Research Article

Stationary Distribution and Dynamic Behaviour of a Stochastic SIVR Epidemic Model with Imperfect Vaccine

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We consider a stochastic SIVR (susceptible-infected-vaccinated-recovered) epidemic model with imperfect vaccine. First, we obtain critical condition under which the disease is persistent in the mean. Second, we establish sufficient conditions for the existence of an ergodic stationary distribution to the model. Third, we study the extinction of the disease. Finally, numerical simulations are given to support the analytical results.

1. Introduction

Mathematical models have been an unavoidable tool in analyzing the mechanisms of infectious diseases. Most modelers were inspired by the works in [1–3]. For controlling the spread of diseases, vaccination is considered the most effective way to reduce both the morbidity and mortality of individuals (see [4–6]). In this paper, we consider the deterministic SIVR (susceptible-infected-vaccinated-recovered) model:

$$\begin{aligned}dS &= [\mu - (\mu + \phi)S - \beta SI] dt, \\dI &= [\beta(S + \eta V)I - (\mu + \gamma)I] dt, \\dV &= [\phi S - \eta\beta VI - \mu V] dt, \\dR &= [\gamma I - \mu R] dt,\end{aligned}\tag{1}$$

where S , I , and R denote the densities of susceptible, infected, and recovered individuals, respectively. V denotes the density of individuals who are immune to an infection as result of vaccination.

The parameters involved in the system are described below:

β : the average number of contacts per infected per unit time

μ : the recruitment rate and the death rate

γ : recovery rate of infected individuals

ϕ : the rate at which susceptible individuals are moved into the vaccination process

η : a positive factor satisfying $0 \leq \eta \leq 1$, and $\eta = 0$ means that the vaccine is perfectly effective and $\eta = 1$ means that the vaccine has no effect

In model (1), the fundamental parameter that governs the spread of the disease into a population is the basic reproduction number denoted by R_0 . It can be thought of as the number of cases one case generates on average over the course of its infectious period in an otherwise uninfected population (see [7]).

Let us denote by $R_0 = \beta/(\mu + \gamma)$ the basic reproduction number of model (1) and by $R_\phi = R_0((\mu + \eta\phi)/(\mu + \phi))$ the basic reproduction number in a population in which a proportion ϕ had been vaccinated. It is known that, in the absence of the disease, there is a unique disease-free equilibrium $(S_0, I_0, V_0, R_0) = (\mu/(\mu + \phi), 0, \phi/(\mu + \phi), 0)$ which is globally asymptotically stable if $R_\phi < 1$. If $R_\phi > 1$ for some parameters values, the model exhibits a backward bifurcation leading to the existence of multiple endemic equilibria and news subthreshold, which may be important when it comes to designing vaccination strategies (see [8, 9]).

It is well known that epidemic models are inevitably affected by the environmental noise that influences the

dynamic behaviour of the epidemic models (see [10–14]). For this, inspired by the works in [15–17], Tornatore et al. [18] have formulated and studied a stochastic version of model (1) by replacing the contact rate β in system (1) by $\beta + \sigma(dW(t)/dt)$, where $W(t)$ denotes the standard Brownian motion and σ^2 denotes the intensity of the white noise. Then, system (1) becomes as follows:

$$\begin{aligned} dS &= [\mu - (\mu + \phi)S - \beta SI] dt - \sigma SI dW(t), \\ dI &= [\beta(S + \eta V)I - (\mu + \gamma)I] dt \\ &\quad + \sigma(S + \eta V)I dW(t), \\ dV &= [\phi S - \eta\beta VI - \mu V] dt - \sigma\eta VI dW(t), \\ dR &= [\gamma I - \mu R] dt. \end{aligned} \quad (2)$$

Dianli Zhao and Sanling Yuan [19] have obtained conditions ensuring the persistence and extinction of model (2) and have found a threshold $R_\phi^S = (1/(\mu + \gamma))[\beta((\mu + \eta\phi)/(\mu + \phi)) - (\sigma^2/2)((\mu + \eta\phi)/(\mu + \phi))^2]$ whose value below 1 or above 1 can determine the extinction and persistence of the epidemic under mild extra conditions.

In this paper, we assume that the multiplicative noise sources are linear in $S(t)$, $I(t)$ and $V(t)$, according to [20]. Note that recovered population has no effect on the dynamics of S , I , and V . Then, following this approach, we obtain the following reduced stochastic SIVR model:

$$\begin{aligned} dS &= [\mu - (\mu + \phi)S - \beta SI] dt + \sigma_1 S dB_1(t), \\ dI &= [\beta(S + \eta V)I - (\mu + \gamma)I] dt + \sigma_2 I dB_2(t), \\ dV &= [\phi S - \eta\beta VI - \mu V] dt + \sigma_3 V dB_3(t), \end{aligned} \quad (3)$$

where $B_1(t)$, $B_2(t)$, and $B_3(t)$ are standard independent Brownian motions and σ_i^2 is a positive constant, for all $i \in \{1, 2, 3\}$.

