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Research Article

An Iterative Method for Finding Common Solution of the Fixed Point Problem of a Finite Family of Nonexpansive Mappings and a Finite Family of Variational Inequality Problems in Hilbert Space

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In this paper, a hybrid iterative algorithm is proposed for finding a common element of the set of common fixed points of finite family of nonexpansive mappings and the set of common solutions of the variational inequality for an inverse strongly monotone mapping on the real Hilbert space. We establish the strong convergence of the proposed method for approximating a common element of the above defined sets under some suitable conditions. The results presented in this paper extend and improve some well-known corresponding results in the earlier and recent literature.

1. Introduction

Throughout, let $\mathscr{V}$ be a real Hilbert space whose inner product and norm are denoted as $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let W be a nonempty closed convex subset of $\mathscr{V}$ and P_W be the metric projection of $\mathscr{V}$ onto W. Let $\mathscr{S} : W \longrightarrow W$ be a nonexpansive mapping if

$$\|\mathscr{S}s - \mathscr{S}t\| \leq \|s - t\| \quad \forall s, t \in W. \tag{1}$$

The set of fixed point of $\mathscr{S}$ is denoted by $Fix\mathscr{S}$.

Let $G : W \longrightarrow \mathscr{V}$ be a nonlinear mapping. The classical variational inequality problem denoted by VIP associated with the set W is to find $t \in W$ such that

$$\langle Gt, s - t\rangle \geq 0, \quad \forall s \in W. \tag{2}$$

The solution set of the VIP (2) is denoted by $VI(W, G)$. The VIP is a fundamental problem in variational analysis which has been extensively studied by many researchers in the past decades, see Yao and Chadli [1], Zeng, Schaible, and Yao [2], and the references therein.

The VIP (2) was first discussed by Lions [3], further many different approaches are given to solve (2) in finite dimensional and infinite dimensional spaces, and the research in this direction is still continued.

Korpelevich [4] proposed a modification of an iterative algorithm for solving the VIP (2) in Euclidean space $\mathbb{R}^n$:

$$\begin{aligned} t_p &= P_W\left(s_p - \varsigma G s_p\right), \\ s_{p+1} &= P_W\left(s_p - \varsigma G t_p\right), \quad \forall p \geq 0, \end{aligned} \tag{3}$$

with $\varsigma > 0$, called the extragradient algorithm under certain assumptions, and it has received great attention. He proved that the sequences $\{s_p\}$ and $\{t_p\}$ converge strongly.

Takahashi and Toyoda [5] introduced an iterative algorithm for finding an element of $F(\mathscr{S}) \cap VI(F, W)$ under the assumption that $\mathscr{S}$ is nonexpansive and F is inverse strongly monotone as

$$\begin{aligned} s_1 &= s \in W \quad \text{choosen arbitrarlily} \\ s_{p+1} &= a_p s_p + \left(1 - a_p\right) \mathscr{S} P_W\left(s_p - \eta_p F s_p\right), \quad p \geq 1 \end{aligned} \tag{4}$$

where a_p is a sequence in $(0, 1)$ and η_p is a sequence in $(0, 2a)$.

Iiduka and Takahashi [6] proposed another iterative algorithm as

$$\begin{aligned} s_1 &= s \in W \quad \text{choosen arbitrarlily} \\ s_{p+1} &= a_p s + \left(1 - a_p\right) \mathcal{S} P_W \left(s_p - \eta_p F s_p\right), \quad p \geq 1 \end{aligned} \tag{5}$$

Marino and Xu [7] introduced the following iterative algorithm based on the viscosity approximation method:

$$s_{p+1} = \left(I - a_p G\right) \mathcal{S} s_p + a_p \zeta g\left(s_p\right), \quad p \geq 0, \tag{6}$$

where G is a strongly positive bounded linear operator on $\mathcal{V}$. They proved that if the sequence $\{a_p\}$ satisfies appropriate conditions, then the sequence $\{s_p\}$ generated by (4) converges strongly to the unique solution of the variational inequality

$$\left\langle \left(G - \zeta g\right) s^*, s - s^* \right\rangle \geq 0, \quad s \in W, \tag{7}$$

Qin and Cho [8] introduced an extended composite iterative algorithm $\{s_p\}$ defined as follows:

$$\begin{aligned} s_1 &= s \in W, \quad \text{choosen arbitrarily} \\ z_p &= \zeta_p s_p + \left(1 - \zeta_p\right) \mathcal{S} s_p, \\ t_p &= b_p s_p + \left(1 - b_p\right) \mathcal{S} z_p, \\ s_{p+1} &= a_p \zeta g\left(s_p\right) + c_p s_p + \left[\left(1 - c_p\right) I - a_p G\right] t_p, \\ & \qquad p \geq 1, \end{aligned} \tag{8}$$

where g is a contraction, $\mathcal{S}$ is a nonexpansive mapping, and G is a strongly positive linear bounded self-adjoint operator. Under certain appropriate assumptions on the parameters, $\{s_p\}$ defined by (8) converges strongly to a fixed point of $\mathcal{S}$, which solves the variational inequality (7).

