Research Article

Generalized Hybrid One-Step Block Method Involving Fifth Derivative for Solving Fourth-Order Ordinary Differential Equation Directly

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A general one-step three-hybrid (off-step) points block method is proposed for solving fourth-order initial value problems of ordinary differential equations directly. A power series approximate function is employed for deriving this method. The approximate function is interpolated at $\{x_n, x_{n+r}, x_{n+s}, x_{n+t}\}$ while its fourth and fifth derivatives are collocated at all points $\{x_i\}$, $i = \{0, r, s, t, 1\}$, in the interval of approximation. Several fourth-order initial value problems of ordinary differential equations are then solved to compare the performance of the proposed method with the derived methods. The analysis of the method reveals that the method is consistent and zero stable concluding that the method is also convergent. The numerical results demonstrate the superiority of the new method over the existing ones in terms of error.

1. Introduction

In this article, we consider the following general fourth-order initial value problems (IVPs) as shown:

$$\begin{aligned} y^{(iv)} &= f(x, y, y', y'', y'''), \\ y(a) &= \alpha_1, \\ y'_0(a) &= \alpha_2, \\ y''_0(a) &= \alpha_3, \\ y'''_0(a) &= \alpha_4, \end{aligned} \quad (1)$$

$$x \in [a, b],$$

with the assumption that $f \in C^5[a, b]$. Several phenomena in physical fields such as neural networks, electric circuits, and ship dynamics can be expressed in differential equations (DEs) forms (see [1–3]).

Block method is one of the efficient methods proposed in 1953 by [4] to enhance the performance of the numerical methods. In 1967, [5] employed block method to provide the

essential starting values needed for the predictor schemes. Subsequently, hybrid methods were initiated by [6], which involve the evaluation of functions off-step (nonstep) points. The introduction of hybrid (off-step) points in block methods has many advantages such as the ability to change step size, utilizing data off-step points, and the most important feature according to [7] is the capability to circumvent zero stability barrier condition ([8]).

To increase the accuracy further and to solve the stiffness problem in ordinary differential equations (ODEs), [9] derived second derivative multistep methods for stiff ODEs. Recently, authors like in [10, 11] proposed high derivative methods for the same reason. The former developed block hybrid-second derivative method for stiff systems, while the latter introduced a Simpson's type second derivative method for the solution of a first-order stiff system of IVP. In addition, [12] proposed a continuous fourth derivative method for third-order boundary value problems. Following these scholars' footsteps a new generalized three-hybrid one-step fifth derivative method for solving fourth-order ODEs directly using the approach of interpolation and collocation will be proposed.

This article consists of five sections: Section 1 is for the introduction, while Section 2 illustrates the method derivation, where we consider three-off-step points and employing the collocation approach. The analysis of the approach is discussed in Section 3 which includes zero stability, order, consistency, and convergence. Section 4 covers the solution of some mathematical problems to show the performance of the developed method. Finally, a brief conclusion is performed in Section 5.

2. Development of the Method

Let the following power series polynomial be the approximate solution of (1)

$$\psi(x) = \sum_{j=0}^{2m+l-1} a_j \left(\frac{x-x_n}{h} \right)^j, \quad (2)$$

where $l = 4$ and $m = 5$, the number of interpolation and collocation points. Now, differentiating (2) four and five times yields

$$\begin{aligned} \psi^{iv}(x) &= \sum_{j=4}^{2m+l-1} \frac{a_j j!}{h^4 (j-4)!} \left(\frac{x-x_n}{h} \right)^{j-4} \\ &= f(x, y, y', y'', y'''), \\ \psi^v(x) &= \sum_{j=5}^{2m+l-1} \frac{a_j j!}{h^5 (j-5)!} \left(\frac{x-x_n}{h} \right)^{j-5} \\ &= g(x, y, y', y'', y'''). \end{aligned} \quad (3)$$

Interpolate (2) at $x_{n+i} = x_n + ih$, $i = \{0, r, s, t\}$, and collocate (3) at all points $x_{n+i} = x_n + ih$, $i = \{0, r, s, t, 1\}$, where $\{r, s, t\} \in (0, 1)$ produces a system of equations in matrix form as below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 & \cdots & r^{\hat{v}} \\ 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 & \cdots & s^{\hat{v}} \\ 1 & t & t^2 & t^3 & t^4 & t^5 & t^6 & \cdots & t^{\hat{v}} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!r}{1!h^4} & \frac{6!r^2}{2!h^4} & \cdots & \frac{\hat{v}!r^{(\hat{v}-4)}}{(\hat{v}-4)!h^4} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!s}{1!h^4} & \frac{6!s^2}{2!h^4} & \cdots & \frac{\hat{v}!s^{(\hat{v}-4)}}{(\hat{v}-4)!h^4} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!t}{1!h^4} & \frac{6!t^2}{2!h^4} & \cdots & \frac{\hat{v}!t^{(\hat{v}-4)}}{(\hat{v}-4)!h^4} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!}{1!h^4} & \frac{6!}{2!h^4} & \cdots & \frac{\hat{v}!}{(\hat{v}-4)!h^4} \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{0!h^5} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{0!h^5} & \frac{6!r}{1!h^5} & \cdots & \frac{(\hat{v})!r^{(\hat{v}-5)}}{(\hat{v}-5)!h^5} \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{0!h^5} & \frac{6!s}{1!h^5} & \cdots & \frac{(\hat{v})!s^{(\hat{v}-5)}}{(\hat{v}-5)!h^5} \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{0!h^5} & \frac{6!t}{1!h^5} & \cdots & \frac{(\hat{v})!t^{(\hat{v}-5)}}{(\hat{v}-5)!h^5} \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{0!h^5} & \frac{6!}{1!h^5} & \cdots & \frac{(\hat{v})!}{(\hat{v}-5)!h^5} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}$$

$$= \begin{bmatrix} y_n \\ y_{n+r} \\ y_{n+s} \\ y_{n+t} \\ f_n \\ f_{n+r} \\ f_{n+s} \\ f_{n+t} \\ f_{n+1} \\ g_n \\ g_{n+r} \\ g_{n+s} \\ g_{n+t} \\ g_{n+1} \end{bmatrix}, \quad (4)$$

where $\hat{v} = 2m + l - 1$. Matrix manipulation is then employed to solve the resulting system (4) for the unknown coefficients $a'_j s$, $j = \{0, 1, \dots, 13\}$. Substituting the obtained values of the coefficients back into (2) yields

$$\begin{aligned} \psi(x) &= \sum_{i=0, r, s, t} \alpha_i y(x + ih) \\ &+ h^4 \left(\sum_{i=r, s, t} \beta_i f(x + ih) + \sum_{i=0}^1 \beta_i f(x + ih) \right) \\ &+ h^5 \left(\sum_{i=r, s, t} \gamma_i g(x + ih) + \sum_{i=0}^1 \gamma_i g(x + ih) \right), \end{aligned} \quad (5)$$

where $n = 0, 1, 2, \dots, N - 1$, $h = x_n - x_{n-1}$ is the constant step size for the partition π_N of the interval $[a, b]$ which is given by $\pi_N = [a = x_0 < x_1 < \dots < x_{N-1} < x_N = b]$, α_i , β_i , and γ_i are undetermined constants listed in Appendix I in the Supplementary Material (see Supplementary Material available online at <https://doi.org/10.1155/2017/7637651>). For simplicity we shall use $y(x + ih) = y_{n+i}$, $f(x + ih) = f_{n+i}$, $g(x + ih) = g_{n+i}$, and $g_{n+i} = f'_{n+i}$.