This paper is organized as follows. In Section 2, we present some lemmas concerning the existence of a global positive solution and ergodic stationary distribution. In Section 3, we prove that the disease is persistent under one condition. In Section 4, we establish sufficient conditions for the existence of a unique ergodic stationary distribution. In Section 5, we determine a condition under which the disease goes to extinction. In the last section, we introduce some examples and numerical simulations to confirm our results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Moreover, let $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i > 0, i = 1, \dots, d\}$. If f is a bounded function on $\mathbb{R}_+$, define $\bar{f} = \sup_{t \in [0, \infty)} f(t)$ and $\underline{f} = \inf_{t \in [0, \infty)} f(t)$. We define also $\langle h \rangle = (1/t) \int_0^t h(s) ds$.

The following theorem concerns the existence and uniqueness of the global positive solution. Since the proof is

almost the same as that in [21, Lemma 2.1 and Lemma 2.2], we omit it here.

Theorem 1. For any initial value $(S(0), I(0), V(0)) \in \mathbb{R}_+^3$, there is a unique solution $(S(t), I(t), v(t)) \in \mathbb{R}_+^3$ of system (3) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one. The solution $(S(t), I(t), v(t))$ has the following properties:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{S(t)}{t} &= 0, \\ \lim_{t \rightarrow \infty} \frac{I(t)}{t} &= 0, \\ \lim_{t \rightarrow \infty} \frac{V(t)}{t} &= 0, \\ &\text{a.s.,} \\ \limsup_{t \rightarrow \infty} \frac{\log S(t)}{t} &\leq 0, \\ \limsup_{t \rightarrow \infty} \frac{\log I(t)}{t} &\leq 0, \\ \limsup_{t \rightarrow \infty} \frac{\log V(t)}{t} &\leq 0, \\ &\text{a.s.} \end{aligned} \quad (4)$$

Furthermore, if $\mu > \max(\sigma_1^2, \sigma_2^2, \sigma_3^2)/2$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_0^t S(r) dB_1(r)}{t} &= 0, \\ \lim_{t \rightarrow \infty} \frac{\int_0^t I(r) dB_2(r)}{t} &= 0, \\ \lim_{t \rightarrow \infty} \frac{\int_0^t V(r) dB_3(r)}{t} &= 0, \\ &\text{a.s.} \end{aligned} \quad (5)$$

Now, we consider the d -dimensional stochastic differential equation:

$$dx(t) = f(x(t), t) dt + g(x(t), t) dB(t) \quad (6)$$

for all $t \geq t_0$,

with initial value $x(0) = x_0 \in \mathbb{R}^d$. $B(t)$ denotes an n -dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let us denote by $\mathcal{C}^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such

that they are continuously twice differentiable in x and once in t . The differential operator L of (6) is defined by [22]

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d [g^T(x, t) g(x, t)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \quad (7)$$

If L acts on a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(x, t) = V_t(x, t) + V_x(x, t) f(x, t) + \frac{1}{2} \text{trace} [g^T(x, t) V_{xx}(x, t) g(x, t)], \quad (8)$$

where

$$\begin{aligned} V_t &= \frac{\partial V}{\partial t}, \\ V_x &= \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_d} \right), \\ V_{xx} &= \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d \times d}. \end{aligned} \quad (9)$$

By Itô's formula, we get

$$dV(x(t), t) = LV(x(t), t) dt + V_x(x(t), t) g(x(t), t) dB(t). \quad (10)$$

Next, we shall present a lemma that gives a criterion for the existence of an ergodic stationary distribution to system (3).

Let $X(t)$ be a homogeneous Markov process in E_d (E_d denotes d -dimensional Euclidian space) and it is described by the following stochastic differential equation:

$$dX(t) = h(X(t)) dt + \sum_{r=1}^k g_r(X(t)) dB_r(t) \quad (11)$$

The diffusion matrix is defined as follows:

$$\begin{aligned} A(X) &= (a_{ij}(X)), \\ a_{ij}(X) &= \sum_{r=1}^k g_r^i(X) g_r^j(X). \end{aligned} \quad (12)$$

Lemma 2 (see [23], Chapter 4). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded domain $D \subset E_d$ with regular boundary Γ and*

(H.1): *there is a positive number λ such that $\sum_{i,j=1}^d a_{ij}(X) \xi_i \xi_j \geq \lambda |\xi|^2$, $X \in D$, $\xi \in \mathbb{R}^d$,*

(H.2): *there exists a nonnegative $\mathcal{C}^2$ -function V such that LV is negative for any $E_d \setminus D$. Then*

$$\begin{aligned} \mathbb{P}_x \left\{ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Phi(X(s)) ds = \int_{E_d} \Phi(x) \pi(x) dx \right\} \\ = 1, \end{aligned} \quad (13)$$

for all $x \in E_d$, where $\Phi(\cdot)$ is a function integrable with respect to the measure π .

Now, we consider the following one-dimensional homogeneous Markov process:

$$\begin{aligned} dX(t) &= b(X(t)) dt + \sigma(X(t)) dB(t), \\ X(t_0) &= X_0, \quad \text{for all } t \geq t_0, \end{aligned} \quad (14)$$

where $b(X)$ and $\sigma(X)$ are measurable functions from $\mathbb{R}$ to $\mathbb{R}$ and $B(t)$ is the Brownian motion. It is assumed that the functions $b(X)$, $\sigma(X)$, and $1/\sigma(X)$ are locally bounded.