Yamada [9] considered the hybrid steepest descent method for solving the VIP over the set of fixed points of a nonexpansive mapping. Moudafi [10] proposed the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping which is also a solution of a variational inequality problem.

In this paper, we introduce and analyze an iterative algorithm by combining Korpelevich's extragradient method, viscosity approximation method, and hybrid steepest method. We prove that under certain conditions the proposed algorithm converges strongly to a common element of $s \in \mathfrak{F} := (\bigcap_{l=1}^{M} Fix(\mathcal{S}_l)) \cap (\bigcap_{l=1}^{M} VI(F_l, W))$, which is a unique solution of variational inequality problem.

2. Preliminaries

Let $\mathcal{V}$ be a real Hilbert space and W be a nonempty closed convex subset of $\mathcal{V}$. For every point $s \in \mathcal{V}$, there exists a unique nearest point in W, denoted by $P_W s$, such that

$$\left\| s - P_W s \right\| \leq \left\| s - t \right\|, \quad \forall t \in W. \tag{9}$$

P_W is called the metric projection of $\mathcal{V}$ onto W. It is well known that P_W is a nonexpansive mapping of $\mathcal{V}$ onto W and satisfies

$$\left\langle s - t, P_W s - P_W t \right\rangle \geq \left\| P_W s - P_W t \right\|^2 \tag{10}$$

for every $s, t \in \mathcal{V}$. It is easy to see that

$$\begin{aligned} & u \in VI(F, W) \Longleftrightarrow \\ & u = P_W (u - \lambda F u), \quad \lambda > 0. \end{aligned} \tag{11}$$

Definition 1. A mapping $A : W \longrightarrow \mathcal{V}$ is said to be

(1) monotone, if

$$\langle As - At, s - t \rangle \geq 0, \quad \forall s, t \in W; \tag{12}$$

(2) L-Lipschitz, if there exists a constant $L > 0$ such that

$$\left\| As - At \right\| \leq L \left\| s - t \right\|, \quad \forall s, t \in W; \tag{13}$$

(3) α-inverse strongly monotone, if there exists a positive real number α such that

$$\langle As - At, s - t \rangle \geq \alpha \left\| As - At \right\|^2, \quad \forall s, t \in W. \tag{14}$$

Remark 2. Any α-inverse strongly monotone mapping A is monotone and $(1/\alpha)$-Lipschitz continuous.

Definition 3. A mapping $g : W \longrightarrow W$ is a contraction if there exists a constant $\tau \in (0, 1)$ such that

$$\left\| g(s) - g(t) \right\| \leq \tau \left\| s - t \right\|, \quad \forall s, t \in W. \tag{15}$$

Lemma 4 (see [7]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space $\mathcal{V}$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Definition 5. A mapping $\mathcal{S} : W \longrightarrow \mathcal{V}$ is said to be k-strictly pseudocontractive mapping if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \left\| \mathcal{S} s - \mathcal{S} t \right\|^2 \leq \left\| s - t \right\|^2 + k \left\| (I - \mathcal{S}) s - (I - \mathcal{S}) t \right\|^2, \\ \forall s, t \in W. \end{aligned} \tag{16}$$

Lemma 6 (see [11]). *Let $\mathcal{V}$ be a Hilbert space and let W be a closed convex subset of $\mathcal{V}$. For any integer $N \geq 1$, assume that for each $1 \leq i \leq N, \mathcal{S}_i : W \longrightarrow \mathcal{V}$ is a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$. Assume that $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$. Then $\sum_{i=1}^N \varphi_i \mathcal{S}_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 7 (see [11]). *Let $\{\mathcal{S}_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ be as in Lemma 6. Suppose that $\{\mathcal{S}_i\}_{i=1}^N$ has a common fixed point in W. Then $F(\sum_{i=1}^N \varphi_i \mathcal{S}_i) = \bigcap_{i=1}^N F(\mathcal{S}_i)$.*

Lemma 8 (see [12]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq \left(1 - \nu_n\right) a_n + \delta_n, \tag{17}$$

where $\{\nu_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence such that

(i) $\sum_{n=1}^{\infty} \nu_n = \infty$;

