Hindawi
Journal of Applied Mathematics
Volume 2018, Article ID 2952791, 17 pages
https://doi.org/10.1155/2018/2952791

Research Article

A Study of a Diseased Prey-Predator Model with Refuge in Prey and Harvesting from Predator

Ahmed Sami Abdulghafour [1] and Raid Kamel Naji [2]

[1] *Scientific Affairs Department, Aliraqia University, Baghdad, Iraq*
[2] *Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq*

Correspondence should be addressed to Ahmed Sami Abdulghafour; alania961@gmail.com

Received 4 September 2018; Accepted 14 November 2018; Published 6 December 2018

Academic Editor: Oluwole D. Makinde

In this paper, a mathematical model of a prey-predator system with infectious disease in the prey population is proposed and studied. It is assumed that there is a constant refuge in prey as a defensive property against predation and harvesting from the predator. The proposed mathematical model is consisting of three first-order nonlinear ordinary differential equations, which describe the interaction among the healthy prey, infected prey, and predator. The existence, uniqueness, and boundedness of the system' solution are investigated. The system's equilibrium points are calculated with studying their local and global stability. The persistence conditions of the proposed system are established. Finally the obtained analytical results are justified by a numerical simulation.

1. Introduction

The interaction between the prey and predator species has a long history since Lotka-Volterra model; see [1]. Similarly, the interaction of the susceptible–infected–recovered population is an interesting subject of research work since the pioneering work of Kermack and McKendrick [2]. The dynamics of disease within the ecological systems now becomes an important subject of research. In fact Anderson and May [3] were the first who combined these two systems, while Chattopadhyay and Arino [4] were the first who used the term "eco-epidemiology" for such type of models. On one hand several studies of prey-predator dynamics have been done within the last decades taking into account the effects of variety of biological factors; see, for example, [5–8] and the references therein. On the other hand variety of mathematical models have been proposed and studied in the field of epidemiology taking into account different types of incidence rates and disease; see, for example, [9–12] and the references therein.

The existence of disease in the prey population, predator population, or both is real situation in the ecological species. It has been observed that this type of incidence occurs through infection by some viral disease, bacterial disease, or parasite disease. Many of these studies were focused on the study of disease in prey population only [13–15]. Other researchers were interested in the study of disease within the predator population only [16–19]. There are also some studies where both the prey and predator populations are infected with the disease [20–23].

It is well known that the harvesting of the species is necessary for the coexistence of the species and hence it took a lot of interest from the researchers in their proposed ecological models. Different types of harvesting have been proposed and studied including constant harvesting, density dependent proportional harvesting, and nonlinear harvesting [8, 24–26]. The refuge of prey species is also a biological factor necessary for the coexistence of the species and hence it is another factor of great interest as defensive properties of the prey against the predation; see [8, 27, 28].

Keeping the above in view, in this paper we propose and study an eco-epidemiological prey-predator model involving a prey refuge and harvesting from the predator. It is assumed that the disease exists only in prey and it will not transfer to predator through the feeding process. The paper is organized as follows. In Section 2, the mathematical model is formulated and its dimensionless variables and parameters are determined; moreover the existence, uniqueness, and uniform boundedness of its solution are discussed. In Section 3, the local stability of all possible equilibrium points and persistence of the system are studied. Section 4 deals with the global stability analysis of the system using suitable Lyapunov functions. However Section 5 provides a numerical simulation of the proposed system for suitable chose of parameters values. Finally Section 6 gives some conclusions and discussed the obtained results.

2. Mathematical Model

In this section, a prey-predator system involving infected disease in prey is proposed for study. It is assumed that there is a harvesting effort applied on the predator individuals only. Accordingly, the following hypotheses are adopted to formulate the mathematical model.

(1) The prey population is divided into two compartments, susceptible with density at time T given by $S(T)$ and infected with density at time T represented by $I(T)$, while the predator consists of only one compartment with density at time T denoted by $Y(T)$.

(2) The prey population grows logistically with intrinsic growth rate $r > 0$ and carrying capacity $K > 0$; it is assumed that the infected cannot reproduce; rather it competes with the susceptible individuals for food and space.

(3) There is a type of protection of prey population from the predation by predator, represented by a constant prey's refuge rate $m \in (0,1)$ that leaves $(1-m)S$ of prey available to be hunted by the predator.

(4) The susceptible prey population becomes infected by contact with the infected prey according to the simple mass action kinetics with $\beta > 0$ as the rate of infection.

(5) The predator population consumes both the prey populations according to modified Holling type II functional response for the predation [29] with half saturation constant $b > 0$ and maximum attack rates $a_1 > 0$ and $a_2 > 0$ for susceptible prey and infected prey, respectively. Since there is a vulnerability of infected prey relative to susceptible prey the vulnerability constant rate $\theta > 0$ is used in the functional response. Moreover the constants $e_1 \in (0,1)$ and $e_2 \in (0,1)$ are the conversion rates from susceptible and infected preys to predator, respectively.

(6) The disease causes a death in the infected population represented by diseased death rate $d_1 > 0$, while in the absence of prey, the predator decays exponentially with natural death rate $d_2 > 0$.

(7) Only the predator population is assumed to be harvested with the Michael-Mentence harvesting function [30], where $E > 0$ represents hunting effort, $c > 0$ is the catchability coefficient of the predator, and l_i, $i = 1, 2$, are suitable positive constants.

According to the above set of hypotheses the dynamics of an eco-epidemic prey-predator model with refuge in prey and harvesting in predator can be described in the following set of first-order nonlinear differential equations.

$$\begin{aligned}
\frac{dS}{dT} &= rS\left(1-\frac{S+I}{K}\right) - \frac{a_1(1-m)SY}{b+(1-m)S+\theta(1-m)I} \\
&\quad - \beta SI \\
\frac{dI}{dT} &= \beta SI - \frac{a_2(1-m)IY}{b+(1-m)S+\theta(1-m)I} - d_1 I \\
\frac{dY}{dT} &= \left(\frac{e_1a_1(1-m)S+e_2a_2(1-m)I}{b+(1-m)S+\theta(1-m)I}\right)Y \\
&\quad - \frac{cEY}{l_1E+l_2Y} - d_2Y
\end{aligned} \tag{1}$$

where $S(0) \geq 0$, $I(0) \geq 0$, and $Y(0) \geq 0$.

Now in order to reduce the number of parameters and specify the control set of parameters the following dimensionless variables and parameters are used:

$$\begin{aligned}
t &= rT, \\
x &= \frac{X}{K}, \\
i &= \frac{I}{K}, \\
y &= \frac{a_1Y}{rK}, \\
w_1 &= \frac{b}{K}, \\
w_2 &= \frac{\beta K}{r}, \\
w_3 &= \frac{a_2}{a_1}, \\
w_4 &= \frac{d_1}{r}, \\
w_5 &= \frac{e_1a_1}{r}, \\
w_6 &= \frac{e_2a_2}{r}, \\
w_7 &= \frac{a_1l_1E}{rl_2K}, \\
w_8 &= \frac{ca_1E}{r^2l_2K}, \\
w_9 &= \frac{d_2}{r}
\end{aligned} \tag{2}$$

System (1) reduces to the following dimensionless system:

$$\begin{aligned}
\frac{ds}{dt} &= s\left[1-(s+i)-\frac{(1-m)\,y}{w_1+(1-m)\,s+\theta(1-m)\,i}-w_2 i\right] \\
&= s f_1(s,i,y) \\
\frac{di}{dt} &= i\left[w_2 s-\frac{w_3(1-m)\,y}{w_1+(1-m)\,s+\theta(1-m)\,i}-w_4\right] \\
&= i f_2(s,i,y) \\
\frac{dy}{dt} &= y\left[\frac{w_5(1-m)\,s+w_6(1-m)\,i}{w_1+(1-m)\,s+\theta(1-m)\,i}-\frac{w_7}{w_8+y}-w_9\right] \\
&= y f_3(s,i,y)
\end{aligned} \tag{3}$$

According to the above dimensionless form, it is easy to verify that the differential equations are continuous and have continuous partial derivatives on the domain $\mathbb{R}^3_+ = \{(s,i,y) \in \mathbb{R}^3 : s(0) \geq 0,\ i(0) \geq 0;\ y(0) \geq 0\}$ and hence they are Lipschitzian functions. Therefore system (3) has a unique solution. Furthermore the solution of system (3) that initials in $\mathbb{R}^3_+$ is uniformly bounded as shown in the following theorem.

Theorem 1. *All solutions of system (3) are uniformly bounded.*

Proof. According to the prey population which is given in the first equation of system (3), it is observed that

$$\frac{ds}{dt} \leq s[1-s] \tag{4}$$

Then by solving this differential inequality, we obtain that $s(t) \leq s(0)/(e^{-t}[1-s(0)]+s(0))$, and then for $t \longrightarrow \infty$, we have $s(t) \leq 1$. Now assume that $\Omega = s+i+y$; then

$$\frac{d\Omega}{dt} = \frac{ds}{dt}+\frac{di}{dt}+\frac{dy}{dt} \xrightarrow{\text{yields}} \frac{d\Omega}{dt} \leq 2-\mu\Omega \tag{5}$$

where $\mu = \min\{1, w_4, w_9\}$. Therefore by solving the last differential inequality we obtain that

$$\Omega(t) \leq \Omega(0)\,e^{-\mu t}-\frac{2}{\mu}\left(e^{-\mu t}-1\right) \tag{6}$$

Hence for $t \longrightarrow \infty$, we get that $\Omega(t) \leq 2/\mu$. Thus all the solutions of system (3) are uniformly bounded. □

3. The Local Stability and Persistence

In the following the existence and local stability of all possible equilibrium points are investigated and then the persistence conditions of the system are established. Obviously system (3) has at most five nonnegative equilibrium points. The vanishing equilibrium point $E_0 = (0,0,0)$ and the axial equilibrium point $E_1 = (1,0,0)$ always exist. The first planar $E_2 = (\overline{s}, 0, \overline{y})$, where $\overline{s}$ represents a unique positive root of the following third-order polynomial equation:

$$A_1 s^3 + A_2 s^2 + A_3 s + A_4 = 0 \tag{7a}$$

where $A_1 = (w_9 - w_5)R^2$, $A_2 = [(w_9-w_5)(w_1-R)+w_9 w_1]R$, $A_3 = (w_5 w_8 - w_7)R^2 + [w_1 w_5 - 2w_1 w_9 - w_8 w_9]R + w_1^{\,2} w_9$, $A_4 = -w_1[w_1 w_9 + (w_7 + w_8 w_9)R] < 0$, while

$$\overline{y} = \frac{(1-\overline{s})(w_1 + R\overline{s})}{R} \tag{7b}$$

from now onward $R = (1-m)$. Straightforward computation shows that E_2 exists uniquely in the interior of positive quadrant of sy −plane provided that the following sufficient conditions hold:

$$\overline{s} < 1 \tag{8a}$$

$$\begin{gathered} w_9 > w_5; \\ A_2 > 0 \\ or\ w_9 > w_5; \\ A_3 < 0 \end{gathered} \tag{8b}$$