(H.3): the functions $b(X)$ and $\sigma(X)$ are such that

$$Y(x) = \int_0^x \exp \left[-2 \int_0^s \frac{b(u)}{\sigma^2(u)} du \right] ds \rightarrow \pm \infty \quad \text{as } x \rightarrow \pm \infty, \quad (15)$$

$$C = \int_{-\infty}^{+\infty} \frac{1}{\sigma^2(x)} \exp \left[2 \int_0^x \frac{b(u)}{\sigma^2(u)} du \right] dx < \infty.$$

Lemma 3 (see [24], Theorem 1.13). *If (H.3) is satisfied, then the stochastic process (14) has ergodic properties with the density given by*

$$\pi(x) = \frac{1}{C \sigma^2(x)} \exp \left[2 \int_0^x \frac{b(u)}{\sigma^2(u)} du \right]. \quad (16)$$

3. Persistence

Definition 4. System (3) is said to be persistent in the mean, if

$$\liminf_{t \rightarrow \infty} \langle I(t) \rangle > 0 \quad \text{a.s.} \quad (17)$$

Theorem 5. *If $R_0^S > 1$, then the disease will be persistent in mean; that is,*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \beta(a + b + \eta c) \langle I(t) \rangle \\ \geq \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1), \quad \text{a.s.}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} a &= \frac{\beta \mu}{(\mu + \phi + \sigma_1^2/2)^2}, \\ b &= \frac{\eta \beta \mu \phi}{(\mu + \phi + \sigma_1^2/2)^2 (\mu + \sigma_3^2/2)}, \\ c &= \frac{\eta \beta \mu \phi}{(\mu + \phi + \sigma_1^2/2) (\mu + \sigma_3^2/2)^2}, \\ R_0^S &= \frac{\beta \mu}{(\mu + \phi + \sigma_1^2/2) (\mu + \gamma + \sigma_2^2/2)} \\ &\quad + \frac{\eta \beta \mu \phi}{(\mu + \phi + \sigma_1^2/2) (\mu + \gamma + \sigma_2^2/2) (\mu + \sigma_3^2/2)}. \end{aligned} \quad (19)$$

Proof. Set

$$V_1 = -\log I - (a + b) \log S - c \log V, \quad (20)$$

where positive constants a, b , and c will be determined later.

We apply Itô's formula on V_1 ; then we get

$$\begin{aligned} dV_1 = & \left[-\frac{a\mu}{S} - \frac{b\mu}{S} + (a + b)(\mu + \phi) + \beta(a + b)I \right. \\ & - \beta S - \eta\beta V + (\mu + \gamma) - \frac{c\phi S}{V} + c\eta\beta I + \mu c + \frac{a}{2}\sigma_1^2 \\ & + \frac{b}{2}\sigma_1^2 + \frac{\sigma_2^2}{2} + \frac{c}{2}\sigma_3^2 \left. \right] dt - [\sigma_1(a + b)dB_1 \\ & + \sigma_2 dB_2 + c\sigma_3 dB_3] = \left[-\beta S - \frac{a\mu}{S} - \frac{b\mu}{S} - \eta\beta V \right. \\ & - \frac{c\phi S}{V} + \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) + \beta(a + b + \eta c)I \\ & + a \left(\mu + \phi + \frac{\sigma_1^2}{2} \right) + b \left(\mu + \phi + \frac{\sigma_1^2}{2} \right) \\ & + c \left(\mu + \frac{\sigma_3^2}{2} \right) \left. \right] dt - [\sigma_1(a + b)dB_1 + \sigma_2 dB_2 \\ & + c\sigma_3 dB_3] \leq \left[-2\sqrt{a\beta\mu} - 3\sqrt{bc\eta\beta\mu\phi} \right. \\ & + \beta(a + b + \eta c)I + \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) \\ & + a \left(\mu + \phi + \frac{\sigma_1^2}{2} \right) + b \left(\mu + \phi + \frac{\sigma_1^2}{2} \right) \\ & + c \left(\mu + \frac{\sigma_3^2}{2} \right) \left. \right] dt - [\sigma_1(a + b)dB_1 + \sigma_2 dB_2 \\ & + c\sigma_3 dB_3] \end{aligned} \quad (21)$$

Let

$$\begin{aligned} a &= \frac{\beta\mu}{(\mu + \phi + \sigma_1^2/2)^2}, \\ b &= \frac{\eta\beta\mu\phi}{(\mu + \phi + \sigma_1^2/2)^2 (\mu + \sigma_3^2/2)}, \\ c &= \frac{\eta\beta\mu\phi}{(\mu + \phi + \sigma_1^2/2) (\mu + \sigma_3^2/2)^2}. \end{aligned} \quad (22)$$

Then

$$\begin{aligned} dV_1 \leq & \left[-\frac{\beta\mu}{\mu + \phi + \sigma_1^2/2} - \frac{\eta\beta\mu\phi}{(\mu + \phi + \sigma_1^2/2)(\mu + \sigma_3^2/2)} \right. \\ & + \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) + \beta(a + b + \eta c)I \left. \right] dt \\ & - [\sigma_1(a + b)dB_1 + \sigma_2 dB_2 + c\sigma_3 dB_3]. \end{aligned} \quad (23)$$