(ii) $\limsup_{n \longrightarrow \infty} (\delta_n / \nu_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 9 (see [13] demiclosed principle). *Let W be a nonempty closed convex subset of a real Hilbert space $\mathscr{V}$. Let $\mathcal{S}$ be a nonexpansive self-mapping on V with $Fix(\mathcal{S}) \neq \emptyset$. Then $I - \mathcal{S}$ is demiclosed. That is, whenever $\{s_p\}$ is a sequence in $\mathscr{V}$ weakly converging to some $s \in \mathscr{V}$ and the sequence $\{(I - \mathcal{S})s_p\}$ strongly converges to some t, it follows that $(I - \mathcal{S})s = t$. Here I is the identity operator of $\mathscr{V}$.*

Rockafellar [14] defined set-valued mapping $T : \mathscr{V} \longrightarrow 2^{\mathscr{V}}$ to be called monotone if for all $s, t \in \mathscr{V}$, $f \in Ts$ and $g \in Tt$ imply

$$\langle s - t, f - g \rangle \geq 0. \tag{18}$$

A set-valued mapping $T : \mathscr{V} \longrightarrow 2^{\mathscr{V}}$ is maximal monotone if the graph of $H(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(s, f) \in \mathscr{V} \times \mathscr{V}$, $\langle s - t, f - g \rangle \geq 0$ for every $(t, g) \in H(T)$ implies $f \in Ts$.

Let $F : W \longrightarrow \mathscr{V}$ be a monotone mapping and let $N_W v$ be the normal cone to W at $v \in W$, that is

$$N_W v = \{w \in \mathscr{V} : \langle u - v, w \rangle \geq 0, \forall u \in W\}. \tag{19}$$

Define

$$Tv = \begin{cases} Fv + N_W v, & v \in W \\ \emptyset, & v \notin W. \end{cases} \tag{20}$$

Then T is the maximal monotone and

$$0 \in Tv \quad \text{if and only if } v \in VI(F, W). \tag{21}$$

3. Main Results

Algorithm 10. Let W be a closed convex subset of a real Hilbert space $\mathscr{V}$, let $g : W \longrightarrow W$ be a ξ-contraction with coefficient $\xi \in [0, 1]$, G be strongly positive linear bounded self-adjoint operator, $F_l : W \longrightarrow \mathscr{V}$ be a ς_l-inverse strongly monotone mappings for each $1 \leq l \leq M$, where M is some positive integer, and $\mathcal{S}_l : W \longrightarrow W$ be a finite family of nonexpansive mappings for all $1 \leq l \leq M$. Let $\{b_{p,l}\}$ for all $1 \leq l \leq M$, $\{a_p\}$ and $\{c_p\}$ be the sequences in $[0, 1]$. For an arbitrarily given $s_1 = s \in W$, we propose the following hybrid iterative algorithm:

$$\begin{aligned} t_{p,1} &= b_{p,1}\mathcal{S}_1 s_p + \left(1 - b_{p,1}\right) s_p, \\ t_{p,l} &= b_{p,l}\mathcal{S}_l s_p + \left(1 - b_{p,l}\right) t_{p,l-1}, \quad l = 1, 2, \ldots, M \\ s_{p+1} &= a_p \zeta g\left(s_p\right) + c_p s_p \\ &\quad + \left[\left(1 - c_p\right) I - a_p G\right] P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right), \\ &\qquad \forall p \geq 1. \end{aligned} \tag{22}$$

The sequence $\{s_p\}$ defined by (22) converges strongly to a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequalities for an inverse strongly monotone mapping which solves the variational inequality problem.

Theorem 11. *Let W be a closed convex subset of a real Hilbert space $\mathscr{V}$, let $g : W \longrightarrow W$ be a ζ-contraction, let $F_l : W \longrightarrow V$ be a ς_l-inverse strongly monotone mapping, let $\mathcal{S}_l : W \longrightarrow W$ be a finite family of nonexpansive mappings with $\mathfrak{F} := (\bigcap_{l=1}^{M} Fix(\mathcal{S}_l)) \cap (\bigcap_{l=1}^{M} VI(F_l, W)) \neq \emptyset$, and let G be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\zeta} > 0$. Assume that $0 < \zeta \leq \bar{\zeta}/\varsigma$ and let $\{\theta_l\}$ be real numbers in $(0, 2\varsigma_l)$. Let $\{a_p\}$, $\{b_{p,l}\}$ and $\{c_p\}$ be sequences in $(0, 1)$ satisfying the following assumptions:*

(i) $\lim_{p\longrightarrow\infty} a_p = 0$ *and* $\sum_{p=1}^{\infty} a_p = \infty$.