The second planar equilibrium point $E_3 = (\hat{s}, \hat{i}, 0)$, where

$$\begin{aligned} \hat{s} &= \frac{w_4}{w_2}; \\ \hat{i} &= \frac{w_2 - w_4}{w_2(1+w_2)} \end{aligned} \tag{9}$$

exists uniquely in the interior of positive quadrant of si −plane provided that

$$w_2 > w_4 \tag{10}$$

Finally the coexistence equilibrium point is denoted by $E_4 = (s^*, i^*, y^*)$ where

$$i^* = \frac{w_3 + w_4 - (w_2 + w_3)\,s^*}{w_3(1+w_2)} \tag{11a}$$

$$y^* = \frac{(w_2 s^* - w_4)\left[(w_1 w_3(1+w_2) + (w_3+w_4)\,\theta R) + (w_3(1+w_2)\,R - (w_2+w_3)\,\theta R)\,s\right]}{w_3^{\,2}(1+w_2)\,R} \tag{11b}$$

while s^* is a unique positive root of the following third-order polynomial equation:

$$D_1 s^3 + D_2 s^2 + D_3 s + D_4 = 0 \tag{11c}$$

Here

$$\begin{aligned} D_1 &= (\sigma_3 - (w_2 + w_3)\sigma_6) w_2 \sigma_5 \\ D_2 &= (w_2\sigma_4 - w_4\sigma_5)\sigma_3 - w_2(w_2 + w_3)\sigma_4\sigma_6 \\ &\quad + [w_2(w_3 + w_4) + w_4(w_2 + w_3)]\sigma_5\sigma_6 \\ &\quad - w_2\sigma_5\sigma_8 \\ D_3 &= [\sigma_1 - \sigma_2(w_2 + w_3)] - w_4\sigma_3\sigma_4 \\ &\quad + [w_2(w_3 + w_4) + w_4(w_2 + w_3)]\sigma_4\sigma_6 \\ &\quad - w_4(w_3 + w_4)\sigma_5\sigma_6 - w_2\sigma_4\sigma_8 + w_4\sigma_5\sigma_8 \\ D_4 &= \sigma_2(w_3 + w_4) - \sigma_7 - w_4(w_3 + w_4)\sigma_4\sigma_6 \\ &\quad + w_4\sigma_4\sigma_8 \end{aligned} \tag{12}$$

with

$$\begin{aligned} \sigma_1 &= [w_5w_8 - (w_7 + w_8w_9)] w_3^3 (1 + w_2)^2 R^2 \\ \sigma_2 &= [w_6w_8 - \theta(w_7 + w_8w_9)] w_3^2 (1 + w_2) R^2 \\ \sigma_3 &= (w_5 - w_9) w_3 (1 + w_2) R^2 \\ \sigma_4 &= [w_1w_3(1 + w_2) + (w_3 + w_4)\theta R] \\ \sigma_5 &= [w_3(1 + w_2)R - (w_2 + w_3)\theta R] \\ \sigma_6 &= (w_6 - w_9\theta) R \\ \sigma_7 &= w_1(w_7 + w_8w_9) w_3^3 (1 + w_2)^2 R \\ \sigma_8 &= w_1w_9w_3(1 + w_2) R \end{aligned} \tag{13}$$

Clearly (11c) has a unique positive root provided that one set of the following sets of conditions holds.

$$\begin{aligned} &D_1 > 0; \\ &D_2 > 0; \\ &D_4 < 0 \end{aligned} \tag{14a}$$

$$\begin{aligned} &D_1 > 0; \\ &D_3 < 0; \\ &D_4 < 0 \end{aligned} \tag{14b}$$

$$\begin{aligned} &D_1 < 0; \\ &D_2 < 0; \\ &D_4 > 0 \end{aligned} \tag{14c}$$

$$\begin{aligned} &D_1 < 0; \\ &D_3 > 0; \\ &D_4 > 0 \end{aligned} \tag{14d}$$

Consequently, the coexistence equilibrium point $E_4 = (s^*, i^*, y^*)$ exists uniquely in the interior of $\mathbb{R}_+^3$ provided that in addition to any one of conditions (14a), (14b), (14c), and (14d) the following sufficient conditions hold:

$$s^* < \frac{w_3 + w_4}{(w_2 + w_3)} \tag{15a}$$

$$(w_2 s^* - w_4)[\sigma_4 + \sigma_5 s] > 0 \tag{15b}$$

Now in order to study the local stability of these equilibrium points, the Jacobian matrix of system (3) at the point $E = (s, i, y)$ is computed as follows:

$$J(E) = (C_{ij})_{3\times 3} \tag{16}$$

where

$$\begin{aligned} C_{11} &= f_1(s, i, y) + s\left(\frac{R^2 y}{N_1^2} - 1\right); \\ C_{12} &= s\left(\frac{\theta R^2 y}{N_1^2} - (1 + w_2)\right); \\ C_{13} &= s\left(-\frac{R}{N_1}\right); \\ C_{21} &= i\left(w_2 + \frac{w_3 R^2 y}{N_1^2}\right); \\ C_{22} &= f_2(s, i, y) + i\left(\frac{\theta w_3 R^2 y}{N_1^2}\right); \\ C_{23} &= i\left(-\frac{w_3 R}{N_1}\right); \\ C_{31} &= y\left(\frac{w_1 w_5 R + (w_5\theta - w_6) R^2 i}{N_1^2}\right); \\ C_{32} &= y\left(\frac{w_1 w_6 R + (w_6 - \theta w_5) R^2 s}{N_1^2}\right); \\ C_{33} &= y\left(\frac{w_7}{N_2^2}\right) + f_3(s, i, y). \end{aligned} \tag{17}$$

Here $N_1 = w_1 + Rs + \theta Ri$ and $N_2 = w_8 + y$. Consequently the Jacobian matrix at vanishing equilibrium point $E_0 = (0, 0, 0)$ is written as

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -w_4 & 0 \\ 0 & 0 & -\left(\dfrac{w_7}{w_8} + w_9\right) \end{bmatrix} \tag{18}$$

Obviously the eigenvalues of $J(E_0)$ are given by $\lambda_{01} = 1 > 0$, $\lambda_{02} = -w_4 < 0$, $\lambda_{03} = -(w_7/w_8 + w_9) < 0$. Therefore E_0 is a saddle point.

The Jacobian matrix at axial equilibrium point $E_1 = (1, 0, 0)$ is written as

$$J(E_1) = \begin{bmatrix} -1 & -(1+w_2) & -\dfrac{R}{w_1 + R} \\ 0 & w_2 - w_4 & 0 \\ 0 & 0 & \dfrac{w_5 R}{w_1 + R} - \dfrac{w_7}{w_8} - w_9 \end{bmatrix} \tag{19}$$

Then the eigenvalues of $J(E_1)$ are given by $\lambda_{11} = -1 < 0$, $\lambda_{12} = w_2 - w_4$ and $\lambda_{13} = w_5 R/(w_1 + R) - w_7/w_8 - w_9$. Accordingly the axial point E_1 is locally asymptotically stable if and only if

$$w_2 < w_4 \tag{20a}$$

$$\frac{w_5 R}{w_1 + R} < \frac{w_7}{w_8} + w_9 \tag{20b}$$

Now the Jacobian matrix at first planar equilibrium point $E_2 = (\bar{s}, 0, \bar{y})$ can be written

$$J(E_2) = \begin{bmatrix} \dfrac{R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} - \bar{s} & \dfrac{\theta R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} - (1 + w_2)\bar{s} & -\dfrac{R\bar{s}}{w_1 + R\bar{s}} \\ 0 & w_2\bar{s} - \dfrac{w_3 R\bar{y}}{w_1 + R\bar{s}} - w_4 & 0 \\ \dfrac{w_1 w_5 R\bar{y}}{(w_1 + R\bar{s})^2} & \dfrac{w_1 w_6 R\bar{y} + (w_6 - \theta w_5) R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} & \dfrac{w_7 \bar{y}}{(w_8 + \bar{y})^2} \end{bmatrix} \tag{21}$$

Clearly The eigenvalues of $J(E_2)$ can be determined as follows:

$$\left(w_2\bar{s} - \frac{w_3 R\bar{y}}{w_1 + R\bar{s}} - w_4 - \lambda\right)\left(\lambda^2 - T_2\lambda + D_2\lambda\right) = 0 \tag{22}$$

where

$$\begin{aligned} T_2 &= \frac{R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} - \bar{s} + \frac{w_7 \bar{y}}{(w_8 + \bar{y})^2} \\ D_2 &= \left(\frac{R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} - \bar{s}\right)\left(\frac{w_7 \bar{y}}{(w_8 + \bar{y})^2}\right) \\ &\quad + \left(\frac{R\bar{s}}{w_1 + R\bar{s}}\right)\left(\frac{w_1 w_5 R\bar{y}}{(w_1 + R\bar{s})^2}\right) \end{aligned} \tag{23}$$

Therefore the eigenvalues of $J(E_2)$ are easily determined by

$$\begin{aligned} \lambda_{21} &= \frac{T_2}{2} - \frac{1}{2}\sqrt{{T_2}^2 - 4D_2} \\ \lambda_{22} &= w_2\bar{s} - \frac{w_3 R\bar{y}}{w_1 + R\bar{s}} - w_4 \\ \lambda_{23} &= \frac{T_2}{2} + \frac{1}{2}\sqrt{{T_2}^2 - 4D_2} \end{aligned} \tag{24}$$

Straightforward computation shows that these eigenvalues have negative real parts and hence the first planar equilibrium point E_2 is locally asymptotically stable if and only if the following sufficient conditions hold:

$$w_2\bar{s} < \frac{w_3 R\bar{y}}{w_1 + R\bar{s}} + w_4 \tag{25a}$$

$$\frac{R^2 \bar{s}\,\bar{y}}{(w_1 + R\bar{s})^2} + \frac{w_7 \bar{y}}{(w_8 + \bar{y})^2} < \bar{s} \tag{25b}$$

$$\begin{aligned} & w_7 R^2 \bar{y}(w_1 + R\bar{s}) + w_1 w_5 R^2 (w_8 + \bar{y})^2 \\ & \quad > w_7 (w_1 + R\bar{s})^3 \end{aligned} \tag{25c}$$

Similarly the Jacobian matrix at second planar equilibrium point $E_3 = (\hat{s}, \hat{i}, 0)$ can be written

$$J(E_3) = \begin{bmatrix} -\hat{s} & -\hat{s}(1 + w_2) & -\dfrac{R\hat{s}}{w_1 + R\hat{s} + \theta R\hat{i}} \\ w_2\hat{i} & 0 & -\dfrac{w_3 R\hat{i}}{w_1 + R\hat{s} + \theta R\hat{i}} \\ 0 & 0 & \dfrac{w_5 R\hat{s} + w_6 R\hat{i}}{w_1 + R\hat{s} + \theta R\hat{i}} - \dfrac{w_7}{w_8} - w_9 \end{bmatrix} \tag{26}$$

Then the eigenvalues of $J(E_3)$ are given by

$$\left(\frac{w_5 R\hat{s} + w_6 R\hat{i}}{w_1 + R\hat{s} + \theta R\hat{i}} - \frac{w_7}{w_8} - w_9 - \lambda\right)\left(\lambda^2 - T_3\lambda + D_3\right) = 0 \tag{27}$$

where

$$\begin{aligned} T_3 &= -\hat{s} < 0 \\ D_3 &= w_2 (1 + w_2)\hat{s}\hat{i} > 0 \end{aligned} \tag{28}$$

Clearly the first two eigenvalues resulting from second term in (27) have negative real parts, while the third eigenvalues in the y −direction which are written as

$$\lambda_{33} = \frac{w_5 R\hat{s} + w_6 R\hat{i}}{w_1 + R\hat{s} + \theta R\hat{i}} - \frac{w_7}{w_8} - w_9 \tag{29}$$

have negative real part and then the second planar equilibrium point is locally asymptotically stable if and only if

$$\frac{w_5 R\hat{s} + w_6 R\hat{i}}{w_1 + R\hat{s} + \theta R\hat{i}} < \frac{w_7}{w_8} + w_9 \tag{30}$$

