k+1)}(u_0) - tV_{k+1}^{1/(k+1)}(u_1), \end{aligned} \quad (27)$$

then $F(0) = F(1) = 0$. By (21), $F(t) \geq 0$; thus $F'(0) \geq 0$; namely,

$$\begin{aligned} V_{k+1}^{1/(k+1)-1}(u_0) V_{k+1}\left(-u_0 + u_1, \overbrace{u_0, \dots, u_0}^k\right) \\ + V_{k+1}^{1/(k+1)}(u_0) - V_{k+1}^{1/(k+1)}(u_1) \geq 0. \end{aligned} \quad (28)$$

Then

$$V_{k+1}^{1/(k+1)-1}(u_0) \int_{\mathbb{S}^n} (-u_0 + u_1) \sigma_k((u_0)_{ij} + u_0 \delta_{ij}) ds \\ + V_{k+1}^{1/(k+1)}(u_0) \geq V_{k+1}^{1/(k+1)}(u_1). \quad (29)$$

By (19),

$$V_{k+1}^{1/(k+1)-1}(u_0) \int_{\mathbb{S}^n} u_1 \sigma_k((u_0)_{ij} + u_0 \delta_{ij}) ds \\ \geq V_{k+1}^{1/(k+1)}(u_1), \quad (30)$$

and then

$$\int_{\mathbb{S}^n} u_1 \sigma_k((u_0)_{ij} + u_0 \delta_{ij}) ds \\ \geq V_{k+1}^{1/(k+1)}(u_1) V_{k+1}^{1-1/(k+1)}(u_0). \quad (31)$$

□

4. Proof of Theorem 2

Now we prove Theorem 2. The main methods are from [7, 12].

Proof. Assuming that (9) has two solutions u and v , then we consider the equation in the following three cases.

Case 1 ($p_0 > k$). Supposing x_0 is the maximum value point of $G = u/v$, then at x_0 , we have

$$0 = \nabla \ln G = \frac{\nabla u}{u} - \frac{\nabla v}{v}, \\ 0 \geq \nabla^2 \ln G = \left(\frac{\nabla^2 u}{u} - \frac{(\nabla u)^2}{u^2} \right) - \left(\frac{\nabla^2 v}{v} - \frac{(\nabla v)^2}{v^2} \right) \quad (32) \\ = \frac{\nabla^2 u}{u} - \frac{\nabla^2 v}{v};$$

that is,

$$\frac{\nabla^2 u}{u} \leq \frac{\nabla^2 v}{v}. \quad (33)$$

Hence

$$fu^{p_0}(x_0) = u^k(x_0) \sigma_k\left(\frac{u_{ij}}{u} + \delta_{ij}\right)(x_0) \\ \leq u^k(x_0) \sigma_k\left(\frac{v_{ij}}{v} + \delta_{ij}\right)(x_0) \quad (34) \\ = \frac{u^k(x_0)}{v^k(x_0)} f v^{p_0}(x_0);$$

therefore

$$u^{p_0-k}(x_0) \leq v^{p_0-k}(x_0) \implies G(x_0) = \frac{u(x_0)}{v(x_0)} \leq 1; \quad (35)$$

then

$$\frac{u}{v} \leq 1. \quad (36)$$

Similarly, we have $v/u \leq 1$. Thus $u \equiv v$.

Case 2 ($0 < p_0 < k$). We have

$$u^{-p_0} \sigma_k(u_{ij} + u \delta_{ij}) = v^{-p_0} \sigma_k(v_{ij} + v \delta_{ij}); \quad (37)$$

then

$$V_{k+1}(u) = \int_{\mathbb{S}^n} u \sigma_k(u_{ij} + u \delta_{ij}) ds \\ = \int_{\mathbb{S}^n} \left(\frac{u}{v}\right)^{p_0+1} v \sigma_k(v_{ij} + v \delta_{ij}) ds \\ \geq \left[\int_{\mathbb{S}^n} u \sigma_k(v_{ij} + v \delta_{ij}) ds \right]^{p_0+1} \quad (38) \\ \cdot \left[\int_{\mathbb{S}^n} v \sigma_k(v_{ij} + v \delta_{ij}) ds \right]^{-p_0} \geq V_{k+1}^{(p_0+1)/(k+1)}(u) \\ \cdot V_{k+1}^{(kp_0+k)/(k+1)}(v) V_{k+1}^{-p_0}(v) = V_{k+1}^{(p_0+1)/(k+1)}(u) \\ \cdot V_{k+1}^{1-(p_0+1)/(k+1)}(v),$$

where we have used Hölder inequality in the first inequality and used (26) in the second one. Hence $V_{k+1}(u) = V_{k+1}(v)$, which forces both the equalities to hold. By the equality condition, there exists a constant $a \in \mathbb{R}$ such that $v = au$. By (9), we know $a = 1$. Therefore, $u \equiv v$.

Case 3 ($p_0 = k$). According to Case 2, when $p_0 = k$, we have

$$V_{k+1}(u) = \int_{\mathbb{S}^n} u \sigma_k(u_{ij} + u \delta_{ij}) ds = \int_{\mathbb{S}^n} \left(\frac{u}{v}\right)^{k+1} \\ \cdot v \sigma_k(v_{ij} + v \delta_{ij}) ds \geq \left[\int_{\mathbb{S}^n} u \sigma_k(v_{ij} + v \delta_{ij}) ds \right]^{k+1} \quad (39) \\ \cdot \left[\int_{\mathbb{S}^n} v \sigma_k(v_{ij} + v \delta_{ij}) ds \right]^{-k} \geq V_{k+1}(u) V_{k+1}^k(v) \\ \cdot V_{k+1}^{-k}(v) = V_{k+1}(u);$$

then all the equalities hold. Thus there exists $a \in \mathbb{R}$, such that $v = au$. Therefore $\{au : \forall a \in \mathbb{R}^+\}$ are the whole solutions of (9).

Now we complete the proof of Theorem 2. □

Competing Interests

The author declares no competing interests.

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