(ii) $\sum_{p=1}^{\infty} |a_p - a_{p-1}| < \infty$ *and* $\sum_{p=1}^{\infty} |c_p - c_{p-1}| < \infty$.

(iii) $\sum_{p=1}^{\infty} |b_{p,l} - b_{p-1,l}| < \infty$ *for each* $l = 1, 2, \ldots, M$.

(iv) $0 < \liminf_{p\longrightarrow\infty} b_{p,l} \leq \limsup_{p\longrightarrow\infty} b_{p,l} < 1$.

Then the sequence $\{s_p\}$ defined by (22) converges strongly to $t \in \mathfrak{F}$, where $t = P_{\mathfrak{F}}(\zeta g + (I - G))t$ is also the unique solution of the following variational inequality:

$$\langle \zeta g(t) - Gq, q - t \rangle \leq 0, \quad \forall q \in \mathfrak{F}. \tag{23}$$

Proof. For any $s, t \in W$ and $\theta_l \in (0, 2\varsigma_l)$, we have

$$\begin{aligned} &\left\|\left(I - \theta_l F_l\right) s - \left(I - \theta_l F_l\right) t\right\|^2 \\ &\quad = \left\|(s - t) - \theta_l\left(F_l s - F_l t\right)\right\|^2 \\ &\quad = \|s - t\|^2 - 2\theta_l \langle F_l s - F_l t, s - t \rangle + \theta_l^2 \left\|F_l s - F_l t\right\|^2 \\ &\quad \leq \|s - t\|^2 - \theta_l\left(2\varsigma_l - \theta_l\right)\left\|F_l s - F_l t\right\|^2 \leq \|s - t\|^2. \end{aligned} \tag{24}$$

From (24), $(I - \theta_l F_l)$ is nonexpansive. Since G is a linear bounded self-adjoint operator, so

$$\|G\| = \sup\{|\langle Gs, s\rangle| : s \in V, \|s\| = 1\}. \tag{25}$$

We assume that

$$\|G\| a_p < 1 - c_p \quad \forall p \geq 0, \tag{26}$$

and since $\lim_{p\longrightarrow\infty} a_p = 0$, so without loss of generality, we have

$$\begin{aligned} &\left\langle\left[\left(1 - c_p\right) I - a_p G\right] s, s\right\rangle = 1 - c_p - a_p \langle Gs, s\rangle \\ &\quad \leq 1 - c_p - a_p \|G\| \leq 0, \end{aligned} \tag{27}$$

so $(1 - c_p)I - a_p G$ is positive. Now, it follows that

$$\begin{aligned} &\left\|\left(1 - c_p\right) I - a_p G\right\| \\ &\quad = \sup\left\{\left\langle\left[\left(1 - c_p\right) I - a_p G\right] s, s\right\rangle : s \in V, \|s\| = 1\right\} \\ &\quad = \sup\left\{1 - c_p - a_p \langle Gs, s\rangle : s \in V, \|s\| = 1\right\} \\ &\quad \leq 1 - c_p - a_p \bar{\zeta}. \end{aligned} \tag{28}$$

Next, we prove that the sequence $\{s_p\}$ is bounded. Fixing $s^* \in \mathfrak{F}$, we have

$$\begin{aligned}\left\|t_{p,1}-s^*\right\| &= \left\|b_{p,1}\mathcal{S}_1 s_p + \left(1-b_{p,1}\right)s_p - s^*\right\| \\ &\le b_{p,1}\left\|s_p - s^*\right\| + \left(1-b_{p,1}\right)\left\|s_p - s^*\right\| \\ &\le \left\|s_p - s^*\right\|.\end{aligned} \tag{29}$$

By induction from $l = 2$ to $l = M$, we have

$$\begin{aligned}\left\|t_{p,l}-s^*\right\| &= \left\|b_{p,l}\mathcal{S}_l s_p + \left(1-b_{p,l}\right)t_{p,l-1} - s^*\right\| \\ &\le b_{p,l}\left\|s_p - s^*\right\| + \left(1-b_{p,l}\right)\left\|t_{p,l-1} - s^*\right\| \\ &\le \left\|s_p - s^*\right\|.\end{aligned} \tag{30}$$