By integrating the last inequality, we obtain

$$\begin{aligned} \frac{V_1(S(t), I(t), V(t)) - V_1(S(0), I(0), V(0))}{t} \\ \leq - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + \beta(a + b + \eta c) \langle I \rangle \\ - \frac{1}{t} \int_0^t \sigma_1(a + b) dB_1(s) + \sigma_2 dB_2(s) \\ + c\sigma_3 dB_3(s). \end{aligned} \quad (24)$$

By applying Theorem 1 and the strong law of large numbers for martingales [22], we get the desired result. $\square$

4. Stationary Distribution

In this section, by using the theory of Has'minski [18], we prove the existence of a unique ergodic stationary distribution, which indicates that the disease is persistent.

Theorem 6. *If $R_0^S > 1$, then system (3) has a unique stationary distribution and it has the ergodic property.*

Proof. To prove Theorem 6, it is sufficient to verify assumptions (H.1) and (H.2).

(i) To verify (H.2), we show that there exists a neighborhood $D \subset \mathbb{R}_+^3$ and a nonnegative $\mathcal{C}^2$ -function V such that $LV(X)$ is negative for any $X \in \mathbb{R}_+^3 \setminus D$. To this end, we define a $\mathcal{C}^2$ -function in the form

$$\begin{aligned} W(S, I, V) = & MV_1(S, I, V) + V_2(S, I, V) - \log S \\ & - \log V, \end{aligned} \quad (25)$$

where V_1 is the same function defined in Section 3, $V_2 = (1/(1 + \theta))(S + I + V)^{1+\theta}$, and M, θ are positive constants that satisfy

$$\begin{aligned} -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} \\ < 0, \end{aligned} \quad (26)$$

$$\mu > \frac{\theta}{2} \max(\sigma_1^2, \sigma_2^2, \sigma_3^2),$$

where

$$A = \sup_{(S, I, V) \in \mathbb{R}_+^3} \left\{ \mu(S + I + V)^\theta - \frac{\rho}{2}(S + I + V)^{\theta+1} \right\}. \quad (27)$$

It is easy to see that

$$\lim_{(S^2+I^2+V^2) \rightarrow \infty} W(S, I, V) = \lim_{(S^2+I^2+V^2) \rightarrow 0} W(S, I, V) = +\infty. \quad (28)$$

Moreover, $W(S, I, V)$ is a continuous function. Hence, $W(S, I, V)$ must have a minimum point $(\bar{S}, \bar{I}, \bar{V})$ in the interior of $\mathbb{R}_+^3$. Then we define a nonnegative $\mathcal{C}^2$ -function V as

$$V(S, I, V) = W(S, I, V) - W(\bar{S}, \bar{I}, \bar{V}). \quad (29)$$

From the proof of Theorem 5, we have

$$LV_1(S, I, V) \leq -\left(\mu + \gamma + \frac{\sigma_2^2}{2}\right)(R_0^S - 1) + \beta(a + b + \eta c)I \quad (30)$$

The operator L defined in Section 2 acts on V_2 , $-\log S$, and $-\log V$ as follows:

$$\begin{aligned} L(-\log S) &= -\frac{\mu}{S} + \mu + \phi + \beta I + \frac{\sigma_1^2}{2}, \\ L(-\log V) &= -\frac{\phi S}{V} + \mu + \eta \beta I + \frac{\sigma_3^2}{2}, \\ LV_2(S, I, V) &= (S + I + V)^\theta [\mu - \mu(S + I + V) - \gamma I] \\ &\quad + \frac{\theta}{2} (S + I + V)^{\theta-1} (\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 V^2) \\ &\leq \mu(S + I + V)^\theta - \mu(S + I + V)^{\theta+1} \\ &\quad + \frac{\theta}{2} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) (S + I + V)^{\theta+1} \\ &= \mu(S + I + V)^\theta \\ &\quad - \left[\mu - \frac{\theta}{2} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) \right] (S + I + V)^{\theta+1} \\ &= \mu(S + I + V)^\theta - \rho(S + I + V)^{\theta+1} \\ &\leq A - \frac{\rho}{2} (S^{1+\theta} + I^{1+\theta} + V^{1+\theta}), \end{aligned} \quad (31)$$

where

$$\rho = \mu - \frac{\theta}{2} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2). \quad (32)$$

Therefore

$$\begin{aligned} LV(S, I, V) &\leq f(S) + g(I) - \frac{\rho}{2} V^{1+\theta} - \frac{\phi S}{V} \\ &\quad - M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} f(S) &= -\frac{\mu}{S} - \frac{\rho}{2} S^{1+\theta}, \\ g(I) &= \beta [M(a + b + \eta c) + 1 + \eta] I - \frac{\rho}{2} I^{1+\theta}. \end{aligned} \quad (34)$$