Finally the Jacobian matrix at coexistence equilibrium point $E_4 = (s^*, i^*, y^*)$ can be written in the form

$$J(E_4) = \left[a_{ij}\right] \quad i, j = 1, 2, 3 \tag{31}$$

where

$$a_{11} = \frac{R^2 s^* y^* - N_1^{*2} s^*}{N_1^{*2}};$$
$$a_{12} = \frac{\theta R^2 s^* y^* - (1 + w_2) N_1^{*2} s^*}{N_1^{*2}};$$
$$a_{13} = -\frac{R s^*}{N_1^*} < 0;$$
$$a_{21} = w_2 i^* + \frac{w_3 R^2 i^* y^*}{N_1^{*2}} > 0;$$
$$a_{22} = \frac{\theta w_3 R^2 i^* y^*}{N_1^{*2}} > 0; \quad (32)$$
$$a_{23} = -\frac{w_3 R i^*}{N_1^*} < 0;$$
$$a_{31} = \frac{w_1 w_5 R y^* + (w_5 \theta - w_6) R^2 i^* y^*}{N_1^{*2}};$$
$$a_{32} = \frac{w_1 w_6 R y^* + (w_6 - \theta w_5) R^2 s^* y^*}{N_1^{*2}};$$
$$a_{33} = \frac{w_7 y^*}{N_2^{*2}} > 0.$$

Here $N_1^* = w_1 + Rs^* + \theta R i^*$ and $N_2^* = w_8 + y^*$. The characteristic equation of $J(E_4)$ can be written in the form

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \quad (33)$$

where

$$\begin{aligned} A_1 &= -(a_{11} + a_{22} + a_{33}) \\ A_2 &= a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} \\ &\quad - a_{23}a_{32} \\ A_3 &= a_{33}(a_{12}a_{21} - a_{11}a_{22}) - a_{23}(a_{12}a_{31} - a_{11}a_{32}) \\ &\quad - a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned} \quad (34)$$

while

$$\begin{aligned} \Delta &= A_1 A_2 - A_3 \\ &= -(a_{11} + a_{22})[a_{11}a_{22} - a_{12}a_{21}] - 2a_{11}a_{22}a_{33} \\ &\quad - (a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] \\ &\quad - (a_{22} + a_{33})[a_{23}a_{32} - a_{22}a_{33}] \end{aligned} \quad (35)$$

Therefore straightforward computation shows that

$$\begin{aligned} A_1 &= -\frac{1}{N_1^{*2} N_2^{*2}} \left[N_2^{*2} s^* \left(R^2 y^* - N_1^{*2} \right) \right. \\ &\quad \left. + \theta w_3 R^2 N_2^{*2} i^* y^* + w_7 N_1^{*2} y^* \right] \\ A_3 &= \frac{w_7 s^* i^* y^*}{N_1^{*2} N_2^{*2}} \left[(w_2 + w_3) \theta R^2 y^* - (1 + w_2) \right. \\ &\quad \left. \cdot \left(w_2 N_1^{*2} + w_3 R^2 y^* \right) \right] + \frac{w_3 R s^* i^* y^*}{N_1^{*5}} \left[(w_5 \theta - w_6) \right. \\ &\quad \cdot R^2 N_1^* \left[R y^* - N_1^* \left(s^* + (1 + w_2) i^* \right) \right] \\ &\quad \left. + w_1 \left(w_6 - w_5 (1 + w_2) \right) R N_1^{*2} \right] \\ &\quad + \frac{R^2 s^{*2} y^*}{N_1^{*4}} \left[w_1 w_2 w_6 N_1^* \right. \\ &\quad + w_6 R \left(w_2 N_1^* s^* + w_3 R y^* \right) \\ &\quad \left. - w_5 \theta R \left(w_2 N_1^* s^* + w_3 R y^* \right) \right] \end{aligned} \quad (36)$$

while

$$\Delta = -M_1^* + M_2^* + M_3^* - M_4^* \quad (37)$$

with

$$\begin{aligned} M_1^* &= \frac{1}{N_1^{*6}} \left[N_1^{*2} s^* i^* y^* \left(R^2 s^* y^* - N_1^{*2} s^* \right. \right. \\ &\quad \left. + w_3 \theta R^2 i^* y^* \right) \left((1 + w_2) \left[w_2 N_1^{*2} + w_3 R^2 y^* \right] \right. \\ &\quad \left. \left. - (w_2 + w_3) \theta R^2 y^* \right) \right] \\ M_2^* &= \frac{w_3 w_7 \theta R^2 s^* i^* y^{*2}}{N_1^{*4} N_2^{*2}} \left[N_1^{*2} - R^2 y^* \right] \\ M_3^* &= \frac{s^* y^*}{N_1^{*6} N_2^{*4}} \left[\left(-R^2 N_2^{*2} s^* y^* + N_1^{*2} N_2^{*2} s^* \right. \right. \\ &\quad \left. - w_7 N_1^{*2} y^* \right) \left(w_7 R^2 N_1^{*2} y^* - w_7 N_1^{*4} \right. \\ &\quad \left. \left. + w_1 w_5 R^2 N_2^{*2} + (w_5 \theta - w_6) R^3 N_2^{*2} i^* \right) \right] \\ M_4^* &= \frac{w_3 R^2 i^* y^{*2}}{N_1^{*6} N_2^{*2}} \left[\left(w_3 \theta R^2 N_2^{*2} i^* + w_7 N_2^{*2} \right) \right. \\ &\quad \left. \cdot \left(w_1 w_6 N_1^* + (w_6 - \theta w_5) R N_1^* s^* + \theta w_7 y^* \right) \right] \end{aligned} \quad (38)$$

Keeping the above in view, it is well known that from Routh-Hurwitz criterion (33) has three roots with negative real parts provided that $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$. Consequently the following theorem can be proved easily.

Theorem 2. *The coexistence equilibrium point E_4 of system (3) is locally asymptotically stable in the interior of positive cone provided that the following sufficient conditions hold:*

$$\max\left\{(1+w_2)w_5, \frac{\theta w_5 R s^*}{w_1+Rs^*}\right\} < w_6 < w_5\theta \tag{39a}$$

$$\max\left\{\frac{(1+w_2)\left(w_2 N_1^{*2}+w_3R^2y^*\right)}{\theta(w_2+w_3)}, RN_1^*\left(s^*+(1+w_2)i^*\right)\right\} < R^2y^* < \frac{(1+w_2)}{\theta} \cdot N_1^{*2} \tag{39b}$$

$$R^2N_2^{*2}s^*y^* + \theta w_3 R^2 N_2^{*2} i^* y^* + w_7 N_1^{*2} y^* < N_1^{*2}N_2^{*2}s^* \tag{39c}$$

$$w_1w_2w_6N_1^* + w_6R\left(w_2N_1^*s^*+w_3Ry^*\right) > w_5\theta R\left(w_2N_1^*s^*+w_3Ry^*\right) \tag{39d}$$

$$w_7R^2N_1^{*2}y^* + w_1w_5R^2N_2^{*2} + (w_5\theta - w_6)R^3N_2^{*2}i^* > w_7N_1^{*4} \tag{39e}$$

$$M_2^* + M_3^* > (M_1^* + M_4^*) \tag{39f}$$

Now before we start studying the global stability of the system, the persistence of system (3), which represents biologically the coexistence of all the species for all the time while mathematically it indicates that the solution of system (3) for all $t>0$ does not have omega limit set on the boundary planes, is investigated in the following theorem when there are no periodic dynamics on the boundary planes.

Clearly system (3) has two subsystems. The first subsystem exists in case of absence of predator species, while the second subsystem exists in case of absence of disease. These subsystems can be written, respectively, as follows:

$$\begin{aligned} \frac{ds}{dt} &= s\left[1-(s+i)-w_2 i\right] = f_{11}(s,i) \\ \frac{di}{dt} &= i\left[w_2 s - w_4\right] = f_{12}(s,i) \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{ds}{dt} &= s\left[1-s-\frac{Ry}{w_1+Rs}\right] = f_{21}(s,y) \\ \frac{dy}{dt} &= y\left[\frac{w_5Rs}{w_1+Rs}-\frac{w_7}{w_8+y}-w_9\right] = f_{23}(s,y) \end{aligned} \tag{41}$$

Note that it is easy to verify that these two subsystems have unique positive equilibrium points in the interior of their positive domains $\Sigma_1 = \{(s,i)\in\mathbb{R}^2 : s(0)\geq 0,\ i(0)\geq 0\}$ and $\Sigma_2 = \{(s,y)\in\mathbb{R}^2 : s(0)\geq 0,\ y(0)\geq 0\}$, respectively. These two positive equilibrium points are given by $e_1 = (\widehat{s},\widehat{i})$ and $e_2 = (\overline{s},\overline{y})$ for subsystems (40) and (41), respectively. In fact they coincide with the predator free equilibrium point E_3 and disease free equilibrium point E_2 of system (3), respectively, and have the same existence conditions.

Therefore according to the Bendixson–Dulac theorem on dynamical system (40) "if there exists a C^1 function $\varphi(s,i)$, called the Dulac function, such that the expression $(\partial/\partial s)(\varphi f_{11}) + (\partial/\partial i)(\varphi f_{12})$ has the same sign and is not identically zero ($\neq 0$) almost everywhere in the simply connected region of the plane, then system (40) has no nonconstant periodic dynamic lying entirely in the Σ_1. Thus by choosing $\varphi(s,i) = 1/si$ we obtain that $(\partial/\partial s)(\varphi f_{11}) + (\partial/\partial i)(\varphi f_{12}) = -1/i < 0$. Therefore subsystem (40) has no periodic dynamic lying entirely in interior Σ_1 and hence $e_1 \equiv E_3$ is a globally asymptotically stable in the interior of Σ_1. Similarly by choosing $\varphi(s,i) = 1/sy$ we can show that the positive equilibrium point of the second subsystem (41) given by $e_2 \equiv E_2$ is globally asymptotically stable in the interior of the Σ_2 provided that one of the following two conditions holds.

$$\frac{R^2}{(w_1+Rs)^2} + \frac{w_7}{s(w_8+y)^2} > \frac{1}{y} \tag{42}$$

$$\frac{R^2}{(w_1+Rs)^2} + \frac{w_7}{s(w_8+y)^2} < \frac{1}{y} \tag{43}$$

Theorem 3. *Assume that condition (42) or (43) holds. Then system (3) is persistent provided that*

$$w_2 > w_4 \tag{44a}$$

$$\frac{w_5R}{w_1+R} > \frac{w_7}{w_8} + w_9 \tag{44b}$$

$$w_2\overline{s} > \frac{w_3R\overline{y}}{w_1+R\overline{s}} + w_4 \tag{44c}$$

$$\frac{w_5R\widehat{s}+w_6R\widehat{i}}{w_1+R\widehat{s}+\theta R\widehat{i}} > \frac{w_7}{w_8} + w_9 \tag{44d}$$

Proof. Consider the following function $\varphi(s,i,y) = s^\alpha i^\beta y^\gamma$, where α, β, and γ are positive constants; clearly $\varphi(s,i,y) > 0$ for all $(s,i,y)\in \operatorname{Int}\mathbb{R}_+^3$ and $\varphi(s,i,y)\longrightarrow 0$ when s,i or $y\longrightarrow 0$. Furthermore, it is clear that

$$\begin{aligned} \frac{\varphi'}{\varphi} &= \alpha\left[1-(s+i)-\frac{Ry}{w_1+Rs+\theta Ri}-w_2 i\right] \\ &\quad + \beta\left[w_2 s - \frac{w_3Ry}{w_1+Rs+\theta Ri} - w_4\right] \\ &\quad + \gamma\left[\frac{w_5Rs+w_6Ri}{w_1+Rs+\theta Ri} - \frac{w_7}{w_8+y} - w_9\right] \end{aligned} \tag{45}$$

Now the proof follows if $\varphi'/\varphi > 0$ for all the boundary equilibrium points, for suitable choice of constants $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

$$\begin{aligned}
\frac{\varphi'}{\varphi}(E_0) &= \alpha - w_4\beta - \left[\frac{w_7}{w_8} + w_9\right]\gamma \\
\frac{\varphi'}{\varphi}(E_1) &= [w_2 - w_4]\beta + \left[\frac{w_5 R}{w_1 + R} - \frac{w_7}{w_8} - w_9\right]\gamma \\
\frac{\varphi'}{\varphi}(E_2) &= \left[w_2\overline{s} - \frac{w_3 R\overline{y}}{w_1 + R\overline{s}} - w_4\right]\beta \\
\frac{\varphi'}{\varphi}(E_3) &= \left[\frac{w_5 R\widehat{s} + w_6 R\widehat{i}}{w_1 + R\widehat{s} + \theta R\widehat{i}} - \frac{w_7}{w_8} - w_9\right]\gamma
\end{aligned} \tag{46}$$

Clearly $(\varphi'/\varphi)(E_0) > 0$ for suitable choice of positive constant α sufficiently large with respect to the constants $\beta > 0$ and $\gamma > 0$, while $(\varphi'/\varphi)(E_1)$ is positive under conditions (44a) and (44b). Moreover $(\varphi'/\varphi)(E_2) > 0$ under condition (44c), while $(\varphi'/\varphi)(E_3)$ is positive too under condition (44d). Hence the proof is complete. □

4. Global Stability

In this section the global stability of each equilibrium point of system (3) is studied using suitable Lyapunov function as given in the following theorems.