Thus, for every $l = 1, 2, \ldots, M$, we get

$$\left\|t_{p,l}-s^*\right\| \le \left\|s_p - s^*\right\|. \tag{31}$$

From (22) and (31), we have

$$\begin{aligned}\left\|s_{p+1}-s^*\right\| &= \left\|a_p\zeta g\left(s_p\right) + c_p s_p \right. \\ &\quad \left. + \left[\left(1-c_p\right)I - a_p G\right]P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right) - s^*\right\| \\ &= \left\|a_p\left(\zeta g\left(s_p\right) - Gs^*\right) + c_p\left(s_p - s^*\right)\right. \\ &\quad \left. + \left[\left(1-c_p\right)I - a_p G\right]P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right) - s^*\right\| \\ &\le a_p\left\|\zeta g\left(s_p\right) - Gs^*\right\| + c_p\left\|s_p - s^*\right\| + \left(1-c_p\right. \\ &\quad \left. - a_p\bar{\zeta}\right)\left\|t_{p,l} - s^*\right\| \le a_p\zeta\left\|g\left(s_p\right) - g\left(s^*\right)\right\| \\ &\quad + a_p\left\|\zeta g\left(s^*\right) - Gs^*\right\| + c_p\left\|s_p - s^*\right\| + \left(1-c_p\right. \\ &\quad \left. - a_p\bar{\zeta}\right)\left\|t_{p,l} - s^*\right\| = a_p\zeta\tau\left\|s_p - s^*\right\| + a_p\left\|\zeta g\left(s^*\right)\right. \\ &\quad \left. - Gs^*\right\| + c_p\left\|s_p - s^*\right\| + \left(1-c_p - a_p\bar{\zeta}\right)\left\|s_p - s^*\right\| \\ &\le \left(1 - a_p\left(\bar{\zeta} - \zeta\tau\right)\right)\left\|s_p - s^*\right\| + a_p\left\|\zeta g\left(s^*\right) - Gs^*\right\| \\ &= \left(1 - a_p\left(\bar{\zeta} - \zeta\tau\right)\right)\left\|s_p - s^*\right\| \\ &\quad + \frac{a_p\left(\bar{\zeta} - \zeta\tau\right)}{\bar{\zeta} - \zeta\tau}\left\|\zeta g\left(s^*\right) - Gs^*\right\| \\ &\le \max\left\{\left\|s_p - s^*\right\|, \frac{1}{\bar{\zeta} - \zeta\tau}\left\|\zeta g\left(s^*\right) - Gs^*\right\|\right\}.\end{aligned} \tag{32}$$

By induction on p, we have

$$\begin{aligned}&\left\|s_p - s^*\right\| \\ &\quad \le \max\left\{\left\|s_o - s^*\right\|, \frac{1}{\bar{\zeta} - \zeta\tau}\left\|\zeta g\left(s^*\right) - Gs^*\right\|\right\}, \\ &\qquad \forall p \ge 0.\end{aligned} \tag{33}$$

Hence, the given sequence $\{s_p\}$ is bounded and so is $\{t_{p,l}\}$ for all $l = 1, 2, \ldots, M$.

Next, we show that

$$\lim_{p\longrightarrow\infty}\left\|s_{p+1} - s_p\right\| = 0. \tag{34}$$

From the definition of the sequence $\{t_{p,l}\}$ for each $l = 1, 2, \ldots, M$ in Algorithm 10, we have

$$\begin{aligned}\left\|t_{p,l} - t_{p-1,l}\right\| &= \left\|b_{p,l}\mathcal{S}_l s_p + \left(1-b_{p,l}\right)t_{p,l-1}\right. \\ &\quad \left. - b_{p-1,l}\mathcal{S}_l s_{p-1} - \left(1-b_{p-1,l}\right)t_{p-1,l-1}\right\| \\ &= \left\|b_{p,l}\left(\mathcal{S}_l s_p - \mathcal{S}_l s_{p-1}\right) + \left(b_{p,l} - b_{p-1,l}\right)\mathcal{S}_l s_{p-1}\right. \\ &\quad + \left(1-b_{p,l}\right)\left(t_{p,l-1} - t_{p-1,l-1}\right) \\ &\quad \left. + \left(b_{p-1,l} - b_{p,l}\right)t_{p-1,l-1}\right\| \le b_{p,l}\left\|s_p - s_{p-1}\right\| + \left(1\right. \\ &\quad \left. - b_{p,l}\right)\left\|t_{p,l-1} - t_{p-1,l-1}\right\| + \left|b_{p,l} - b_{p-1,l}\right|\left(\left\|\mathcal{S}_l s_{p-1}\right\|\right. \\ &\quad \left. + \left\|t_{p-1,l-1}\right\|\right).\end{aligned} \tag{35}$$