Next, we construct the following compact subset:

$$D = \left\{ \epsilon \leq S \leq \frac{1}{\epsilon}, \epsilon^2 \leq V \leq \frac{1}{\epsilon^2}, \epsilon \leq I \leq \frac{1}{\epsilon} \right\}, \quad (35)$$

where ϵ is a sufficiently small positive number, satisfying the following inequalities:

$$\check{g} - \frac{\mu}{\epsilon} < 0, \quad (36)$$

$$\check{f} + \beta [M(a + b + \eta c) + 1 + \eta] \epsilon < 0, \quad (37)$$

$$\check{f} + \check{g} - \frac{\phi}{\epsilon} < 0, \quad (38)$$

$$\check{g} - \frac{\rho}{2} \frac{1}{\epsilon^{1+\theta}} < 0, \quad (39)$$

$$\check{f} + B - \frac{\rho}{4} \frac{1}{\epsilon^{1+\theta}} < 0, \quad (40)$$

$$\check{f} + \check{g} - \frac{\rho}{2} \frac{1}{\epsilon^{2(1+\theta)}} < 0, \quad (41)$$

where the constant B will be determined later.

Then

$$\mathbb{R}_+^3 \setminus D = D_1^c \cup D_2^c \cup D_3^c \cup D_4^c \cup D_5^c \cup D_6^c, \quad (42)$$

with

$$\begin{aligned} D_1^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : S < \epsilon \right\}, \\ D_2^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : I < \epsilon \right\}, \\ D_3^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : V < \epsilon^2, S > \epsilon \right\}, \\ D_4^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : \frac{1}{S} < \epsilon \right\}, \\ D_5^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : \frac{1}{I} < \epsilon \right\}, \\ D_6^c &= \left\{ (S, I, V) \in \mathbb{R}_+^3 : \frac{1}{V} < \frac{1}{\epsilon^2} \right\}. \end{aligned} \quad (43)$$

Now, we will show that LV is negative for any $(S, I, V) \in \mathbb{R}_+^3 \setminus D$.

Case 1. If $(S, I, V) \in D_1^c$, we obtain from (36) that

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{g} - \frac{\mu}{\epsilon} < 0. \end{aligned} \quad (44)$$

Case 2. If $(S, I, V) \in D_2^c$, (37) implies that

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{f} \\ &\quad + \beta [M(a + b + \eta c) + 1 + \eta] \epsilon < 0. \end{aligned} \quad (45)$$

Case 3. If $(S, I, V) \in D_3^c$, from (38) it follows that

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{f} + \check{g} - \frac{\phi}{\epsilon} < 0. \end{aligned} \quad (46)$$

Case 4. If $(S, I, V) \in D_4^c$, (39) implies that

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{g} - \frac{\rho}{2} \frac{1}{\epsilon^{1+\theta}} < 0. \end{aligned} \quad (47)$$

Case 5. If $(S, I, V) \in D_5^c$, from (40) we get

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{f} + B - \frac{\rho}{4} \frac{1}{\epsilon^{1+\theta}} < 0, \end{aligned} \quad (48)$$

where

$$B = \sup_{I \in (0, \infty)} \left\{ \beta [M(a + b + \eta c) + 1 + \eta] I - \frac{\rho}{4} I^{1+\theta} \right\}. \quad (49)$$

Case 6. If $(S, I, V) \in D_6^c$, from (41) we obtain

$$\begin{aligned} LV(S, I, V) &\leq -M \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) (R_0^S - 1) + A + 2\mu \\ &\quad + \phi + \frac{\sigma_1^2}{2} + \frac{\sigma_3^2}{2} + \check{f} + \check{g} - \frac{\rho}{2} \frac{1}{\epsilon^{2(1+\theta)}} \\ &< 0. \end{aligned} \quad (50)$$

From the previous discussion, we have

$$LV(S, I, V) < 0, \quad (S, I, V) \in \mathbb{R}_+^3 \setminus D. \quad (51)$$

(ii) Now, we verify assumption (H.1). The diffusion matrix of system (3) is

$$\tilde{A} = \begin{pmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 \\ 0 & 0 & \sigma_3^2 V^2 \end{pmatrix}. \quad (52)$$

There is $\lambda = \min(\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 V^2) > 0$ such that

$$\begin{aligned} \sum_{i,j=1}^3 \tilde{a}_{ij}(X) \xi_i \xi_j &= \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 I^2 \xi_2^2 + \sigma_3^2 V^2 \xi_3^2 \\ &\geq \lambda |\xi|^2, \end{aligned} \quad (53)$$

for $(S, I, V) \in \bar{D}$ and $\xi \in \mathbb{R}_+^3$. That is to say, assumption (H.1) holds.

Consequently, system (3) has an ergodic stationary distribution. $\square$

5. Extinction

Theorem 7. We assume that $\mu > \max(\sigma_1^2, \sigma_2^2, \sigma_3^2)/2$. Let $(S(t), I(t), V(t)) \in \mathbb{R}_+^3$ be the solution of system (3). If $R_0^S < 1$, then the disease dies out with probability one.