Theorem 4. *Assume that the axial equilibrium point $E_1 = (1, 0, 0) = (s_0, 0, 0)$ is locally asymptotically stable in $\mathbb{R}^3_+$. Then, it is globally asymptotically stable, provided that the following conditions are outstanding:*

$$\frac{w_6}{w_3}w_2 < w_5(1 + w_2) < \frac{w_6}{w_3}w_4 \tag{47a}$$

$$w_9 > w_5\frac{R}{w_1} \tag{47b}$$

Proof. Recognize the following function:

$$V_1 = c_1\int_{s_0}^{s}\frac{\tau - s_0}{\tau}d\tau + c_2 i + c_3 y \tag{48}$$

where c_i, $i = 1, 2, 3$, are positive constants to be determined later on. Clearly $V_1(s, i, y) > 0$ is a continuously differentiable real valued function for all $(s, i, y) \in \mathbb{R}^3_+$ with $(s, i, y) \neq (s_0, 0, 0)$ and $V_1(s_0, 0, 0) = 0$. Moreover we have that

$$\begin{aligned}
\frac{dV_1}{dt} &= c_1\left(\frac{s - s_0}{s}\right)\frac{ds}{dt} + c_2\frac{di}{dt} + c_3\frac{dy}{dt} \\
&= -c_1(s - s_0)^2 - c_1\frac{Rsy}{N_1} + c_1\frac{Rs_0 y}{N_1} \\
&\quad - c_1(1 + w_2)si + c_1(1 + w_2)s_0 i + c_2 w_2 si \\
&\quad - c_2\frac{w_3 Riy}{N_1} - c_2 w_4 i + c_3\frac{w_5 Rs + w_6 Ri}{N_1}y \\
&\quad - \frac{c_3 w_7 y}{w_8 + y} - c_3 w_9 y
\end{aligned} \tag{49}$$

Here $R = (1 - m)$, $N_1 = w_1 + Rs + \theta Ri$. Further simplification gives that

$$\begin{aligned}
\frac{dV_1}{dt} &\le -c_1(s - s_0)^2 - [c_1 - c_3 w_5]\frac{Rsy}{N_1} \\
&\quad - [c_1(1 + w_2) - c_2 w_2]si - [c_2 w_3 - c_3 w_6]\frac{Riy}{N_1} \\
&\quad - [c_2 w_4 - c_1(1 + w_2)s_0]i \\
&\quad - \left[c_3 w_9 - c_1\frac{Rs_0}{w_1}\right]y
\end{aligned} \tag{50}$$

So by choosing the positive constants c_i, $i = 1, 2, 3$, as given below, we get

$$c_1 = w_5;$$

$$c_2 = \frac{w_6}{w_3};$$

$$c_3 = 1$$

$$\begin{aligned}
\frac{dV_1}{dt} &\le -w_5(s - 1)^2 - \left[w_5(1 + w_2) - \frac{w_6}{w_3}w_2\right]si \\
&\quad - \left[\frac{w_6}{w_3}w_4 - w_5(1 + w_2)\right]i \\
&\quad - \left[w_9 - w_5\frac{R}{w_1}\right]y
\end{aligned} \tag{51}$$

Clearly conditions (47a)-(47b) guarantee that $dV_1/dt < 0$; hence dV_1/dt is negative definite, and hence the axial equilibrium point E_1 is globally asymptotically stable and the proof is complete. □

Theorem 5. *Assume that the disease free equilibrium point $E_2 = (\overline{s}, 0, \overline{y})$ is locally asymptotically stable in $\mathbb{R}^3_+$. Then, it is a globally asymptotically stable, provided that the following conditions hold:*

$$w_4 w_1(1 + w_2)\overline{N}_1 > w_4 R^2\theta\overline{y} + w_1 w_2(1 + w_2)\overline{N}_1\overline{s} \tag{52a}$$

$$w_5\theta\left(\frac{R\overline{s}}{\overline{N}_1}\right)^2 < w_6 < \frac{w_1 w_3 w_5(1 + w_2)\overline{s} + w_4 w_5\theta R\overline{s}}{w_4\overline{N}_1} \tag{52b}$$

$$w_1\overline{N}_1 > R^2\overline{y} \tag{52c}$$

$$\frac{w_7\overline{N}_1}{w_1 w_5 w_8\overline{N}_2}(y - \overline{y})^2 < \left[1 - \frac{R^2\overline{y}}{w_1\overline{N}_1}\right](s - \overline{s})^2 \tag{52d}$$

where $\overline{N}_1 = w_1 + R\overline{s}$.

Proof. Consider the function

$$V_2 = \overline{c}_1\int_{\overline{s}}^{s}\frac{\tau - \overline{s}}{\tau}d\tau + \overline{c}_2 i + \overline{c}_3\int_{\overline{y}}^{y}\frac{\tau - \overline{y}}{\tau}d\tau \tag{53}$$

where $\overline{c}_i$, $i = 1, 2, 3$, are positive constants to be determined later on. Clearly $V_2(s, i, y) > 0$ is a continuously differentiable real valued function for all $(s, i, y) \in \mathbb{R}_+^3$ with $(s, i, y) \neq (\overline{s}, 0, \overline{y})$ and $V_2(\overline{s}, 0, \overline{y}) = 0$. Moreover we have that

$$\begin{aligned}\frac{dV_2}{dt} &= \overline{c}_1 (s - \overline{s}) \left[1 - (s + i) - \frac{Ry}{N_1} - w_2 i\right] \\ &+ \overline{c}_2 i \left[w_2 s - \frac{w_3 R y}{N_1} - w_4\right] \\ &+ \overline{c}_3 (y - \overline{y}) \left[\frac{w_5 R s + w_6 R i}{N_1} - \frac{w_7}{w_8 + y} - w_9\right]\end{aligned} \tag{54}$$

Further simplification gives

$$\begin{aligned}\frac{dV_2}{dt} &= \overline{c}_1 (s - \overline{s}) \left[-(s - \overline{s}) - (1 + w_2) i - \frac{R}{N_1} (y - \overline{y}) \right. \\ &\left. + \frac{R^2 \overline{y}}{N_1 \overline{N}_1} (s - \overline{s}) + \frac{R^2 \theta \overline{y}}{N_1 \overline{N}_1} i\right] + w_2 \overline{c}_2 i s - \frac{w_3 R \overline{c}_2}{N_1} i y \\ &- w_4 \overline{c}_2 i + \overline{c}_3 (y - \overline{y}) \left[\frac{w_5 R s}{N_1} - \frac{w_5 R \overline{s}}{\overline{N}_1} + \frac{w_6 R i}{N_1} \right. \\ &\left. + \frac{w_7}{N_2 \overline{N}_2} (y - \overline{y})\right]\end{aligned} \tag{55}$$

Here $N_2 = w_8 + y$ and $\overline{N}_2 = w_8 + \overline{y}$.

Now rearranging the terms of the last equation further gives

$$\begin{aligned}\frac{dV_2}{dt} &\leq -\overline{c}_1 \left[1 - \frac{R^2 \overline{y}}{w_1 \overline{N}_1}\right] (s - \overline{s})^2 \\ &- \frac{R}{N_1} \left(\overline{c}_1 - \overline{c}_3 \frac{w_1 w_5}{\overline{N}_1}\right) (s - \overline{s}) (y - \overline{y}) \\ &+ \frac{\overline{c}_3 w_7}{w_8 \overline{N}_2} (y - \overline{y})^2 \\ &- \left[\overline{c}_1 \left((1 + w_2) - \frac{R^2 \theta \overline{y}}{w_1 \overline{N}_1}\right) - \overline{c}_2 w_2\right] s i \\ &- \frac{R}{N_1} \left[\overline{c}_2 w_3 + \overline{c}_3 \left(\frac{w_5 \theta R \overline{s}}{\overline{N}_1} - w_6\right)\right] i y \\ &- \frac{R}{N_1} \left[\overline{c}_1 \frac{R \theta \overline{s} \overline{y}}{\overline{N}_1} - \overline{c}_3 \left(\frac{w_5 R \theta \overline{s} \overline{y}}{\overline{N}_1} - w_6 \overline{y}\right)\right] i \\ &- (\overline{c}_2 w_4 - \overline{c}_1 (1 + w_2) \overline{s}) i\end{aligned} \tag{56}$$

So by choosing the positive constants as

$$\begin{aligned}\overline{c}_1 &= 1, \\ \overline{c}_2 &= \frac{(1 + w_2) \overline{s}}{w_4}, \\ \overline{c}_3 &= \frac{\overline{N}_1}{w_1 w_5}\end{aligned} \tag{57}$$

we get that

$$\begin{aligned}\frac{dV_2}{dt} &\leq -\left[1 - \frac{R^2 \overline{y}}{w_1 \overline{N}_1}\right] (s - \overline{s})^2 + \frac{w_7 \overline{N}_1}{w_1 w_5 w_8 \overline{N}_2} (y - \overline{y})^2 \\ &- \frac{R \overline{y}}{w_1 N_1} \left[\frac{w_6 \overline{N}_1^{\,2} - w_5 \theta (R \overline{s})^2}{w_5 \overline{N}_1}\right] i \\ &- \left[\frac{w_4 \left[w_1 (1 + w_2) \overline{N}_1 - R^2 \theta \overline{y}\right] - w_1 w_2 (1 + w_2) \overline{N}_1 \overline{s}}{w_1 w_4 \overline{N}_1}\right] \\ &\cdot s i - \frac{R}{N_1} \left[\frac{w_1 w_3 w_5 (1 + w_2) \overline{s} + w_4 w_5 \theta R \overline{s} - w_4 w_6 \overline{N}_1}{w_1 w_4 w_5}\right] \\ &\cdot i y\end{aligned} \tag{58}$$

Accordingly, using the given conditions (52a)–(52d) we obtain

$$\begin{aligned}\frac{dV_2}{dt} &\leq -\left[1 - \frac{R^2 \overline{y}}{w_1 \overline{N}_1}\right] (s - \overline{s})^2 \\ &+ \frac{w_7 \overline{N}_1}{w_1 w_5 w_8 \overline{N}_2} (y - \overline{y})^2 \\ &- \frac{R \overline{y}}{w_1 N_1} \left[\frac{w_6 \overline{N}_1^{\,2} - w_5 \theta (R \overline{s})^2}{w_5 \overline{N}_1}\right] i\end{aligned} \tag{59}$$