When l=1, we have

$$\begin{aligned}\left\|t_{p,1} - t_{p-1,1}\right\| &= \left\|b_{p,1}\mathcal{S}_1 s_p + \left(1-b_{p,1}\right)s_p\right. \\ &\quad \left. - b_{p-1,1}\mathcal{S}_1 s_{p-1} - \left(1-b_{p-1,1}\right)s_{p-1}\right\| \\ &= \left\|b_{p,1}\left(\mathcal{S}_1 s_p - \mathcal{S}_1 s_{p-1}\right) + \left(b_{p,1} - b_{p-1,1}\right)\mathcal{S}_1 s_{p-1}\right. \\ &\quad \left. + \left(1-b_{p,1}\right)\left(s_p - s_{p-1}\right) + \left(b_{p-1,1} - b_{p,1}\right)s_{p-1}\right\| \\ &\le b_{p,1}\left\|s_p - s_{p-1}\right\| + \left(1-b_{p,1}\right)\left\|s_p - s_{p-1}\right\| + \left|b_{p,1}\right. \\ &\quad \left. - b_{p-1,1}\right|\left(\left\|\mathcal{S}_1 s_{p-1}\right\| + \left\|s_{p-1}\right\|\right) \le \left\|s_p - s_{p-1}\right\| + \left|b_{p,1}\right. \\ &\quad \left. - b_{p-1,1}\right|\left(\left\|\mathcal{S}_1 s_{p-1}\right\| + \left\|s_{p-1}\right\|\right).\end{aligned} \tag{36}$$

From (35) and (36), for l=2,...,M, we get

$$\begin{aligned}\left\|t_{p,l} - t_{p-1,l}\right\| &\le \left\|s_p - s_{p-1}\right\| \\ &\quad + \sum_{i=2}^{l}\left\|\mathcal{S}_i s_{p-1} - t_{p-1,i-1}\right\|\left|b_{p,i} - b_{p-1,i}\right| \\ &\quad + \left(\left\|\mathcal{S}_1 s_{p-1}\right\| + \left\|s_{p-1}\right\|\right)\left|b_{p,1} - b_{p-1,1}\right|.\end{aligned} \tag{37}$$

Next,

$$\begin{aligned}\left\|s_{p+1} - s_p\right\| &= \left\|a_p\zeta g\left(s_p\right) + c_p s_p + \left[\left(1-c_p\right)I - a_p G\right]\right. \\ &\quad \cdot P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right) - a_{p-1}\zeta g\left(s_{p-1}\right) - c_{p-1}s_{p-1} \\ &\quad \left. - \left[\left(1-c_{p-1}\right)I - a_{p-1}G\right]P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\| \\ &\le \left\|a_p\zeta\left(g\left(s_p\right) - g\left(s_{p-1}\right)\right) + \left(a_p - a_{p-1}\right)\zeta g\left(s_{p-1}\right)\right. \\ &\quad + c_p\left(s_p - s_{p-1}\right) + \left(c_p - c_{p-1}\right)s_{p-1} + \left[\left(1-c_p\right)I\right. \\ &\quad \left. - a_p G\right]\left(P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&- P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\big) + \left[\left(1 - c_p\right) I - a_p G\right] \\
&\cdot P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right) - \left[\left(1 - c_{p-1}\right) I - a_{p-1} G\right] \\
&\cdot P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\big\| \le a_p \zeta \tau \left\| s_p - s_{p-1}\right\| + \left|a_p\right. \\
&\left. - a_{p-1}\right| \left\|\zeta g\left(s_{p-1}\right)\right\| + c_p \left\| s_p - s_{p-1}\right\| + \left|c_p - c_{p-1}\right| \\