The threshold $\tilde{R}_0^S$ is defined as follows:

$$\begin{aligned} \tilde{R}_0^S &= \frac{\beta \mu}{(\mu + \phi)(\mu + \gamma + \sigma_2^2/2)} \\ &\quad + \frac{\eta \beta \phi}{(\mu + \phi)(\mu + \gamma + \sigma_2^2/2)} \end{aligned} \quad (54)$$

Proof. By integrating system (3), we obtain

$$\begin{aligned} \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{V(t) - V(0)}{t} &= \mu \\ &\quad - \mu \langle S \rangle - (\mu + \gamma) \langle I \rangle - \mu \langle V \rangle \\ &\quad + \frac{1}{t} \left[\sigma_1 \int_0^t S(r) dB_1(r) + \sigma_2 \int_0^t I(r) dB_2(r) \right. \\ &\quad \left. + \sigma_3 \int_0^t V(r) dB_3(r) \right]. \end{aligned} \quad (55)$$

It follows that

$$\langle V \rangle = 1 - \langle S \rangle - \frac{\mu + \gamma}{\mu} \langle I \rangle + \psi_1(t), \quad (56)$$

where

$$\begin{aligned} \psi_1(t) = & - \left(\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} \right. \\ & + \frac{V(t) - V(0)}{t} \Big) + \frac{1}{\mu t} \left[\sigma_1 \int_0^t S(r) dB_1(r) \right. \\ & \left. + \sigma_2 \int_0^t I(r) dB_2(r) + \sigma_3 \int_0^t V(r) dB_3(r) \right]. \end{aligned} \quad (57)$$

From the first equation of system (3), we obtain

$$\langle S \rangle = \frac{\mu}{\mu + \phi} - \frac{\beta}{\mu + \phi} \langle SI \rangle + \psi_2(t), \quad (58)$$

where

$$\psi_2(t) = - \frac{S(t) - S(0)}{(\mu + \phi)t} + \frac{\sigma_1}{(\mu + \phi)t} \int_0^t S(s) dB_1(s). \quad (59)$$

Applying Itô's formula to system (3), one obtains

$$\begin{aligned} \frac{\log I(t) - \log I(0)}{t} = & \beta \langle S \rangle + \eta \beta \langle V \rangle - (\mu + \gamma) - \frac{\sigma_2^2}{2} \\ & + \frac{\sigma_2 B_2(t)}{t}. \end{aligned} \quad (60)$$

By injecting (56) and (58) into (60), we have

$$\begin{aligned} \frac{\log I(t) - \log I(0)}{t} = & \beta \frac{\mu + \eta \phi}{\mu + \phi} - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) \\ & + \beta(1 - \eta) \psi_1(t) + \eta \beta \psi_2(t) \\ & - \beta^2 \frac{1 - \eta}{\mu + \phi} \langle SI \rangle \\ & - \beta \frac{\eta(\mu + \gamma)}{\mu} \langle I \rangle + \frac{\sigma_2 B_2(t)}{t} \\ \leq & \beta \frac{\mu + \eta \phi}{\mu + \phi} - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) \\ & + \beta(1 - \eta) \psi_1(t) + \eta \beta \psi_2(t) \\ & + \frac{\sigma_2 B_2(t)}{t}. \end{aligned} \quad (61)$$

From the strong law of large numbers for martingales [22], we get

$$\lim_{t \rightarrow \infty} \frac{B_2(t)}{t} = 0, \quad \text{a.s.} \quad (62)$$

By Theorem 1, we get

$$\lim_{t \rightarrow \infty} \psi_1(t) = \lim_{t \rightarrow \infty} \psi_2(t) = 0. \quad (63)$$

By taking the superior limit of both sides of (61), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log I(t)}{t} \leq & \frac{\beta \mu}{\mu + \phi} + \frac{\eta \beta \phi}{\mu + \phi} - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) \\ = & \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) \left(\tilde{R}_0^S - 1 \right). \end{aligned} \quad (64)$$

This finishes the proof. $\square$

Theorem 8. We assume that $\sigma_1 = \sigma_3 = \sigma$ and $B_1(t) = B_3(t) = B(t)$. Let $(S(t), I(t), V(t)) \in \mathbb{R}_+^3$ be the solution of system (3). If $\sigma^2 > 2\eta\phi$, then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log I(t)}{t} \leq & (\mu + \gamma) \left(\frac{\beta}{\mu + \gamma} \int_0^\infty x \pi(x) dx - 1 \right) \quad \text{a.s.} \end{aligned} \quad (65)$$

If $(\beta/(\mu + \gamma)) \int_0^\infty x \pi(x) dx < 1$, then

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{a.s.}, \quad (66)$$

and the distribution of $S(t) + \eta V(t)$ converges weakly to the measure that has the density given by

$$\pi(x) = \frac{Q}{\sigma^2} x^{-2(1-\eta\phi/\sigma^2)} e^{-2\mu/\sigma^2 x}, \quad x \in (0, \infty), \quad (67)$$

where

$$Q = \sigma^2 \left(\frac{2\mu}{\sigma^2} \right)^{1-2(\eta\phi/\sigma^2)} \Gamma^{-1} \left(1 - 2\frac{\eta\phi}{\sigma^2} \right). \quad (68)$$

Proof. We assume that $\sigma_1 = \sigma_3 = \sigma$ and $B_1(t) = B_3(t) = B(t)$. From system (3), we have

$$\begin{aligned} d(S(t) + \eta V(t)) \leq & [\mu + \eta \phi (S(t) + \eta V(t))] dt \\ & + \sigma_1 S(t) dB_1(t) \\ & + \eta V(t) \sigma_3 dB_3(t), \\ = & dX(t), \end{aligned} \quad (69)$$

where

$$dX(t) = [\mu + \eta \phi X(t)] dt + \sigma X(t) dB(t), \quad (70)$$

with the initial value $X(0) = S(0) + \eta V(0)$.