Calculating the first, second, and third derivatives of (5) produces

$$\begin{aligned} \psi'(x) &= \frac{d}{dx} \left(\sum_{i=0, r, s, t} \alpha_i y_{n+i} \right) \\ &+ h^4 \frac{d}{dx} \left(\sum_{i=r, s, t} \beta_i f_{n+i} + \sum_{i=0}^1 \beta_i f_{n+i} \right) \\ &+ h^5 \frac{d}{dx} \left(\sum_{i=r, s, t} \gamma_i g_{n+i} + \sum_{i=0}^1 \gamma_i g_{n+i} \right), \end{aligned} \quad (6)$$

$$\psi''(x) = \frac{d^2}{dx^2} \left(\sum_{i=0,r,s,t} \alpha_i y_{n+i} \right) + h^4 \frac{d^2}{dx^2} \left(\sum_{i=r,s,t} \beta_i f_{n+i} + \sum_{i=0}^1 \beta_i f_{n+i} \right) \quad (7)$$

$$+ h^5 \frac{d^2}{dx^2} \left(\sum_{i=r,s,t} \gamma_i g_{n+i} + \sum_{i=0}^1 \gamma_i g_{n+i} \right),$$

$$\psi'''(x) = \frac{d^3}{dx^3} \left(\sum_{i=0,r,s,t} \alpha_i y_{n+i} \right) + h^4 \frac{d^3}{dx^3} \left(\sum_{i=r,s,t} \beta_i f_{n+i} + \sum_{i=0}^1 \beta_i f_{n+i} \right) \quad (8)$$

$$+ h^5 \frac{d^3}{dx^3} \left(\sum_{i=r,s,t} \gamma_i g_{n+i} + \sum_{i=0}^1 \gamma_i g_{n+i} \right).$$

Evaluating (5) at the noninterpolating points $\{x_{n+t}, x_{n+1}\}$ with (6) and (7) at all points $\{x_{n+i}\}$, $i = \{0, r, s, t, 1\}$, gives the following general equations in block form:

$$A^{[0]} Y_m^{[1]} = A^{[1]} Y_m^{[0]} + \sum_{i=0}^1 B^{[i]} F_m^{[i]} + \sum_{i=0}^1 D^{[i]} G_m^{[i]}, \quad (9)$$

where $A^{[0]}$ is an identity matrix of order 12 and

$A^{[1]}$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & rh & 0 & 0 & 0 & \frac{r^2 h^2}{2} & 0 & 0 & 0 & \frac{r^3 h^3}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & sh & 0 & 0 & 0 & \frac{s^2 h^2}{2} & 0 & 0 & 0 & \frac{s^3 h^3}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & th & 0 & 0 & 0 & \frac{t^2 h^2}{2} & 0 & 0 & 0 & \frac{t^3 h^3}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & h & 0 & 0 & 0 & \frac{h^2}{2} & 0 & 0 & 0 & \frac{h^3}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & rh & 0 & 0 & 0 & \frac{r^2 h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & sh & 0 & 0 & 0 & \frac{s^2 h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & th & 0 & 0 & 0 & \frac{t^2 h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & h & 0 & 0 & 0 & \frac{h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & rh \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & sh \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & th \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{[0]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{116}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{216}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{316}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{416}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{516}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{616}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{716}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{816}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{916}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1016}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1116}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1216}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1316}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1416}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1516}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{1616}^{[0]} \end{bmatrix},$$

$$B^{[1]} = \begin{bmatrix} B_{11}^{[1]} & B_{12}^{[1]} & B_{13}^{[1]} & B_{14}^{[1]} \\ B_{21}^{[1]} & B_{22}^{[1]} & B_{23}^{[1]} & B_{24}^{[1]} \\ B_{31}^{[1]} & B_{32}^{[1]} & B_{33}^{[1]} & B_{34}^{[1]} \\ B_{41}^{[1]} & B_{42}^{[1]} & B_{43}^{[1]} & B_{44}^{[1]} \\ B_{51}^{[1]} & B_{52}^{[1]} & B_{53}^{[1]} & B_{54}^{[1]} \\ B_{61}^{[1]} & B_{62}^{[1]} & B_{63}^{[1]} & B_{64}^{[1]} \\ B_{71}^{[1]} & B_{72}^{[1]} & B_{73}^{[1]} & B_{74}^{[1]} \\ B_{81}^{[1]} & B_{82}^{[1]} & B_{83}^{[1]} & B_{84}^{[1]} \\ B_{91}^{[1]} & B_{92}^{[1]} & B_{93}^{[1]} & B_{94}^{[1]} \\ B_{101}^{[1]} & B_{102}^{[1]} & B_{103}^{[1]} & B_{104}^{[1]} \\ B_{111}^{[1]} & B_{112}^{[1]} & B_{113}^{[1]} & B_{114}^{[1]} \\ B_{121}^{[1]} & B_{122}^{[1]} & B_{123}^{[1]} & B_{124}^{[1]} \\ B_{131}^{[1]} & B_{132}^{[1]} & B_{133}^{[1]} & B_{134}^{[1]} \\ B_{141}^{[1]} & B_{142}^{[1]} & B_{143}^{[1]} & B_{144}^{[1]} \\ B_{151}^{[1]} & B_{152}^{[1]} & B_{153}^{[1]} & B_{154}^{[1]} \\ B_{161}^{[1]} & B_{162}^{[1]} & B_{163}^{[1]} & B_{164}^{[1]} \end{bmatrix},$$

$$D^{[0]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{116}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{216}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{316}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{416}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{516}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{616}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{716}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{816}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{916}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1016}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1116}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1216}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1316}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1416}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1516}^{[0]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{1616}^{[0]} \end{bmatrix},$$