Clearly $dV_2/dt < 0$ is negative definite, and hence the disease free equilibrium point E_2 is globally asymptotically stable under the given conditions and hence the proof is complete. □

Theorem 6. *Assume that the free predator equilibrium point $E_3 = (\widehat{s}, \widehat{i}, 0)$ is locally asymptotically stable in $\mathbb{R}_+^3$. Then it is globally asymptotically stable, provided that the following condition holds:*

$$\frac{w_9}{w_5} > \frac{R\left[w_2 \widehat{s} + w_3 (1 + w_2) \widehat{i}\right]}{w_1 w_2} \tag{60a}$$

$$w_3 w_5 (1 + w_2) > w_2 w_6 \tag{60b}$$

Proof. Consider the next function

$$V_3 = \widehat{c}_1 \int_{\widehat{s}}^{s} \frac{\tau - \widehat{s}}{\tau} d\tau + \widehat{c}_2 \int_{\widehat{i}}^{i} \frac{\tau - \widehat{i}}{\tau} d\tau + \widehat{c}_3 y \tag{61}$$

where $\widehat{c}_i$, $i = 1, 2, 3$, are positive constants to be determined later on. Clearly $V_3(s, i, y) > 0$ is a continuously differentiable real valued function for all $(s, i, y) \in \mathbb{R}_+^3$ with $(s, i, y) \neq (\widehat{s}, \widehat{i}, 0)$ and $V_3(\widehat{s}, \widehat{i}, 0) = 0$. Moreover we have that

$$\begin{aligned}\frac{dV_3}{dt} &= \widehat{c}_1 (s - \widehat{s}) \left[1 - (s + i) - \frac{Ry}{N_1} - w_2 i\right] \\ &+ \widehat{c}_2 \left(i - \widehat{i}\right) \left[w_2 s - \frac{w_3 R y}{N_1} - w_4\right] \\ &+ \widehat{c}_3 y \left[\frac{w_5 R s + w_6 R i}{N_1} - \frac{w_7}{N_2} - w_9\right]\end{aligned} \tag{62}$$

Here R, N_1, and N_2 are given above. Now straightforward computations give

$$\begin{aligned}\frac{dV_3}{dt} \leq & -\hat{c}_1 (s-\hat{s})^2 \\ & - [\hat{c}_1 (1+w_2) - \hat{c}_2 w_2] (i-\hat{i}) (s-\hat{s}) - \hat{c}_3 \frac{w_7}{N_2} y \\ & - \left[\hat{c}_3 w_9 - \hat{c}_2 \frac{w_3 R}{w_1} \hat{i} - \hat{c}_1 \frac{R\hat{s}}{w_1}\right] y \\ & - \frac{R}{N_1} [\hat{c}_1 - \hat{c}_3 w_5] sy - \frac{R}{N_1} [\hat{c}_2 w_3 - \hat{c}_3 w_6] iy\end{aligned} \tag{63}$$

By choosing the positive constants as

$$\hat{c}_1 = 1, \quad \hat{c}_2 = \frac{(1+w_2)}{w_2}, \quad \hat{c}_3 = \frac{1}{w_5} \tag{64}$$

then we obtain that

$$\begin{aligned}\frac{dV_3}{dt} \leq & -(s-\hat{s})^2 \\ & - \left[\frac{w_9}{w_5} - \frac{R\left(w_2 \hat{s} + w_3 (1+w_2) \hat{i}\right)}{w_1 w_2}\right] y \\ & - \frac{R}{N_1} \left[\frac{w_3 w_5 (1+w_2) - w_2 w_6}{w_2 w_5}\right] iy\end{aligned} \tag{65}$$

Accordingly, using the given conditions (60a) and (60b) we obtain

$$\begin{aligned}\frac{dV_3}{dt} \leq & -(s-\hat{s})^2 \\ & - \left[\frac{w_9}{w_5} - \frac{R\left(w_2 \hat{s} + w_3 (1+w_2) \hat{i}\right)}{w_1 w_2}\right] y\end{aligned} \tag{66}$$

Clearly $dV_3/dt \leq 0$, which means it is negative semidefinite, and hence the predator free equilibrium point E_3 is globally stable (but not asymptotically stable) under the given conditions. Moreover, since system (3) has the maximum invariant set for $dV_3/dt = 0$ if and only if conditions (60a)-(60b) hold and $(s,i,y) = (\hat{s},\hat{i},0)$, by Lyapunov-Lasalle's theorem, all the solutions starting in $\mathbb{R}^3_+$ approach the singleton set $\{E_3\}$, which is the positively invariant subset of the set where $dV_3/dt = 0$. Hence E_3 becomes attracting too; hence it is globally asymptotically stable and that completes the proof. □

Theorem 7. *Assume that the coexistence equilibrium point $E_4 = (s^*, i^*, y^*)$ of system (3) is locally asymptotically stable in $\mathbb{R}^3_+$. Then it is globally asymptotically stable, provided that the following conditions hold:*

$$R^2 y^* < \min\left\{w_1 N_1^*, \frac{(1+w_2) w_1 N_1^*}{\theta}\right\} \tag{67a}$$

$$\frac{\theta w_5 R s^*}{(w_1 + Rs^*)} < w_6 < \frac{w_5 (w_1 + \theta R i^*)}{R i^*} \tag{67b}$$

$$\begin{aligned}\Lambda_1 &= c_1^* \left[(1+w_2) N_1 N_1^* - \theta R^2 y^*\right] \\ &\quad - c_2^* \left[w_2 N_1 N_1^* + w_3 R^2 y^*\right] > 0\end{aligned} \tag{67c}$$

$$2\Lambda_2 - \Lambda_1 > 0 \tag{67d}$$

$$\begin{aligned}&\left[\frac{2\Lambda_3 + \Lambda_1}{N_1 N_1^*}\right] (i-i^*)^2 + \frac{w_7}{N_2 N_2^*} (y-y^*)^2 \\ &\quad < \left[\frac{2\Lambda_2 - \Lambda_1}{N_1 N_1^*}\right] (s-s^*)^2\end{aligned} \tag{67e}$$

Here Λ_i, $i = 1, 2, 3$, c_1^*, and c_2^* are given in the proof.

Proof. Consider the real valued function

$$\begin{aligned}V_4 = & c_1^* \int_{s^*}^{s} \frac{\tau - s^*}{\tau} d\tau + c_2^* \int_{i^*}^{i} \frac{\tau - i^*}{\tau} d\tau \\ & + c_3^* \int_{y^*}^{y} \frac{\tau - y^*}{\tau} d\tau.\end{aligned} \tag{68}$$

Here c_i^*, $i = 1, 2, 3$, are positive constants to be determined. Clearly $V_4(s,i,y) > 0$ is a continuously differentiable real valued function for all $(s,i,y) \in \mathbb{R}^3_+$ with $(s,i,y) \neq (s^*, i^*, y^*)$ and $V_3(s^*, i^*, y^*) = 0$. Moreover we have that

$$\begin{aligned}\frac{dV_4}{dt} = & c_1^* (s-s^*) \\ & \cdot \left[-(s-s^*) - (1+w_2)(i-i^*) - \frac{Ry}{N_1} + \frac{Ry^*}{N_1^*}\right] \\ & + c_2^* (i-i^*) \left[w_2 (s-s^*) - \frac{w_3 R y}{N_1} + \frac{w_3 R y^*}{N_1^*}\right] \\ & + c_3^* (y-y^*) \left[\frac{N_3}{N_1} - \frac{N_3^*}{N_1^*} - \frac{w_7}{N_2} + \frac{w_7}{N_2^*}\right]\end{aligned} \tag{69}$$

Here R, N_1, and N_2 are given above, while $N_1^* = w_1 + Rs^* + \theta R i^*$, $N_2^* = w_8 + y^*$, $N_3 = w_5 Rs + w_6 Ri$, and $N_3^* = w_5 R s^* + w_6 R i^*$. Further simplification for the above equation leads to

$$\begin{aligned}\frac{dV_4}{dt} \leq & -c_1^* \left(\frac{w_1 N_1^* - R^2 y^*}{w_1 N_1^*}\right) (s-s^*)^2 + c_2^* \\ & \cdot \frac{\theta w_3 R^2 y^*}{w_1 N_1^*} (i-i^*)^2 + c_3^* \frac{w_7}{w_8 N_2^*} (y-y^*)^2 - (s \\ & - s^*) (i-i^*) \left[c_1^* \left(\frac{(w_2+1) N_1 N_1^* - \theta R^2 y^*}{N_1 N_1^*}\right)\right. \\ & \left. - c_2^* \left(\frac{w_2 N_1 N_1^* + w_3 R^2 y^*}{N_1 N_1^*}\right)\right] - (s-s^*)(y-y^*)\end{aligned}$$

$$\cdot \frac{R}{N_1}\left[c_1^* - \frac{c_3^*}{N_1^*}\left(w_1 w_5 + \theta w_5 R i^* - w_6 R i^*\right)\right] - (i - i^*)(y - y^*)\frac{R}{N_1}\left[c_2^* w_3 - \frac{c_3^*}{N_1^*}\left(w_1 w_6 - \theta w_5 R s^* + w_6 R s^*\right)\right] \quad (70)$$

Consequently, by choosing the positive constants c_i^*, $i = 1, 2, 3$, as

$$c_1^* = \frac{w_1 w_5 + \theta w_5 R i^* - w_6 R i^*}{N_1^*}; \quad c_2^* = \frac{w_1 w_6 - \theta w_5 R s^* + w_6 R s^*}{w_3 N_1^*} \quad \text{and } c_3^* = 1 \quad (71)$$

the following is obtained:

$$\frac{dV_4}{dt} = -\frac{\Lambda_2}{N_1 N_1^*}(s - s^*)^2 + \frac{\Lambda_3}{N_1 N_1^*}(i - i^*)^2 + \frac{w_7}{N_2 N_2^*}(y - y^*)^2 - \frac{\Lambda_1}{N_1 N_1^*}(s - s^*)(i - i^*) \quad (72)$$

where $\Lambda_2 = c_1^*(N_1 N_1^* - R^2 y^*) > 0$ and $\Lambda_3 = c_2^* \theta w_3 R^2 y^* > 0$. Further simplification for the last equation gives

$$\frac{dV_4}{dt} = -\left[\frac{2\Lambda_2 - \Lambda_1}{N_1 N_1^*}\right](s - s^*)^2 + \left[\frac{2\Lambda_3 + \Lambda_1}{N_1 N_1^*}\right](i - i^*)^2 + \frac{w_7}{N_2 N_2^*}(y - y^*)^2 - \left[\sqrt{\frac{\Lambda_1}{2N_1 N_1^*}}(s - s^*) + \sqrt{\frac{\Lambda_1}{2N_1 N_1^*}}(i - i^*)\right]^2 \quad (73)$$

Clearly, $dV_4/dt < 0$ is negative definite function under the given conditions and hence the coexistence equilibrium point $E_4 = (s^*, i^*, y^*)$ is globally asymptotically stable and this completes the proof. □

5. Numerical Simulation

In this section the global dynamics of system (3) is studied numerically to verify the obtained analytical results in addition to specifying the control set of parameters. For the following hypothetical set of parameters, system (3) is solved numerically and the obtained trajectories are drawn in the form of phase portrait and time series.

$$\begin{aligned} w_1 &= 0.4, \\ w_2 &= 0.5, \\ w_3 &= 1, \\ w_4 &= 0.1, \\ w_5 &= 0.4 \\ w_6 &= 0.7, \\ w_7 &= 0.2, \\ w_8 &= 1, \\ w_9 &= 0.1, \\ m &= 0.4, \\ \theta &= 1 \end{aligned} \quad (74)$$

It is observed that, for the set of data (74), system (3) has a globally asymptotically stable positive equilibrium point $E_4 = (0.45, 0.28, 0.17)$ as shown in Figure 1.