&\cdot \left\| s_{p-1}\right\| + \left(1 - c_p - a_p \bar{\zeta}\right) \left\| P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right)\right. \\
&\left. - P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\| + \left|c_p - c_{p-1}\right| \left\| P_W\left(t_{p-1,l}\right.\right. \\
&\left.\left. - \theta_l F_l t_{p-1,l}\right)\right\| + \left|a_p - a_{p-1}\right| \left\| G P_W\left(t_{p-1,l}\right.\right. \\
&\left.\left. - \theta_l F_l t_{p-1,l}\right)\right\| \le a_p \zeta \tau \left\| s_p - s_{p-1}\right\| + c_p \left\| s_p - s_{p-1}\right\| \\
&+ \left(1 - c_p - a_p \bar{\zeta}\right) \left\| t_{p,l} - t_{p-1,l}\right\| + \left[\left|a_p - a_{p-1}\right|\right. \\
&\left.\cdot \left(\left\|\zeta g\left(s_{p-1}\right)\right\| + \left\| G P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right)\right] \\
&+ \left[\left|c_p - c_{p-1}\right| \left(\left\| s_{p-1}\right\| + \left\| P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right)\right] \\
&\le a_p \zeta \tau \left\| s_p - s_{p-1}\right\| + c_p \left\| s_p - s_{p-1}\right\| + \left(1 - c_p\right. \\
&\left. - a_p \bar{\zeta}\right) \left\{\left\| s_p - s_{p-1}\right\|\right. \\
&+ \sum_{i=2}^{l} \left\| \mathcal{S}_i s_{p-1} - t_{p-1,i-1}\right\| \left|b_{p,i} - b_{p-1,i}\right| \\
&\left. + \left(\left\| \mathcal{S}_1 s_{p-1}\right\| + \left\| s_{p-1}\right\|\right) \left|b_{p,1} - b_{p-1,1}\right|\right\} + \left[\left|a_p\right.\right. \\
&\left.\left. - a_{p-1}\right| \left(\left\|\zeta g\left(s_{p-1}\right)\right\| + \left\| G P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right)\right] \\
&+ \left|c_p - c_{p-1}\right| \left(\left\| s_{p-1}\right\| + \left\| P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right) \\
&\le \left[1 - a_p\left(\bar{\zeta} - \zeta \tau\right)\right] \left\| s_p - s_{p-1}\right\| \\
&+ M_1 \left[\sum_{i=1}^{l} \left|b_{p,i} - b_{p,l,i}\right| + \left|a_p - a_{p-1}\right| + \left|c_p - c_{p-1}\right|\right]
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
M_1 = \max \Bigg\{ &\sup_{p \ge 1} \left(\sum_{i=2}^{l} \left\| \mathcal{S}_i s_p - t_{p,i-1}\right\| + \left\| \mathcal{S}_l s_{p-1}\right\| + \left\| t_{p-1,l-1}\right\| \right), \\
&\sup_{p \ge 1} \left(\left\|\zeta g\left(s_{p-1}\right)\right\| + \left\| G P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right), \\
&\sup_{p \ge 1} \left(\left\| s_{p-1}\right\| + \left\| P_W\left(t_{p-1,l} - \theta_l F_l t_{p-1,l}\right)\right\|\right) \Bigg\}.
\end{aligned}
\tag{39}
$$