$$D^{[1]} = \begin{bmatrix} D_{11}^{[1]} & D_{12}^{[1]} & D_{13}^{[1]} & D_{14}^{[1]} \\ D_{21}^{[1]} & D_{22}^{[1]} & D_{23}^{[1]} & D_{24}^{[1]} \\ D_{31}^{[1]} & D_{32}^{[1]} & D_{33}^{[1]} & D_{34}^{[1]} \\ D_{41}^{[1]} & D_{42}^{[1]} & D_{43}^{[1]} & D_{44}^{[1]} \\ D_{51}^{[1]} & D_{52}^{[1]} & D_{53}^{[1]} & D_{54}^{[1]} \\ D_{61}^{[1]} & D_{62}^{[1]} & D_{63}^{[1]} & D_{64}^{[1]} \\ D_{71}^{[1]} & D_{72}^{[1]} & D_{73}^{[1]} & D_{74}^{[1]} \\ D_{81}^{[1]} & D_{82}^{[1]} & D_{83}^{[1]} & D_{84}^{[1]} \\ D_{91}^{[1]} & D_{92}^{[1]} & D_{93}^{[1]} & D_{94}^{[1]} \\ D_{101}^{[1]} & D_{102}^{[1]} & D_{103}^{[1]} & D_{104}^{[1]} \\ D_{111}^{[1]} & D_{112}^{[1]} & D_{113}^{[1]} & D_{114}^{[1]} \\ D_{121}^{[1]} & D_{122}^{[1]} & D_{123}^{[1]} & D_{124}^{[1]} \\ D_{131}^{[1]} & D_{132}^{[1]} & D_{133}^{[1]} & D_{134}^{[1]} \\ D_{141}^{[1]} & D_{142}^{[1]} & D_{143}^{[1]} & D_{144}^{[1]} \\ D_{151}^{[1]} & D_{152}^{[1]} & D_{153}^{[1]} & D_{154}^{[1]} \\ D_{161}^{[1]} & D_{162}^{[1]} & D_{163}^{[1]} & D_{164}^{[1]} \end{bmatrix}.$$

The entries of $B^{[0]}$, $B^{[1]}$, $D^{[0]}$, and $D^{[1]}$ are listed in Appendix II in the Supplementary Material section, while the vectors $Y_m^{[0]}$, $Y_m^{[1]}$, $F_m^{[0]}$, $F_m^{[1]}$, $G_m^{[0]}$, $G_m^{[1]}$ are defined as follows:

$$Y_m^{[0]} = \begin{bmatrix} y_{n-t} \\ y_{n-s} \\ y_{n-r} \\ y_n \\ y'_{n-t} \\ y'_{n-s} \\ y'_{n-r} \\ y'_n \\ y''_{n-t} \\ y''_{n-s} \\ y''_{n-r} \\ y''_n \\ y'''_{n-t} \\ y'''_{n-s} \\ y'''_{n-r} \\ y'''_n \end{bmatrix},$$

$$Y_m^{[1]} = \begin{bmatrix} y_{n+r} \\ y_{n+s} \\ y_{n+t} \\ y_{n+1} \\ y'_{n+r} \\ y'_{n+s} \\ y'_{n+t} \\ y'_{n+1} \\ y''_{n+r} \\ y''_{n+s} \\ y''_{n+t} \\ y''_{n+1} \\ y'''_{n+r} \\ y'''_{n+s} \\ y'''_{n+t} \\ y'''_{n+1} \end{bmatrix},$$

$$F_m^{[0]} = \begin{bmatrix} f_{n-15} \\ f_{n-14} \\ f_{n-13} \\ f_{n-12} \\ f_{n-11} \\ f_{n-10} \\ f_{n-9} \\ f_{n-8} \\ f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix},$$

$$F_m^{[1]} = \begin{bmatrix} f_{n+r} \\ f_{n+s} \\ f_{n+t} \\ f_{n+1} \end{bmatrix}, \quad G_m^{[1]} = \begin{bmatrix} g_{n+r} \\ g_{n+s} \\ g_{n+t} \\ g_{n+1} \end{bmatrix},$$

$$G_m^{[0]} = \begin{bmatrix} g_{n-15} \\ g_{n-14} \\ g_{n-13} \\ g_{n-12} \\ g_{n-11} \\ g_{n-10} \\ g_{n-9} \\ g_{n-8} \\ g_{n-7} \\ g_{n-6} \\ g_{n-5} \\ g_{n-4} \\ g_{n-3} \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix}, \quad (11)$$

3. Analysis of the Method

3.1. Zero Stability

Definition 1. The hybrid block method formula (9) is said to be zero stable if no root R_m of the first characteristic equation $\rho(R)$ has modulus greater than one; that is, $|R_m| \leq 1$, and if $R_m = 1$ then the multiplicity of R_m must not exceed four.

To prove that the roots of the first characteristic equation satisfy the previous definition

$$\rho(R) = \det [RA^{[0]} - A^{[1]}] = \begin{vmatrix} R & 0 & 0 & -1 & 0 & 0 & 0 & -rh & 0 & 0 & 0 & -\frac{r^2h^2}{2} & 0 & 0 & 0 & -\frac{r^3h^3}{6} \\ 0 & R & 0 & -1 & 0 & 0 & 0 & -sh & 0 & 0 & 0 & -\frac{s^2h^2}{2} & 0 & 0 & 0 & -\frac{s^3h^3}{6} \\ 0 & 0 & R & -1 & 0 & 0 & 0 & -th & 0 & 0 & 0 & -\frac{t^2h^2}{2} & 0 & 0 & 0 & -\frac{t^3h^3}{6} \\ 0 & 0 & 0 & R-1 & 0 & 0 & 0 & -h & 0 & 0 & 0 & -\frac{h^2}{2} & 0 & 0 & 0 & -\frac{h^3}{6} \\ 0 & 0 & 0 & 0 & R & 0 & 0 & -1 & 0 & 0 & 0 & -rh & 0 & 0 & 0 & -\frac{r^2h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & R & 0 & -1 & 0 & 0 & 0 & -sh & 0 & 0 & 0 & -\frac{s^2h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & R & -1 & 0 & 0 & 0 & -th & 0 & 0 & 0 & -\frac{t^2h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 & 0 & 0 & 0 & -h & 0 & 0 & 0 & -\frac{h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & -1 & 0 & 0 & 0 & -rh \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & -1 & 0 & 0 & 0 & -sh \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & -1 & 0 & 0 & 0 & -th \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 & 0 & 0 & -h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{vmatrix} = 0, \quad (12)$$

which imply that

$$R^{12}(R-1)^4 = 0 \implies$$

$$R_m = \begin{cases} 0, & \text{if } m = 1(1)12, \\ 1, & \text{if } m = 13(1)16. \end{cases} \quad (13)$$

As a result, the developed method is zero stable.