Now in order to discover the impact of varying the parameters values on the dynamics of system (3), the system is solved numerically with varying one parameter each time and then the attractors of the obtained trajectories are present in the form of figures as shown in Figures 2–7.

It is observed that for the set of data (74) with $w_1 \geq 0.47$ the trajectory of system (3) approaches asymptotically predator free equilibrium point, while it approaches disease free point when $0.15 < w_1 \leq 0.25$; see Figure 2. Further the trajectory of system (3) approaches periodic dynamics in the sy −plane for data (74) with $w_1 \leq 0.15$ as shown in Figure 3; however it approaches asymptotically the positive equilibrium point otherwise.

On the other hand, for the data (74) with $w_2 \geq 0.88$ and $0.1 < w_2 \leq 0.17$, the solution of system (3) approaches asymptotically $E_3 = (\hat{s}, \hat{i}, 0)$ as shown in Figures 4(a)-4(b), while it approaches asymptotically $E_1 = (1, 0, 0)$ when $w_2 \leq 0.1$ as shown in Figure 4(c). Otherwise the system still has a globally asymptotically stable positive equilibrium point. Note that although it looks confusing as the trajectory of system (3) approaches the predator free equilibrium point E_3 too with decreasing the infection rate w_2, that depends on our hypothetical set of data in which we assumed that the conversion rate of predator from susceptible (w_5) is less than that from infected species (w_6) and once (w_5) enter the range $w_5 \geq 0.47$ the system approaches disease free point $E_2 = (\overline{s}, 0, \overline{y})$ as shown in the typical figure given by Figure 4(d).

Moreover it is observed that varying the parameter w_3 with the rest of parameters as in (74) has a quantitative effect on the dynamics of system (3) and the solution still approaches a positive equilibrium point that depends on the value of w_3. Now for the parameters values given by (74) with $0.2 \leq w_4 < 0.5$ and $0.5 \leq w_4$ the trajectory of system (3) approaches asymptotically $E_3 = (\hat{s}, \hat{i}, 0)$ and $E_1 = (1, 0, 0)$,

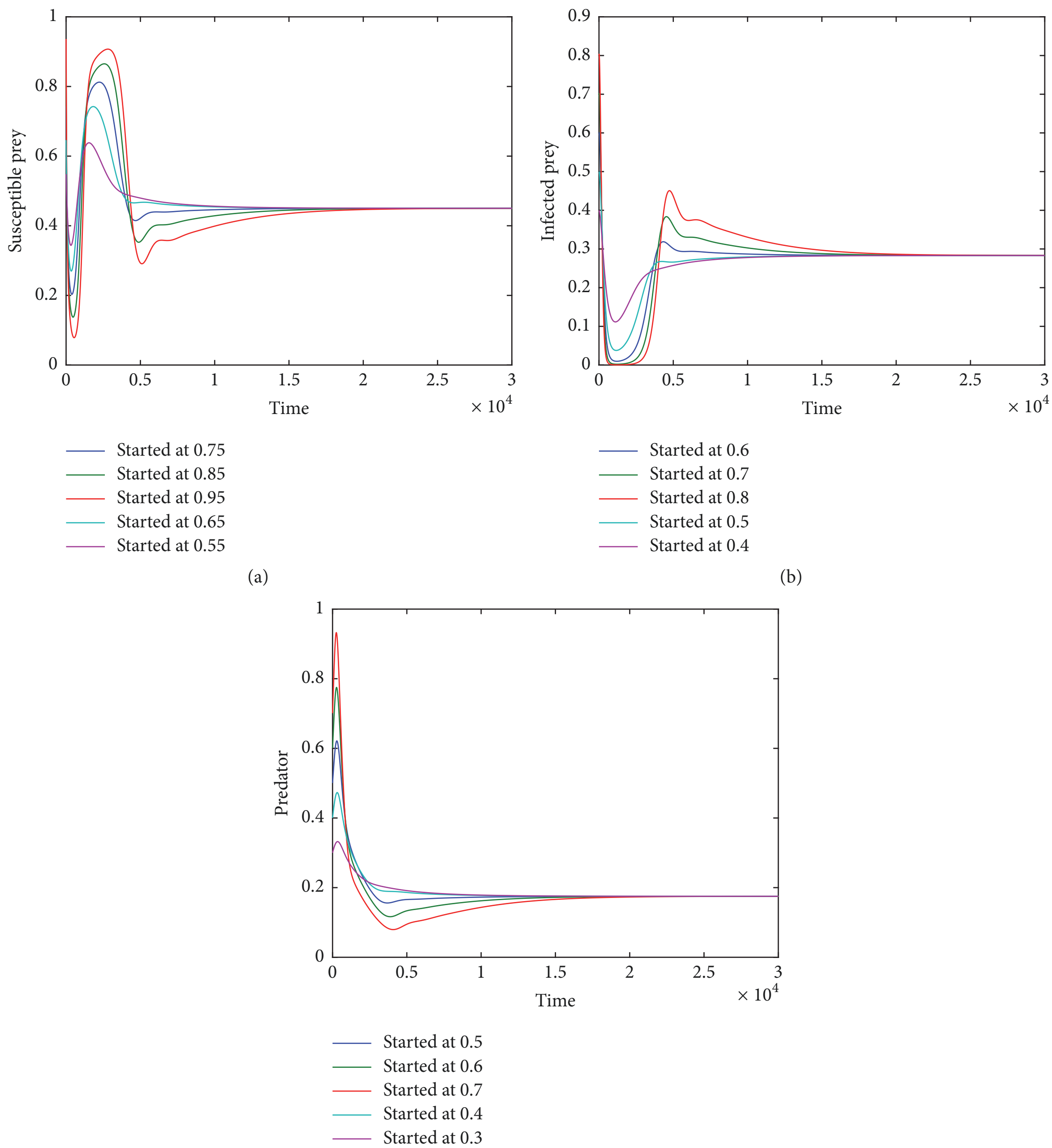

Figure 1: Time series for the trajectories of system (3) using data (74) with different sets of initial points. (a) Trajectories of Susceptible prey. (b) Trajectories of infected prey. (c) Trajectories of predator.

respectively, as given in Figure 5. Otherwise the system still is stable at positive equilibrium point. There is similar behavior of the parameter θ in the range $\theta \geq 1.3$ to that of system (3) using data (74) with $0.2 \leq w_4 < 0.5$, while the system still is stable at positive equilibrium point otherwise.

For the parameters (74) with $w_5 \geq 0.48$ (similarly when $w_8 \geq 1.5$) the trajectory of system (3) approaches asymptotically $E_2 = (\overline{s}, 0, \overline{y})$, while it approaches $E_3 = (\widehat{s}, \widehat{i}, 0)$ for data (74) with $w_5 \leq 0.25$ (similarly when $w_6 \leq 0.65$ or $w_8 \leq 0.8$) as shown in Figure 6. Otherwise the system has a globally asymptotically stable positive point.

Finally for the parameters (74) with $w_7 \geq 0.23$ (similarly when $w_9 \geq 0.13$ or $m \geq 0.49$) the trajectory of system (3) approaches asymptotically $E_3 = (\widehat{s}, \widehat{i}, 0)$, while it approaches $E_2 = (\overline{s}, 0, \overline{y})$ for data (74) with $w_7 \leq 0.15$ (similarly when $w_9 \leq 0.06$ or $m \leq 0.06$) as shown in Figure 7. Otherwise the system has a globally asymptotically stable positive point.

6. Discussion and Conclusions

The dynamics of a refuged prey-predator system, involving infectious disease in prey species and harvesting from a

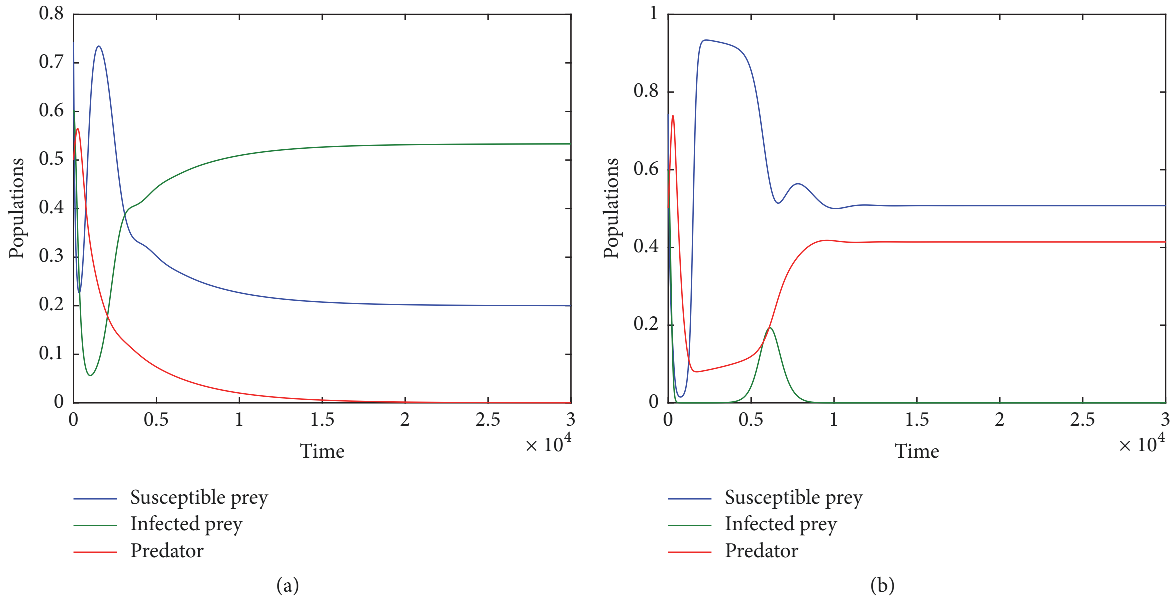

Figure 2: Time series for the trajectory of the system (3) using data (74) with typical values of w_1. (a) Trajectories of three species approach asymptotically to $E_3 = (0.2, 0.53, 0)$ when $w_1 = 0.55$. (b) Trajectories of three species approach asymptotically to $E_2 = (0.5, 0, 0.41)$ when $w_1 = 0.2$.