We compute that

$$\begin{aligned} \int \frac{\mu + \eta \phi x}{\sigma^2 x^2} dx = & \frac{1}{\sigma^2} \int \left(\frac{\mu}{x^2} + \frac{\eta \phi}{x} \right) dx \\ = & \frac{1}{\sigma^2} \left[-\frac{\mu}{x} + \eta \phi \ln x \right] + B, \end{aligned} \quad (71)$$

where B is a constant.

One can see that

$$\begin{aligned} & \int_0^\infty x^{-2(1-\eta\phi/\sigma^2)} e^{-2\mu/\sigma^2 x} dx \\ &= \left(\frac{2\mu}{\sigma^2}\right)^{2(\eta\phi/\sigma^2)-1} \int_0^\infty y^{(1-2(\eta\phi/\sigma^2))-1} e^{-y} dy \\ &= \left(\frac{2\mu}{\sigma^2}\right)^{2(\eta\phi/\sigma^2)-1} \Gamma\left(1-2\frac{\eta\phi}{\sigma^2}\right), \end{aligned} \quad (72)$$

where Γ is the Gamma function defined by

$$\Gamma(Z) = \int_0^\infty x^{Z-1} e^{-x} dx. \quad (73)$$

If $\sigma^2 > 2\eta\phi$, then

$$\int_0^\infty x^{-2(1-\eta\phi/\sigma^2)} e^{-2\mu/\sigma^2 x} dx < \infty. \quad (74)$$

Thus, condition (H.3) in Lemma 3 is verified. So, system (70) has the ergodic property and the invariant density given by

$$\pi(x) = \frac{Q}{\sigma^2} x^{-2(1-\eta\phi/\sigma^2)} e^{-2\mu/\sigma^2 x}, \quad x \in (0, \infty). \quad (75)$$

Let us compute Q .

According to

$$\int_0^\infty \pi(x) dx = 1, \quad (76)$$

one has

$$Q = \sigma^2 \left(\frac{2\mu}{\sigma^2}\right)^{1-2(\eta\phi/\sigma^2)} \Gamma^{-1}\left(1-2\frac{\eta\phi}{\sigma^2}\right). \quad (77)$$

From the ergodic theorem, it follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X(r) dr = \int_0^\infty x \pi(x) dx \quad \text{a.s.} \quad (78)$$

On the other hand, by applying Itô's formula to $\log I(t)$ and then integrating, one has

$$\begin{aligned} & \frac{\log I(t) - \log I(0)}{t} \\ &= \frac{\beta}{t} \int_0^t (S(r) + \eta V(r)) dr - (\mu + \gamma) \\ & \quad - \frac{\sigma^2}{2} \int_0^t (S(r) + \eta V(r))^2 dr + \frac{1}{t} M(t), \end{aligned} \quad (79)$$

where $M(t) = \sigma \int_0^t (S(r) + \eta V(r)) dB(r)$ whose quadratic variation is

$$\langle M, M \rangle(t) = \sigma^2 \int_0^t (S(r) + \eta V(r))^2 dr. \quad (80)$$

In view of the exponential martingales inequality [22], for any positive constants T, α , and ν , we have

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left[M(t) - \frac{\alpha}{2} \langle M, M \rangle(t) \right] > \nu \right\} \leq e^{-\alpha \nu}. \quad (81)$$

Choosing $T = k, \alpha = 1$ and $\nu = 2 \log k$, one has

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq k} \left[M(t) - \frac{1}{2} \langle M, M \rangle(t) \right] > 2 \log k \right\} \leq \frac{1}{k^2}. \quad (82)$$

Applying the Borel-Cantelli Lemma [22] leads to the fact that, for almost all $\omega \in \Omega$, there exists a random integer $k_0 = k_0(\omega)$ such that, for any $k \geq k_0$, we obtain

$$\sup_{0 \leq t \leq k} \left[M(t) - \frac{1}{2} \langle M, M \rangle(t) \right] \leq 2 \log k. \quad (83)$$

That is,

$$M(t) \leq \frac{1}{2} \langle M, M \rangle(t) + 2 \log k \quad \text{for all } t \leq k. \quad (84)$$

Substituting this inequality into (79) yields

$$\begin{aligned} & \frac{\log I(t) - \log I(0)}{t} \\ & \leq \frac{\beta}{t} \int_0^t (S(r) + \eta V(r)) dr - (\mu + \gamma) + 2 \frac{\log k}{t} \end{aligned} \quad (85)$$

for all $t \leq k$.