3.2. Order of the Method. The linear operator Θ associated with the hybrid block methods formula (9) is defined as

$$\Theta \{y(x); h\} = A^{[0]}Y_m^{[1]} - A^{[1]}Y_m^{[0]} - \sum_{i=0}^1 B^{[i]}F_m^{[i]} - \sum_{i=0}^1 D^{[i]}G_m^{[i]}. \quad (14)$$

Expanding the above equation in Taylor series and combining like terms imply

$$\Theta \{y(x); h\} = \widehat{C}_0 h^0 y(x) + \widehat{C}_1 h^1 y'(x) + \widehat{C}_2 h^2 y''(x) + \dots + \widehat{C}_{p+4} h^{p+4} y^{(p+4)}(x) + \dots \quad (15)$$

According to [7, 13] method (9) is said to be of order p if

$$\widehat{C}_0 = \widehat{C}_1 = \dots = \widehat{C}_{p+3} = 0,$$

$$\widehat{C}_{p+4} \neq 0. \quad (16)$$

The term $\widehat{C}_{p+4}$ is called the error constant and the local truncation error is given by

$$t_{n+k} = \widehat{C}_{p+4} y^{(p+4)} h^{p+4} + O(h^{p+5}). \quad (17)$$

Comparing like terms of $y^{(i)}$ and h^i in (15) produces the coefficients $\widehat{C}_0 = \widehat{C}_1 = \dots = \widehat{C}_{13} = 0$ with vector of error constants

$$\widehat{C}_{14} = [\widehat{C}_{14}^{[1]} \ \widehat{C}_{14}^{[2]} \ \widehat{C}_{14}^{[3]} \ \widehat{C}_{14}^{[4]} \ \widehat{C}_{14}^{[5]} \ \widehat{C}_{14}^{[6]} \ \widehat{C}_{14}^{[7]} \ \widehat{C}_{14}^{[8]} \ \widehat{C}_{14}^{[9]} \ \widehat{C}_{14}^{[10]} \ \widehat{C}_{14}^{[11]} \ \widehat{C}_{14}^{[12]} \ \widehat{C}_{14}^{[13]} \ \widehat{C}_{14}^{[14]} \ \widehat{C}_{14}^{[15]} \ \widehat{C}_{14}^{[16]}]^T, \quad (18)$$

where

$$\begin{aligned} \widehat{C}_{14}^{[1]} = \frac{r^8}{7846046208000} & (20r^6 - 70r^5s - 70r^5t - 70r^5 \\ & + 65r^4s^2 + 260r^4st + 260r^4s + 65r^4t^2 + 260r^4t \\ & + 65r^4 - 260r^3s^2t - 260r^3s^2 - 260r^3st^2 \\ & - 1040r^3st - 260r^3s - 260r^3t^2 - 260r^3t \\ & + 286r^2s^2t^2 + 1144r^2s^2t + 286r^2s^2 + 1144r^2st^2 \\ & + 1144r^2st + 286r^2t^2 - 1430rs^2t^2 - 1430rs^2t \\ & - 1430rst^2 + 2145s^2t^2), \end{aligned}$$

$$\begin{aligned} \widehat{C}_{14}^{[2]} = \frac{s^8}{7846046208000} & (65r^2s^4 - 260r^2s^3t - 260r^2s^3 \\ & + 286r^2s^2t^2 + 1144r^2s^2t + 286r^2s^2 - 1430r^2st^2 \\ & - 1430r^2st + 2145r^2t^2 - 70rs^5 + 260rs^4t + 260rs^4 \\ & - 260rs^3t^2 - 1040rs^3t - 260rs^3 + 1144rs^2t^2 \\ & + 1144rs^2t - 1430rst^2 + 20s^6 - 70s^5t - 70s^5 \\ & + 65s^4t^2 + 260s^4t + 65s^4 - 260s^3t^2 - 260s^3t \\ & + 286s^2t^2), \end{aligned}$$

$$\begin{aligned} \widehat{C}_{14}^{[3]} = \frac{t^8}{7846046208000} & (286r^2s^2t^2 - 1430r^2s^2t \\ & + 2145r^2s^2 - 260r^2st^3 + 1144r^2st^2 - 1430r^2st \\ & + 65r^2t^4 - 260r^2t^3 + 286r^2t^2 - 260rst^3 \\ & + 1144rst^2 - 1430rst + 260rst^4 - 1040rst^3 \\ & + 1144rst^2 - 70rt^5 + 260rt^4 - 260rt^3 + 65s^2t^4 \\ & - 260s^2t^3 + 286s^2t^2 - 70st^5 + 260st^4 - 260st^3 \\ & + 20t^6 - 70t^5 + 65t^4), \end{aligned}$$

$$\begin{aligned} \widehat{C}_{14}^{[4]} = \frac{1}{7846046208000} & (2145r^2s^2t^2 - 1430r^2s^2t \\ & + 286r^2s^2 - 1430r^2st^2 + 1144r^2st - 260r^2s \\ & + 286r^2t^2 - 260r^2t + 65r^2 - 1430rs^2t^2 + 1144rs^2t \\ & - 260rs^2 + 1144rst^2 - 1040rst + 260rs - 260rt^2 \\ & + 260rt - 70r + 286s^2t^2 - 260s^2t + 65s^2 - 260st^2 \\ & + 260st - 70s + 65t^2 - 70t + 20), \end{aligned}$$

$$\begin{aligned} \widehat{C}_{14}^{[5]} = \frac{r^7}{1307674368000} & (28r^6 - 91r^5s - 91r^5t - 91r^5 \\ & + 78r^4s^2 + 312r^4st + 312r^4s + 78r^4t^2 + 312r^4t \\ & + 78r^4 - 286r^3s^2t - 286r^3s^2 - 286r^3st^2 \\ & - 1144r^3st - 286r^3s - 286r^3t^2 - 286r^3t \\ & + 286r^2s^2t^2 + 1144r^2s^2t + 286r^2s^2 + 1144r^2st^2 \\ & + 1144r^2st + 286r^2t^2 - 1287rs^2t^2 - 1287rs^2t \\ & - 1287rst^2 + 1716s^2t^2), \end{aligned}$$