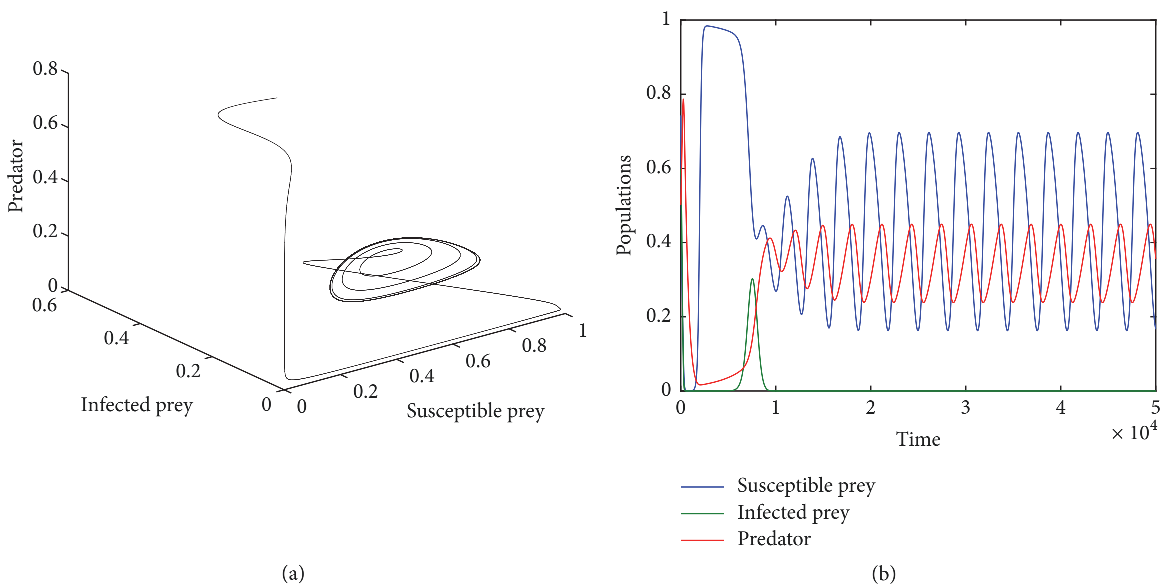

Figure 3: System (3) approaches asymptotically to periodic dynamics using data (74) with $w_1 = 0.1$. (a) Periodic attractor in the sy −plane. (b) Trajectories of the three species approach to periodic in the sy −plane.

predator species, is mathematically simulated through a mathematical model consisting of three nonlinear ordinary differential equations of the first order. The existence, uniqueness, and boundedness of the solution of the proposed model are discussed analytically. All feasible equilibrium points are determined and then the local stability analysis for them is carried out. The persistence conditions of the system are established. Suitable Lyapunov functions are used to show the global stability of the system's equilibrium points. Finally the proposed dynamical system is solved numerically in order to confirm the obtained analytical results and specify the control set of parameters too. It is observed that for the hypothetical set of parameters given by (74) the following results are obtained; different sets of parameters values may be used too.

(1) For the data (74), system (3) has a globally asymptotically stable positive equilibrium point in the interior of $\mathbb{R}^3_+$.

(2) The system has no periodic dynamics lying in the interior of $\mathbb{R}^3_+$; rather it either persists at the positive

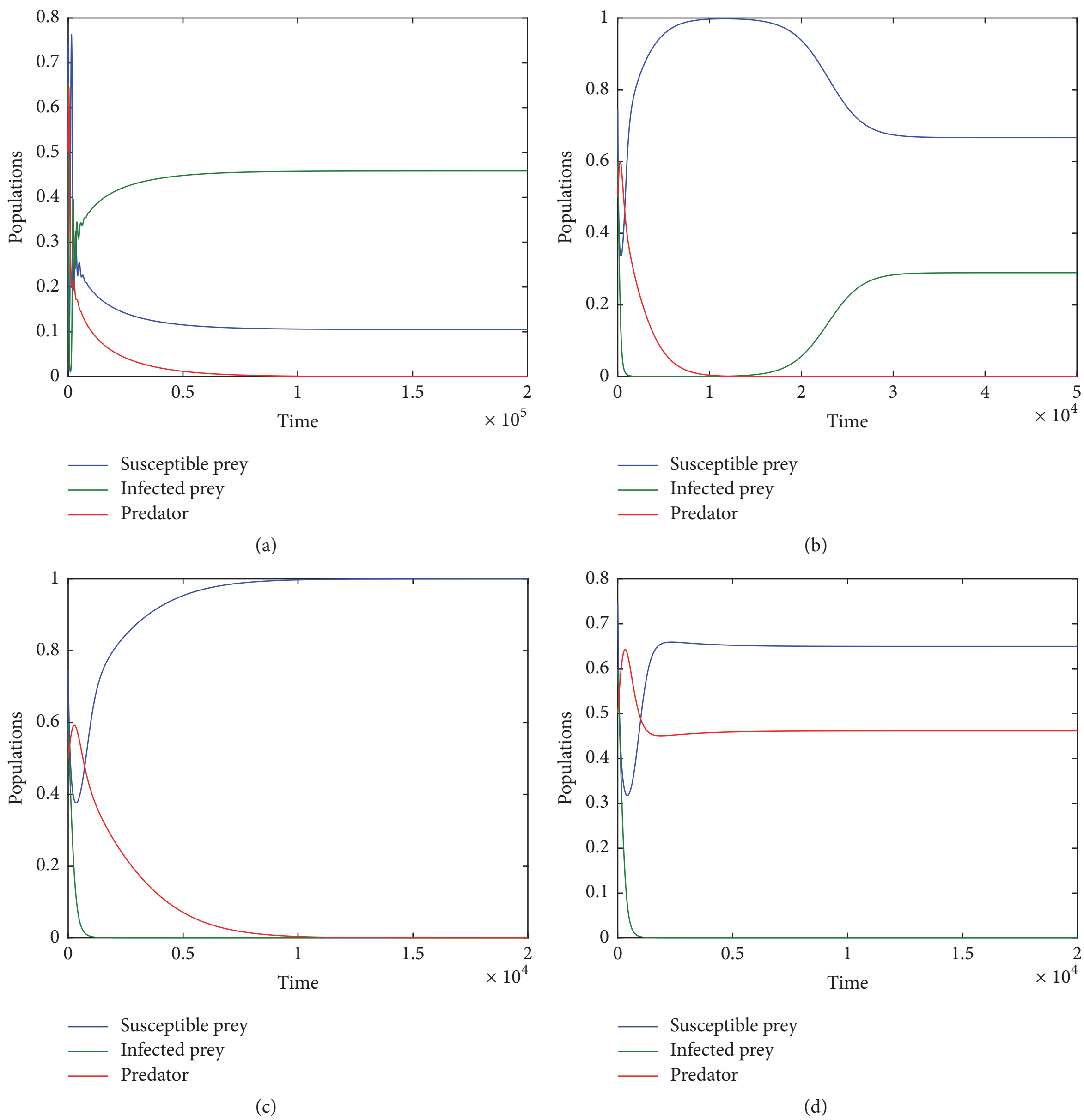

FIGURE 4: Time series for the trajectory of the system (3) using data (74) with typical values of w_2. (a) Trajectories of three species approach asymptotically to $E_3 = (0.1, 0.45, 0)$ when $w_2 = 0.95$. (b) Trajectories of three species approach asymptotically to $E_3 = (0.66, 0.29, 0)$ when $w_2 = 0.15$. (c) Trajectories of three species approach asymptotically to $E_1 = (1, 0, 0)$ when $w_2 = 0.05$. (d) Trajectories of three species approach asymptotically to $E_2 = (0.65, 0, 0.46)$ when $w_2 = 0.15$ and $w_5 = 0.48$.

equilibrium point or else loses its persistence and the system approaches asymptotically a specific attractor in the boundary planes.

(3) Increasing the half saturation constant represented by w_1 above 0.47 leads to losing the persistence of system (3) and the system approaches asymptotically the predator free equilibrium point, while decreasing half saturation constant in the range $0.15 < w_1 \leq 0.25$ makes the solution approaches asymptotically the disease free equilibrium point. Finally further decreasing of w_1 below the value 0.15 leads to losing the stability of the disease free point and the solution approaches periodic dynamics in the interior of sy −plane.

(4) Increasing the infection rate parameter (w_2) above the value 0.88 leads to losing the persistence of system (3) too and the solution of system (3) approaches asymptotically the predator free equilibrium point. However decreasing the value of this parameter to the range $0{,}1 < w_2 \leq 0.17$ makes the system approaches asymptotically either disease free equilibrium point or predator free equilibrium point depending on the values of conversion rates from susceptible prey and

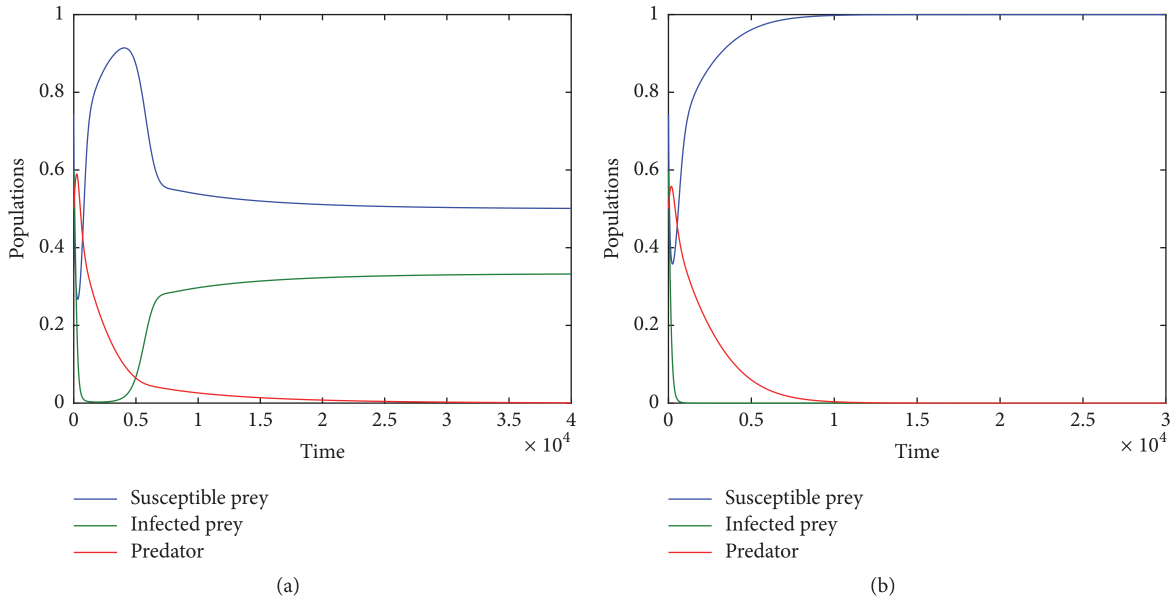

FIGURE 5: Time series for the trajectory of the system (3) using data (74) with typical values of w_4. (a) Trajectories of three species approach asymptotically to $E_3 = (0.5, 0.33, 0)$ when $w_4 = 0.25$. (b) Trajectories of three species approach asymptotically to $E_2 = (1, 0, 0)$ when $w_4 = 0.55$.

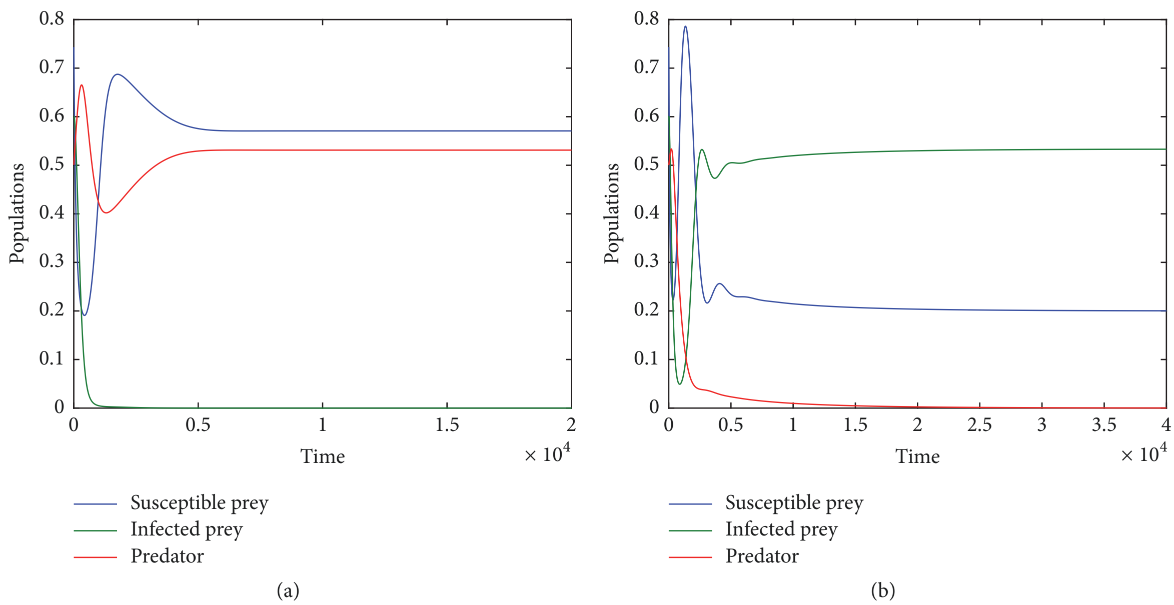

FIGURE 6: Time series for the trajectory of the system (3) using data (74) with typical values of w_5. (a) Trajectories of three species approach asymptotically to $E_2 = (0.57, 0, 0.53)$ when $w_5 = 0.5$. (b) Trajectories of three species approach asymptotically to $E_3 = (0.2, 0.53, 0)$ when $w_5 = 0.15$.

infected prey represented by w_5 and w_6, respectively. Finally decreasing the infection rate further ($w_2 \le 0.1$) leads to approaches to axial equilibrium point E_1.