Applying the comparison theorem of stochastic differential equations on inequality (69) gives

$$S(t) + \eta V(t) \leq X(t) \quad \text{a.s.} \quad (86)$$

It follows that

$$\begin{aligned} & \frac{\log I(t) - \log I(0)}{t} \leq \frac{\beta}{t} \int_0^t X(r) dr - (\mu + \gamma) \\ & \quad + 2 \frac{\log k}{t}, \quad \text{for all } t \leq k. \end{aligned} \quad (87)$$

Taking the superior limit on both sides of the previous inequality leads to

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\log I(t) - \log I(0)}{t} \\ & \leq \beta \int_0^\infty x \pi(x) dx - (\mu + \gamma) \quad \text{a.s.} \end{aligned} \quad (88)$$

If $(\beta/(\mu + \gamma)) \int_0^\infty x \pi(x) dx < 1$, we conclude that

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{a.s.} \quad (89)$$

As a result, for any small $\epsilon > 0$, there exist t_0 and a set $\Omega_\epsilon \subset \Omega$ such that $\mathbb{P}(\Omega_\epsilon) > 1 - \epsilon$ and $\beta(S + \eta^2 V)I \leq \epsilon(S + \eta^2 V) \leq \epsilon(S + \eta V)$ for all $t \geq t_0$ and $\omega \in \Omega_\epsilon$. From

$$\begin{aligned} & [\mu - \mu(S(t) + \eta V(t)) - \epsilon(S(t) + \eta V(t))] dt \\ & \quad + \sigma(S(t) + \eta V(t)) dB(t) \leq d(S(t) + \eta V(t)) \\ & \leq [\mu - \mu(S(t) + \eta V(t))] dt \\ & \quad + \sigma(S(t) + \eta V(t)) dB(t), \end{aligned} \quad (90)$$

it follows that the distribution of the process $S(t) + \eta V(t)$ converges weakly to the measure that has the density π .

This completes the proof. $\square$

6. Numerical Simulations

In this section, we present the numerical simulations to support the above analytical results, illustrating persistence in mean and extinction of the disease.

In the two following examples, we choose the initial value as $(S(0), I(0), V(0)) = (0.2, 0.2, 0.2)$.

Example 9. In model (3), we choose the parameters as follows:

$$\begin{aligned}\sigma_1 &= 0.2, \\ \sigma_2 &= 0.2, \\ \sigma_3 &= 0.2, \\ \beta &= 10, \\ \mu &= 1, \\ \phi &= 15, \\ \gamma &= 4, \\ \eta &= 0.6.\end{aligned}\tag{91}$$

We compute that

$$R_0^S = 1.2215 > 1.\tag{92}$$

Then, according to Theorem 5, the disease is persistent (see Figure 1). Furthermore, by Theorem 6, system (3) has a unique stationary distribution (see Figure 2).

Example 10. We choose the parameters as follows:

$$\begin{aligned}\sigma_1 &= 0.6, \\ \sigma_2 &= 1, \\ \sigma_3 &= 0.8, \\ \beta &= 8, \\ \mu &= 1, \\ \phi &= 15, \\ \gamma &= 4, \\ \eta &= 0.6.\end{aligned}\tag{93}$$

Then

$$\tilde{R}_0^S = 0.9091 < 1\tag{94}$$

Theorem 7 implies that system (3) has disease extinction (see Figure 3).

FIGURE 1: Simulation of the paths $S(t)$, $I(t)$, and $V(t)$ for model (3).

Example 11. We choose the parameters as follows:

$$\begin{aligned}\sigma_1 &= \sigma_3 = \sigma = 1.15, \\ \beta &= 10, \\ \mu &= 1, \\ \phi &= 1, \\ \gamma &= 4, \\ \eta &= 0.2.\end{aligned}\tag{95}$$

Then

$$\begin{aligned}\sigma^2 &> 2\eta\phi, \\ \frac{\beta}{\mu + \gamma} \int_0^\infty x\pi(x) dx &< 1.\end{aligned}\tag{96}$$

According to Theorem 8, we get

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{a.s.}\tag{97}$$

It follows that the distribution of the process $S(t) + \eta V(t)$ converges weakly to the measure that has the density π (see Figure 4).

7. Conclusion

This paper is concerned with the dynamics of a stochastic SIVR epidemic model with multiplicative noise sources. By constructing a convenient positive function, we establish sufficient conditions for the existence of a unique ergodic stationary distribution to model (3) and we prove that the disease will be permanent. In addition, we also establish sufficient conditions for extinction of the disease.

Many works have been done to study the continuous stochastic models by using the white noise. Sometimes, population systems may suffer sudden environmental perturbations, such as SARS, floods, and toxic pollutants. These phenomena cannot be modeled by stochastic continuous

FIGURE 2: Distribution of $S(300)$, $I(300)$, and $V(300)$. FIGURE 3: Simulation of the paths $S(t)$, $I(t)$, and $V(t)$ for model (3). FIGURE 4: Distribution of $S(t) + \eta V(t)$ at time $t = 10\,000$.

models. During the last years, there was an increasing interest in Lévy noise that has non-Gaussian statistics. A lot of works have been realized and have shown the effectiveness of the Lévy noise in studying such phenomena (see [25–27]). We investigate in our future works the impact of the Lévy noise on the dynamic of complicated population systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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