$$\begin{aligned} \widehat{C}_{14}^{[6]} = \frac{s^7}{1307674368000} & (78r^2s^4 - 286r^2s^3t - 286r^2s^3 \\ & + 286r^2s^2t^2 + 1144r^2s^2t + 286r^2s^2 - 1287r^2st^2 \\ & - 1287r^2st + 1716r^2t^2 - 91rs^5 + 312rs^4t + 312rs^4 \\ & - 286rs^3t^2 - 1144rs^3t - 286rs^3 + 1144rs^2t^2 \\ & + 1144rs^2t - 1287rst^2 + 28s^6 - 91s^5t - 91s^5 \\ & + 78s^4t^2 + 312s^4t + 78s^4 - 286s^3t^2 - 286s^3t \\ & + 286s^2t^2), \end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[7]} = & \frac{t^7}{1307674368000} \left(286r^2s^2t^2 - 1287r^2s^2t \right. \\ & + 1716r^2s^2 - 286r^2st^3 + 1144r^2st^2 - 1287r^2st \\ & + 78r^2t^4 - 286r^2t^3 + 286r^2t^2 - 286rs^2t^3 \\ & + 1144rs^2t^2 - 1287rs^2t + 312rst^4 - 1144rst^3 \\ & + 1144rst^2 - 91rt^5 + 312rt^4 - 286rt^3 + 78s^2t^4 \\ & - 286s^2t^3 + 286s^2t^2 - 91st^5 + 312st^4 - 286st^3 \\ & \left. + 28t^6 - 91t^5 + 78t^4 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[8]} = & \frac{1}{1307674368000} \left(1716r^2s^2t^2 - 1287r^2s^2t \right. \\ & + 286r^2s^2 - 1287r^2st^2 + 1144r^2st - 286r^2s \\ & + 286r^2t^2 - 286r^2t + 78r^2 - 1287rs^2t^2 + 1144rs^2t \\ & - 286rs^2 + 1144rst^2 - 1144rst + 312rs - 286rt^2 \\ & + 312rt - 91r + 286s^2t^2 - 286s^2t + 78s^2 - 286st^2 \\ & \left. + 312st - 91s + 78t^2 - 91t + 28 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[9]} = & \frac{r^6}{100590336000} \left(14r^6 - 42r^5s - 42r^5t - 42r^5 \right. \\ & + 33r^4s^2 + 132r^4st + 132r^4s + 33r^4t^2 + 132r^4t \\ & + 33r^4 - 110r^3s^2t - 110r^3s^2 - 110r^3st^2 - 440r^3st \\ & - 110r^3s - 110r^3t^2 - 110r^3t + 99r^2s^2t^2 + 396r^2s^2t \\ & + 99r^2s^2 + 396r^2st^2 + 396r^2st + 99r^2t^2 - 396rs^2t^2 \\ & \left. - 396rs^2t - 396rst^2 + 462s^2t^2 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[10]} = & \frac{s^6}{100590336000} \left(33r^2s^4 - 110r^2s^3t - 110r^2s^3 \right. \\ & + 99r^2s^2t^2 + 396r^2s^2t + 99r^2s^2 - 396r^2st^2 \\ & - 396r^2st + 462r^2t^2 - 42rs^5 + 132rs^4t + 132rs^4 \\ & - 110rs^3t^2 - 440rs^3t - 110rs^3 + 396rs^2t^2 \\ & + 396rs^2t - 396rst^2 + 14s^6 - 42s^5t - 42s^5 \\ & + 33s^4t^2 + 132s^4t + 33s^4 - 110s^3t^2 - 110s^3t \\ & \left. + 99s^2t^2 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[11]} = & \frac{t^6}{100590336000} \left(99r^2s^2t^2 - 396r^2s^2t \right. \\ & + 462r^2s^2 - 110r^2st^3 + 396r^2st^2 - 396r^2st \\ & \left. + 33r^2t^4 - 110r^2t^3 + 99r^2t^2 - 110rs^2t^3 + 396rs^2t^2 \right.\end{aligned}$$

$$\begin{aligned}& - 396rs^2t + 132rst^4 - 440rst^3 + 396rst^2 - 42rt^5 \\ & + 132rt^4 - 110rt^3 + 33s^2t^4 - 110s^2t^3 + 99s^2t^2 \\ & \left. - 42st^5 + 132st^4 - 110st^3 + 14t^6 - 42t^5 + 33t^4 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[12]} = & \frac{1}{100590336000} \left(462r^2s^2t^2 - 396r^2s^2t \right. \\ & + 99r^2s^2 - 396r^2st^2 + 396r^2st - 110r^2s + 99r^2t^2 \\ & - 110r^2t + 33r^2 - 396rs^2t^2 + 396rs^2t - 110rs^2 \\ & + 396rst^2 - 440rst + 132rs - 110rt^2 + 132rt \\ & - 42r + 99s^2t^2 - 110s^2t + 33s^2 - 110st^2 + 132st \\ & \left. - 42s + 33t^2 - 42t + 14 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[13]} = & \frac{r^5}{50295168000} \left(28r^6 - 77r^5s - 77r^5t - 77r^5 \right. \\ & + 55r^4s^2 + 220r^4st + 220r^4s + 55r^4t^2 + 220r^4t \\ & + 55r^4 - 165r^3s^2t - 165r^3s^2 - 165r^3st^2 - 660r^3st \\ & - 165r^3s - 165r^3t^2 - 165r^3t + 132r^2s^2t^2 \\ & + 528r^2s^2t + 132r^2s^2 + 528r^2st^2 + 528r^2st \\ & + 132r^2t^2 - 462rs^2t^2 - 462rs^2t - 462rst^2 \\ & \left. + 462s^2t^2 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[14]} = & \frac{s^5}{50295168000} \left(55r^2s^4 - 165r^2s^3t - 165r^2s^3 \right. \\ & + 132r^2s^2t^2 + 528r^2s^2t + 132r^2s^2 - 462r^2st^2 \\ & - 462r^2st + 462r^2t^2 - 77rs^5 + 220rs^4t + 220rs^4 \\ & - 165rs^3t^2 - 660rs^3t - 165rs^3 + 528rs^2t^2 \\ & + 528rs^2t - 462rst^2 + 28s^6 - 77s^5t - 77s^5 \\ & + 55s^4t^2 + 220s^4t + 55s^4 - 165s^3t^2 - 165s^3t \\ & \left. + 132s^2t^2 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[15]} = & \frac{t^5}{50295168000} \left(132r^2s^2t^2 - 462r^2s^2t \right. \\ & + 462r^2s^2 - 165r^2st^3 + 528r^2st^2 - 462r^2st \\ & + 55r^2t^4 - 165r^2t^3 + 132r^2t^2 - 165rs^2t^3 \\ & + 528rs^2t^2 - 462rs^2t + 220rst^4 - 660rst^3 \\ & + 528rst^2 - 77rt^5 + 220rt^4 - 165rt^3 + 55s^2t^4 \\ & - 165s^2t^3 + 132s^2t^2 - 77st^5 + 220st^4 - 165st^3 \\ & \left. + 28t^6 - 77t^5 + 55t^4 \right),\end{aligned}$$

$$\begin{aligned}\widehat{C}_{14}^{[16]} = & \frac{1}{50295168000} (462r^2s^2t^2 - 462r^2s^2t \\ & + 132r^2s^2 - 462r^2st^2 + 528r^2st - 165r^2s \\ & + 132r^2t^2 - 165r^2t + 55r^2 - 462rs^2t^2 + 528rs^2t \\ & - 165rs^2 + 528rst^2 - 660rst + 220rs - 165rt^2 \\ & + 220rt - 77r + 132s^2t^2 - 165s^2t + 55s^2 - 165st^2 \\ & + 220st - 77s + 55t^2 - 77t + 28),\end{aligned}\quad (19)$$

which concludes that the order of the developed method is $p = 10$.

3.3. Consistency

Definition 2. A block method is said to be consistent if its order p is greater than one.

Consistency property is achieved for the hybrid block method from the above analysis since the order $p = 10 \geq 1$.

3.4. Convergence

Theorem 3 (see [14]). *Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.*

The hybrid block method (9) is convergent since it fulfils both the consistency and zero stability conditions.