(5) It is observed that varying the parameter w_3, which represents the ratio of the predator's attack rate of infected prey to predator's attack rate of susceptible prey, has a quantitative effect on the dynamics of system (3) and the system still persists at the positive equilibrium point that depends on the value of w_3.

(6) Increasing the death rate of the infected prey (w_4) so that $0.2 \le w_4 < 0.5$ and $0.5 \le w_4$ leads to losing the persistence of system (3) and the trajectory of system approaches asymptotically the predator free equilibrium point and axial equilibrium

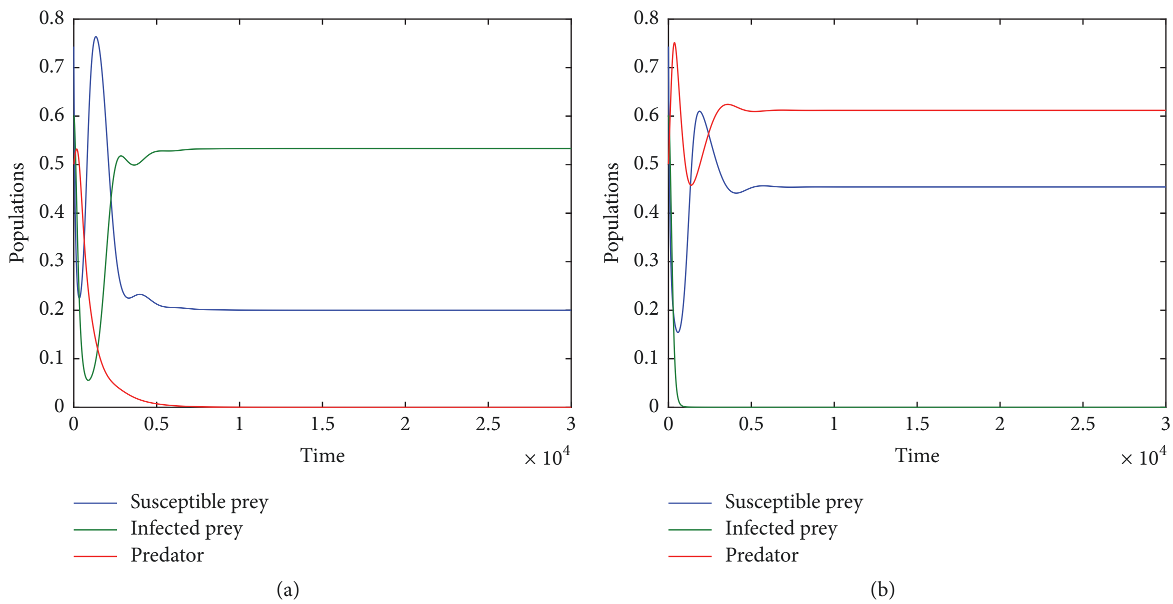

FIGURE 7: Time series for the trajectory of the system (3) using data (74) with typical values of w_7. (a) Trajectories of three species approach asymptotically to $E_3 = (0.2, 0.53, 0)$ when $w_7 = 0.3$. (b) Trajectories of three species approach asymptotically to $E_2 = (0.45, 0, 0.61)$ when $w_7 = 0.1$.

point, respectively. However the system still persists at the positive equilibrium point otherwise. Similar behavior of the dynamics has been obtained when the vulnerability constant rate increases above 1.3 with the rest of parameters as in (74).

(7) Now increasing the conversion rate from the susceptible prey above $w_5 \geq 0.48$ or decreasing it below $w_5 \leq 0.25$ causes losing of persistence of the system and the trajectory approaches asymptotically the disease free equilibrium point and predator free equilibrium point, respectively. Similar dynamical behavior has been obtained when increasing or decreasing the parameter w_8, which stands for the half saturation constant of harvesting in Michael-Mentence harvesting function, as that obtained in case of increasing or decreasing w_5. The decreasing of the conversion rate from the infected prey below 0.65 has the same dynamical effects on system (3) as that obtained with decreasing w_5 too.

(8) Finally increasing the parameter w_7, which stands for the maximum harvesting rate in Michael-Mentence harvesting function, above 0.23 leads to losing the persistence of system (3) and the solution approaches asymptotically the predator free equilibrium point, while it approaches asymptotically the disease free equilibrium point with decreasing the parameter w_7 below 0.15. Similar effect on the dynamical behavior of system (3) is obtained in case of increasing or decreasing the refuge rate m and predator death rate w_9 as that effect obtained with varying w_7.

Keeping the above in view, it is easy to verify that all the analytical stability conditions are satisfied for each case in the above-mentioned point. Furthermore, the refuge represented by parameter m and harvesting represented by parameters w_7 and w_8 have a vital effect on the dynamical behavior of system (3) including losing the persistence and moving between the disease free equilibrium point and predator free equilibrium point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. R. May, *Stability and Complexity in Model Ecosystems*, Princeton University, Princeton, NJ, USA, 1973.

[2] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics," *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, vol. 115, no. 772, pp. 700–721, 1927.

[3] R. M. Anderson and R. M. May, "The invasion, persistence and spread of infectious diseases within animal and plant communities," *Philosophical Transactions of the Royal Society B: Biological Sciences*, vol. 314, no. 1167, pp. 533–570, 1986.

[4] J. Chattopadhyay and O. Arino, "A predator-prey model with disease in the prey," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 36, pp. 747–766, 1999.

[5] Z. Z. Ma, F. D. Chen, C. Q. Wu, and W. L. Chen, "Dynamic behaviors of a Lotka-Volterra predator-prey model incorporating a prey refuge and predator mutual interference," *Applied*

Mathematics and Computation, vol. 219, no. 15, pp. 7945–7953, 2013.

[6] P. Jyoti Pal and P. K. Mandal, "Bifurcation analysis of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and strong Allee effect," *Mathematics and Computers in Simulation*, vol. 97, pp. 123–146, 2014.

[7] R. K. Naji and S. J. Majeed, "The dynamical analysis of a prey-predator model with a refuge-stage structure prey population," *International Journal of Differential Equations*, vol. 2016, Article ID 2010464, 10 pages, 2016.

[8] A. S. Abdulghafour and R. K. Naji, "The impact of refuge and harvesting on the dynamics of prey-predator system," *Science International (Lahore)*, vol. 30, no. 2, pp. 315–323, 2018.

[9] D. Greenhalgh, Q. J. Khan, and F. I. Lewis, "Hopf bifurcation in two SIRS density dependent epidemic models," *Mathematical and Computer Modelling*, vol. 39, no. 11-12, pp. 1261–1283, 2004.

[10] T. Zhang, J. Liu, and Z. Teng, "Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 1, pp. 293–306, 2010.

[11] P. Yongzhen, S. Li, C. Li, and S. Chen, "The effect of constant and pulse vaccination on an SIR epidemic model with infectious period," *Applied Mathematical Modelling*, vol. 35, no. 8, pp. 3866–3878, 2011.

[12] R. K. Naji and B. H. Abdulateef, "The dynamics of SICIR model with nonlinear incidence rate and saturated treatment function," *Science International (Lahore)*, vol. 29, no. 6, pp. 1223–1236, 2017.

[13] M. S. Rahman and S. Chakravarty, "A predator-prey model with disease in prey," *Nonlinear Analysis: Modelling and Control*, vol. 18, no. 2, pp. 191–209, 2013.

[14] S. Jana and T. K. Kar, "Modeling and analysis of a prey-predator system with disease in the prey," *Chaos, Solitons & Fractals*, vol. 47, pp. 42–53, 2013.

[15] S. Kant and V. Kumar, "Dynamics of a prey-predator system with infection in prey," *Electronic Journal of Differential Equations*, vol. 2017, no. 209, pp. 1–27, 2017.

[16] M. Haque and E. Venturino, "Increase of the prey may decrease the healthy predator population in presence of a disease in predator," *HERMIS*, vol. 7, pp. 38–59, 2006.

[17] M. Haque, "A predator-prey model with disease in the predator species only," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2224–2236, 2010.

[18] K. P. Das, "A mathematical study of a predator-prey dynamics with disease in predator," *ISRN Applied Mathematics*, vol. 2011, Article ID 807486, 16 pages, 2011.

[19] M. V. R. Murthy and D. K. Bahlool, "Modeling and Analysis of a Prey-Predator System with Disease in Predator," *IOSR Journal of Mathematics*, vol. 12, no. 1, pp. 21–40, 2016.

[20] Y. Hsieh and C. Hsiao, "Predator-prey model with disease infection in both populations," *Mathematical Medicine and Biology*, vol. 25, no. 3, pp. 247–266, 2008.

[21] K. P. Das, K. Kundu, and J. Chattopadhyay, "A predator-prey mathematical model with both the populations affected by diseases," *Ecological Complexity*, vol. 8, no. 1, pp. 68–80, 2011.

[22] K. P. Das, S. K. Sasmal, and J. Chattopadhyay, "Disease control through harvesting - conclusion drawn from a mathematical study of a predator-prey model with disease in both the population," *International Journal of Biomathematics and Systems Biology*, vol. 1, no. 1, pp. 1–29, 2014.

[23] S. Kant and V. Kumar, "Stability analysis of predator-prey system with migrating prey and disease infection in both species," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 42, pp. 509–539, 2017.

[24] N. Bairagi, S. Chaudhuri, and J. Chattopadhyay, "Harvesting as a disease control measure in an eco-epidemiological system—a theoretical study," *Mathematical Biosciences*, vol. 217, no. 2, pp. 134–144, 2009.

[25] R. Bhattacharyya and B. Mukhopadhyay, "On an eco-epidemiological model with prey harvesting and predator switching: local and global perspectives," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3824–3833, 2010.

[26] S. Gakkhar and K. B. Agnihotri, "The dynamics of disease transmission in a prey predator system with harvesting of prey," *International Journal of Advanced Research in Computer Engineering and Technology*, vol. 1, pp. 229–239, 2012.

[27] Q. Yue, "Dynamics of a modified Leslie–Gower predator–prey model with Holling-type II schemes and a prey refuge," *SpringerPlus*, vol. 5, no. 1, 2016.

[28] J. Ghosh, B. Sahoo, and S. Poria, "Prey-predator dynamics with prey refuge providing additional food to predator," *Chaos, Solitons & Fractals*, vol. 96, pp. 110–119, 2017.

[29] S. Gakkhar and R. K. Naji, "Existence of chaos in two-prey, one-predator system," *Chaos, Solitons & Fractals*, vol. 17, no. 4, pp. 639–649, 2003.

[30] C. W. Clark, "Aggregation and fishery dynamics: a theoretical study of schooling and the Purse Seine tuna fisheries," *NOAA Fisheries Bulletin*, vol. 77, no. 2, pp. 317–337, 1979.