3.5. Region of Absolute Stability. Stability region of the hybrid block (9) is discussed in spirit of [7]. The test problem of the forms $y' = \lambda y$, $y'' = \lambda^2 y$, $y''' = \lambda^3 y$, $y^{iv} = \lambda^4 y$, and $y^v = \lambda^5 y$, where $\lambda < 0$ are substituted into the main methods of block (9), is shown below

$$\overline{A}^{[0]} \overline{Y}_{m+1} = \sum_{i=1}^4 \overline{A}^{[i]} \overline{Y}_m + \sum_{i=0}^1 \overline{B}^{[i]} \overline{F}_m + \sum_{i=0}^1 \overline{D}^{[i]} \overline{G}_m, \quad (20)$$

where

$$\overline{A}^{[0]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{A}^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{A}^{[2]} = \begin{bmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\overline{A}^{[3]} = \begin{bmatrix} 0 & 0 & 0 & \frac{r^2}{2} \\ 0 & 0 & 0 & \frac{s^2}{2} \\ 0 & 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$\overline{A}^{[4]} = \begin{bmatrix} 0 & 0 & 0 & \frac{r^3}{6} \\ 0 & 0 & 0 & \frac{s^3}{6} \\ 0 & 0 & 0 & \frac{t^3}{6} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix},$$

$$\overline{B}^{[0]} = \frac{1}{h^4} \begin{bmatrix} 0 & 0 & 0 & B_{116}^{[0]} \\ 0 & 0 & 0 & B_{216}^{[0]} \\ 0 & 0 & 0 & B_{316}^{[0]} \\ 0 & 0 & 0 & B_{416}^{[0]} \end{bmatrix},$$

$$\overline{B}^{[1]} = \frac{1}{h^4} \begin{bmatrix} B_{11}^{[1]} & B_{12}^{[1]} & B_{13}^{[1]} & B_{14}^{[1]} \\ B_{21}^{[1]} & B_{22}^{[1]} & B_{23}^{[1]} & B_{24}^{[1]} \\ B_{31}^{[1]} & B_{32}^{[1]} & B_{33}^{[1]} & B_{34}^{[1]} \\ B_{41}^{[1]} & B_{42}^{[1]} & B_{43}^{[1]} & B_{44}^{[1]} \end{bmatrix},$$

$$\overline{D}^{[0]} = \frac{1}{h^5} \begin{bmatrix} 0 & 0 & 0 & D_{116}^{[0]} \\ 0 & 0 & 0 & D_{216}^{[0]} \\ 0 & 0 & 0 & D_{316}^{[0]} \\ 0 & 0 & 0 & D_{416}^{[0]} \end{bmatrix},$$

$$\overline{D}^{[1]} = \frac{1}{h^5} \begin{bmatrix} D_{11}^{[1]} & D_{12}^{[1]} & D_{13}^{[1]} & D_{14}^{[1]} \\ D_{21}^{[1]} & D_{22}^{[1]} & D_{23}^{[1]} & D_{24}^{[1]} \\ D_{31}^{[1]} & D_{32}^{[1]} & D_{33}^{[1]} & D_{34}^{[1]} \\ D_{41}^{[1]} & D_{42}^{[1]} & D_{43}^{[1]} & D_{44}^{[1]} \end{bmatrix},$$

(21)

while the vectors $\overline{Y}_{m+1}$, $\overline{Y}_m^{[1]}$, $\overline{Y}_m^{[2]}$, $\overline{Y}_m^{[3]}$, $\overline{F}_m^{[0]}$, $\overline{F}_m^{[1]}$, $\overline{G}_m^{[0]}$, and $\overline{G}_m^{[1]}$ are defined as follows:

$$\overline{Y}_{m+1} = \begin{bmatrix} y_{n+r} \\ y_{n+s} \\ y_{n+t} \\ y_{n+1} \end{bmatrix},$$

$$\overline{Y}_m^{[1]} = \begin{bmatrix} y_{n-t} \\ y_{n-s} \\ y_{n-r} \\ y_n \end{bmatrix},$$

$$\bar{Y}_m^{[2]} = \begin{bmatrix} y'_{n-t} \\ y'_{n-s} \\ y'_{n-r} \\ y'_n \end{bmatrix},$$

$$\bar{Y}_m^{[3]} = \begin{bmatrix} y''_{n-t} \\ y''_{n-s} \\ y''_{n-r} \\ y''_n \end{bmatrix},$$

$$\bar{Y}_m^{[4]} = \begin{bmatrix} y'''_{n-t} \\ y'''_{n-s} \\ y'''_{n-r} \\ y'''_n \end{bmatrix},$$

$$\bar{F}_m^{[0]} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix},$$

$$\bar{F}_m^{[1]} = \begin{bmatrix} f_{n+r} \\ f_{n+s} \\ f_{n+t} \\ f_{n+1} \end{bmatrix},$$

$$\bar{G}_m^{[0]} = \begin{bmatrix} g_{n-3} \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix},$$

$$\bar{G}_m^{[1]} = \begin{bmatrix} g_{n+r} \\ g_{n+s} \\ g_{n+t} \\ g_{n+1} \end{bmatrix},$$

(22)

substituting $q = \lambda h$ in (20) yields

$$\bar{Y}_{m+1} = M(q) \bar{Y}_m^{[1]}, \quad (23)$$

where the amplification matrix $M(q)$ is given by

$$M(q) = \left(\bar{A}^{[0]} - q^4 \bar{B}^{[1]} - q^5 \bar{D}^{[1]} \right)^{-1} \left(\bar{A}^{[1]} + q \bar{A}^{[2]} + q^2 \bar{A}^{[3]} + q^3 \bar{A}^{[4]} + q^4 \bar{B}^{[0]} + q^5 \bar{D}^{[0]} \right). \quad (24)$$

The analysis shows that the matrix $M(q)$ has eigenvalues $\{\eta_1, \eta_2, \eta_3, \eta_4\} = \{0, 0, 0, \eta_4\}$, where the dominant eigenvalue

FIGURE 1: Region of absolute stability.

η_4 is a function of q subject to the values $\{r, s, t\} = \{1/4, 1/2, 3/4\}$ given by

$$\eta_4 = \frac{\sum_{i=0}^{20} c_i q^i}{K \sum_{j=0}^{21} c_j q^j}, \quad (25)$$

where $K = 10376293541461622784$ and the values c_i and c_j are listed in Table 1.

The region of absolute stability is depicted in the dark area in Figure 1.

3.6. Pseudocode of the Developed Method. The following pseudocode illustrates how the developed method is implemented for solving fourth-order initial value problems.

Step 1. The values $y(x_0), y'(x_0), y''(x_0), y'''(x_0)$ on the interval $[a, b]$ from the problem are given.

Step 2. Set the values of the hybrid points $\{r, s, t\}$ and step size h .

Step 3. For $j = a : h : b$, set $x_{n+j} = x_n + jh, j = \{r, s, t, 1\}$.

Step 4. Evaluate the approximate values $y(x_{n+j}), y'(x_{n+j}), y''(x_{n+j}), y'''(x_{n+j})$ using the direct hybrid block method.

Step 5. Find the solution for the resultant system using the built in function `fsolve` in Matlab.

Step 6. Calculate the maximum error = |Exact solution – approximate solution|.

4. Numerical Examples

The general three-hybrid one-step hybrid block method (9) with order $p = 10$ in this section was employed to solve five problems from the literature. In order to show the performance of the proposed method, the problems are

TABLE 1

c-value	q^i coefficients	q^j coefficients
c_0	-19991259433786613699768237287038031044280320	0
c_1	-19991259433786613699768237287038031044280320	192662816967402505371648000
c_2	-99956297168933068498841186435190155221401600	0
c_3	-33318765722977689499613728811730051740467200	0
c_4	-78083522390790487043756727396404132472422400	0
c_5	-12270936643443632883203625005043197725900800	502432963738164264960000
c_6	-994813550529796720162894578620563036569600	-79502926099016318976000
c_7	59773830578760679822424316692208078028800	0
c_8	31534047193242772464338416945297159618560	0
c_9	4794949827573668397142419401567455150080	133171866570211983360
c_{10}	359049264395827172053492731789485015040	-14458011284982988800
c_{11}	-3356754007384253296917217569574748160	268452685173657600
c_{12}	-4527291309580589178991255608586403840	0
c_{13}	-632549152682975031987854772642562048	5775969118252032
c_{14}	-50983394698341254755895386446446592	-659482112163840
c_{15}	-2347009723724983385107341899462400	26223671235440
c_{16}	3581359970953878842628978864128	-529590387200
c_{17}	8897056249188627790299370721984	67083520896
c_{18}	741826115832694210392858425664	-8638140672
c_{19}	30585824514148316274163930311	506275974
c_{20}	582915589719914249658541800	-15909210
c_{21}	0	189567

solved with different values of step size h and hybrid points $\{r, s, t\}$.

$$\begin{aligned}y''(0) &= 3, \\ y'''(0) &= 1.\end{aligned}$$

Problem 1 (linear fourth-order problem).

(27)

$$\begin{aligned}y^{(iv)} &= y''' + y'' + y' + 2y, \\ 0 &\leq x \leq 2,\end{aligned}$$

$$\begin{aligned}y(0) &= 0, \\ y'(0) &= 0, \\ y''(0) &= 0, \\ y'''(0) &= 30.\end{aligned}\tag{26}$$

Exact Solution. $y = x^2 + e^x$ with $h = 1/10$.

Source. See [15].

Problem 3 (application to problem from ship dynamics). In this example, the derived method is implemented to solve a physical problem from ship dynamics. As stated by [3], when a sinusoidal wave of frequency passes along a ship or off-shore structure, the resultant fluid actions vary with time. In a particular case study by [3], the fourth-order problem is defined as

$$y^{(iv)} + 3y'' + y(2 + \epsilon \cos(\omega(x))) = 0,$$

Exact Solution. $y = 2e^{2x} - 5e^{-x} + 3 \cos(x) - 9 \sin(x)$.

$x > 0$,

Source. See [15].

$$y(0) = 1,$$

(28)

Problem 2 (nonlinear fourth-order problem).

$$y'(0) = y''(\rho) = y'''(0) = 0.$$

$$y^{(iv)} = (y')^2 - yy''' - 4x^2 + e^x(1 + x^2 - 4x),$$

$$0 \leq x \leq 1,$$

$$y(0) = 1,$$

$$y'(0) = 1,$$

Exact Solution. $y(x) = 2 \cos(x) + \cos(\sqrt{2}x)$.

Problem 4 (system of linear fourth-order IVPs).

$$y_1^{(iv)} = y_1 + y_4 - e^{2x},$$

$$y(0) = 1,$$

$$y'(0) = 1,$$

$$y''(0) = 1,$$

$$y'''(0) = 1,$$

$$0 \leq x \leq 2,$$

$$y_2^{(iv)} = y_2 + y_3 - xe^x,$$

$$y(0) = 1,$$

$$y'(0) = -1,$$

$$y''(0) = 1,$$

$$y'''(0) = -1,$$

$$0 \leq x \leq 2,$$

$$y_3^{(iv)} = 4y_1 + y_3,$$

$$y(0) = 0,$$

$$y'(0) = 1,$$

$$y''(0) = 2,$$

$$y'''(0) = 3,$$

$$0 \leq x \leq 2,$$

$$y_4^{(iv)} = y_1 + y_2 + 16y_4 - e^x - e^{-x},$$

$$y(0) = 1,$$

$$y'(0) = 2,$$

$$y''(0) = 4,$$

$$y'''(0) = 8,$$

$$0 \leq x \leq 2,$$

(29)

Exact Solution. $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = xe^x$, and $y_4 = e^{2x}$ with $h = 1/5$.

Problem 5 (system of nonlinear fourth-order IVPs).

$$y_1^{(iv)} = y_1 + y_2 - \cos(x),$$

$$y(0) = 0,$$

$$y'(0) = 1,$$

$$y''(0) = 0,$$

$$y'''(0) = -1,$$

$$0 \leq x \leq \pi,$$

$$y_2^{(iv)} = y_1 + y_2 - \sin(x),$$

$$y(0) = 1,$$

$$y'(0) = 0,$$

$$y''(0) = -1,$$

$$y'''(0) = 0,$$

$$0 \leq x \leq \pi,$$

$$y_3^{(iv)} = 32y_1y_2,$$

$$y(0) = 0,$$

$$y'(0) = 2,$$

$$y''(0) = 0,$$

$$y'''(0) = -8,$$

$$0 \leq x \leq \pi,$$

$$y_4^{(iv)} = 16(y_2^2 - y_1^2),$$

$$y(0) = 1,$$

$$y'(0) = 0,$$

$$y''(0) = -2,$$

$$y'''(0) = 0,$$

$$0 \leq x \leq \pi,$$

(30)

Exact Solution. $y_1 = \sin(x)$, $y_2 = \cos(x)$, $y_3 = \sin(2x)$, and $y_4 = x^2 + \cos(2x)$, with $h = \pi/5$.

5. Conclusion

A hybrid one-step block method of order $p = 10$ with generalized three-off-step points has been derived successfully. The developed method was employed to solve general fourth-order IVPs of ODEs directly. The numerical analysis performed shows that the developed method is consistent and zero stable which leads to the convergence of the method. The numerical results were then compared with the results obtained by the existing methods in terms of error. The new method is found to have better performance over the other methods (refer to Tables 2–12).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

TABLE 2: Comparison of the derived method with [15, 16] and Adams Bashforth-Adams Moulton method.

Step size	Method employed	Absolute error at $x = 2$
0.1	Derived method	$1.34E(-11)$
	BHCM4	$1.74E(-8)$
	Jator	$1.26E(-4)$
	Adams	$2.11E(-3)$
0.05	Derived method	$1.95E(-11)$
	BHCM4	$8.45E(-11)$
	Jator	$1.91E(-6)$
	Adams	$5.37E(-4)$

TABLE 3: Comparison of the derived method with [15, 17] and Adams Bashforth-Adams Moulton method.

Step size	Method employed	Absolute error at $x = 1$
0.2	Derived method	$2.22E(-15)$
	BHCM4	$2.38E(-12)$
	Adams	$5.01E(-7)$
	Awoyemi	$5.84E(-4)$
0.1	Derived method	$1.77E(-15)$
	BHCM4	$1.95E(-14)$
	Adams	$2.44E(-6)$
	Awoyemi	$9.26E(-5)$

TABLE 4: Comparison of the proposed method with [15] for the case $\epsilon = 0$.

Step size	Method employed	Absolute error at $x = 15$
0.25	Derived method	$1.1E(-16)$
	BHCM4	$5.2E(-7)$
	Twizell	$1.9E(-4)$
	Adams	$4.9E(-3)$
0.1	Derived method	$1.1E(-14)$
	BHCM4	$2.8E(-10)$
	Cortell	$3.7E(-5)$
	Adams	$8.4E(-5)$

TABLE 5: Exact and approximate solutions for solving y_1 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_1
0.20	1.221402758160169900	1.221402758160169900	$0.000000E(+00)$
0.40	1.491824697641270300	1.491824697641270300	$0.000000E(+00)$
0.60	1.822118800390509100	1.822118800390508900	$2.220446E(-16)$
0.80	2.225540928492467900	2.225540928492467400	$4.440892E(-16)$
1.00	2.718281828459045500	2.718281828459045100	$4.440892E(-16)$
1.20	3.320116922736547200	3.320116922736547700	$4.440892E(-16)$
1.40	4.055199966844674500	4.055199966844674500	$0.000000E(+00)$
1.60	4.953032424395114000	4.953032424395114900	$8.881784E(-16)$
1.80	6.049647464412944800	6.049647464412945700	$8.881784E(-16)$
2.00	7.389056098930648600	7.389056098930650400	$1.776357E(-15)$

TABLE 6: Exact and approximate solutions for solving y_2 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_2
0.20	0.818730753077981820	0.818730753077981820	0.000000E(+00)
0.40	0.670320046035639330	0.670320046035639330	0.000000E(+00)
0.60	0.548811636094026390	0.548811636094026390	0.000000E(+00)
0.80	0.449328964117221560	0.449328964117221620	5.551115E(-17)
1.00	0.367879441171442330	0.367879441171442330	0.000000E(+00)
1.20	0.301194211912202080	0.301194211912202080	0.000000E(+00)
1.40	0.246596963941606490	0.246596963941606460	2.775558E(-17)
1.60	0.201896517994655440	0.201896517994655380	5.551115E(-17)
1.80	0.165298888221586560	0.165298888221586500	5.551115E(-17)
2.00	0.135335283236612730	0.135335283236612650	8.326673E(-17)

TABLE 7: Exact and approximate solutions for solving y_3 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_3
0.20	0.244280551632033990	0.244280551632033970	2.775558E(-17)
0.40	0.596729879056508210	0.596729879056508090	1.110223E(-16)
0.60	1.093271280234305600	1.093271280234305400	2.220446E(-16)
0.80	1.780432742793974300	1.780432742793974100	2.220446E(-16)
1.00	2.718281828459045500	2.718281828459045100	4.440892E(-16)
1.20	3.984140307283856600	3.984140307283857000	4.440892E(-16)
1.40	5.677279953582544000	5.677279953582544000	0.000000E(+00)
1.60	7.924851879032181600	7.924851879032183400	1.776357E(-15)
1.80	10.889365435943299000	10.889365435943303000	3.552714E(-15)
2.00	14.778112197861295000	14.778112197861301000	5.329071E(-15)

TABLE 8: Exact and approximate solutions for solving y_4 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_4
0.20	1.491824697641270300	1.491824697641270300	0.000000E(+00)
0.40	2.225540928492467900	2.225540928492467400	4.440892E(-16)
0.60	3.320116922736548100	3.320116922736547700	4.440892E(-16)
0.80	4.953032424395114900	4.953032424395114900	0.000000E(+00)
1.00	7.389056098930650400	7.389056098930650400	0.000000E(+00)
1.20	11.023176380641601000	11.023176380641601000	0.000000E(+00)
1.40	16.444646771097048000	16.444646771097048000	0.000000E(+00)
1.60	24.532530197109342000	24.532530197109349000	7.105427E(-15)
1.80	36.598234443677974000	36.598234443677988000	1.421085E(-14)
2.00	54.598150033144215000	54.598150033144236000	2.131628E(-14)

TABLE 9: Exact and approximate solutions for solving y_1 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_1
0.6283185	0.587785252292473140	0.587785252292475800	2.664535E(-15)
1.2566371	0.951056516295153530	0.951056516295303300	1.497691E(-13)
1.8849556	0.951056516295153640	0.951056516296290070	1.136424E(-12)
2.5132741	0.587785252292473250	0.587785252297075340	4.602096E(-12)
3.1415927	0.000000000000000122	0.00000000013537650	1.353753E(-11)

TABLE 10: Exact and approximate solutions for solving y_2 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_2
0.6283185	0.809016994374947450	0.809016994374955000	7.549517E(-15)
1.2566371	0.309016994374947450	0.309016994375326810	3.793632E(-13)
1.8849556	-0.309016994374947340	-0.309016994372823590	2.123746E(-12)
2.5132741	-0.809016994374947340	-0.809016994368452200	6.495138E(-12)
3.1415927	-1.000000000000000000	-0.999999999984998780	1.500122E(-11)

TABLE 11: Exact and approximate solutions for solving y_3 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_3
0.6283185	0.951056516295153530	0.951056516302223540	7.070011E(-12)
1.2566371	0.587785252292473250	0.587785252378536070	8.606282E(-11)
1.8849556	-0.587785252292473030	-0.587785251895384220	3.970888E(-10)
2.5132741	-0.951056516295153640	-0.951056515173170910	1.121983E(-09)
3.1415927	-0.0000000000000000245	0.000000002364014064	2.364014E(-09)

TABLE 12: Exact and approximate solutions for solving y_4 using the developed method.

X-value	Exact solution	Approximate solution	Error in solving y_4
0.6283185	0.703801170418521750	0.703801170427732160	9.210410E(-12)
1.2566371	0.770119709799349850	0.770119709892383430	9.303358E(-11)
1.8849556	2.744040590017221300	2.744040590323378700	3.061573E(-10)
2.5132741	6.625563811072136200	6.625563811678808300	6.066720E(-10)
3.1415927	10.869604401089358000	10.869604401982164000	8.928058E(-10)

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