From conditions (i), (ii), and (iii), we have

$$
\lim_{p \longrightarrow \infty} \left\| s_{p+1} - s_p\right\| = 0. \tag{40}
$$

Setting $w_{p,l} = P_W(t_{p,l} - \theta_l F_l t_{p,l})$, we have

$$
s_{p+1} = a_p \zeta g\left(s_p\right) + c_p s_p + \left[\left(1 - c_p\right) I - a_p G\right] w_{p,l}. \tag{41}
$$

Next for $s^* \in \mathfrak{F}$, we have

$$
\begin{aligned}
&\left\| s_{p+1} - s^*\right\|^2 = \left\| a_p \zeta g\left(s_p\right) + c_p s_p\right. \\
&\quad \left. + \left[\left(1 - c_p\right) I - a_p G\right] w_{p,l} - s^*\right\|^2 \\
&\quad = \left\| a_p\left(\zeta g\left(s_p\right) - G s^*\right) + c_p\left(s_p - s^*\right)\right. \\
&\quad \left. + \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right)\right\|^2 = a_p^2 \left\|\zeta g\left(s_p\right)\right. \\
&\quad \left. - G s^*\right\|^2 + \left\| c_p\left(s_p - s^*\right)\right. \\
&\quad \left. + \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right)\right\|^2 \\
&\quad + 2 a_p \left\langle \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right)\right. \\
&\quad \left. + c_p\left(s_p - s^*\right), \zeta g\left(s_p\right) - G s^*\right\rangle = a_p^2 \left\|\zeta g\left(s_p\right)\right. \\
&\quad \left. - G s^*\right\|^2 + \left\| c_p\left(s_p - s^*\right)\right. \\
&\quad \left. + \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right)\right\|^2 + 2 a_p c_p \left\langle s_p\right. \\
&\quad \left. - s^*, \zeta g\left(s_p\right) - G s^*\right\rangle \\
&\quad + 2 a_p \left\langle \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right), \zeta g\left(s_p\right)\right. \\
&\quad \left. - G s^*\right\rangle \le a_p^2 \left\|\zeta g\left(s_p\right) - G s^*\right\|^2 + \left[c_p \left\| s_p - s^*\right\|\right. \\
&\quad \left. + \left(1 - c_p - a_p \bar{\zeta}\right) \left\| w_{p,l} - s^*\right\|\right]^2 + 2 a_p c_p \left\langle s_p\right. \\
&\quad \left. - s^*, \zeta g\left(s_p\right) - G s^*\right\rangle \\
&\quad + 2 a_p \left\langle \left[\left(1 - c_p\right) I - a_p G\right]\left(w_{p,l} - s^*\right), \zeta g\left(s_p\right)\right. \\
&\quad \left. - G s^*\right\rangle = \left(1 - c_p - a_p \bar{\zeta}\right)^2 \left\| w_{p,l} - s^*\right\|^2 + c_p^2 \left\| s_p\right. \\
&\quad \left. - s^*\right\|^2 + 2\left(1 - c_p - a_p \bar{\zeta}\right) c_p \left\| w_{p,l} - s^*\right\| \left\| s_p - s^*\right\| \\
&\quad + d_p \le \left(1 - c_p - a_p \bar{\zeta}\right)^2 \left\| w_{p,l} - s^*\right\|^2 + c_p^2 \left\| s_p - s^*\right\|^2 \\
&\quad + \left(1 - c_p - a_p \bar{\zeta}\right) c_p \left[\left\| w_{p,l} - s^*\right\|^2 + \left\| s_p - s^*\right\|^2\right] \\
&\quad + d_p = \left[\left(1 - a_p \bar{\zeta}\right)^2 - 2\left(1 - a_p \bar{\zeta}\right) c_p + c_p^2\right] \left\| w_{p,l}\right. \\
&\quad \left. - s^*\right\|^2 + c_p^2 \left\| s_p - s^*\right\|^2 + \left[\left(1 - a_p \bar{\zeta}\right) c_p - c_p^2\right] \\
&\quad \cdot \left(\left\| w_{p,l} - s^*\right\|^2 + \left\| s_p - s^*\right\|^2\right) + d_p = \left(1 - a_p \bar{\zeta}\right)^2 \\
&\quad \cdot \left\| w_{p,l} - s^*\right\|^2 - \left(1 - a_p \bar{\zeta}\right) c_p \left\| w_{p,l} - s^*\right\|^2 + \left(1\right. \\
&\quad \left. - a_p \bar{\zeta}\right) c_p) \left\| s_p - s^*\right\|^2 + d_p = \left(1 - a_p \bar{\zeta}\right)\left(1 - c_p\right.
\end{aligned}
$$

$$
\begin{aligned}
&- a_p\bar{\zeta})\left\|w_{p,l}-s^*\right\|^2+\left(1-a_p\bar{\zeta}\right)c_p\left\|s_p-s^*\right\|^2+d_p\\
&\le\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\\
&\cdot\left[\left\|\left(t_{p,l}-\theta_lF_lt_{p,l}\right)-\left(s^*-\theta_lF_ls^*\right)\right\|^2\right]+\left(1-a_p\bar{\zeta}\right)\\
&\cdot c_p\left\|s_p-s^*\right\|^2+d_p\le\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\\
&\cdot\left[\left\|t_{p,l}-s^*\right\|^2+\theta_l\left(\theta_l-2\varsigma_l\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2\right]+\left(1\right.\\
&\left.-a_p\bar{\zeta}\right)c_p\left\|s_p-s^*\right\|^2+d_p\le\left\|s_p-s^*\right\|^2+\theta_l\left(\theta_l\right.\\
&\left.-2\varsigma_l\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2+d_p,
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
d_p&=a_p^2\left\|\zeta g\left(s_p\right)-Gs^*\right\|^2+2c_pa_p\left\langle s_p-s^*,\zeta g\left(s_p\right)\right.\\
&\left.-Gs^*\right\rangle\\
&+2a_p\left\langle\left[\left(1-c_p\right)I-a_pG\right]\left(w_{p,l}-s^*\right),\zeta g\left(s_p\right)\right.\\
&\left.-Gs^*\right\rangle.
\end{aligned}
\tag{43}
$$

From (42), we get

$$
\begin{aligned}
&-\theta_l\left(\theta_l-2\varsigma_l\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2\\
&\le\left\|s_p-s^*\right\|^2-\left\|s_{p+1}-s^*\right\|^2+d_p\\
&\le\left\|s_p-s_{p+1}\right\|\left(\left\|s_p-s^*\right\|+\left\|s_{p+1}-s^*\right\|\right)+d_p.
\end{aligned}
\tag{44}
$$

From condition (i), we get

$$
\lim_{p\longrightarrow\infty}d_p=0.
\tag{45}
$$

From (34), (44), and (45), we get

$$
\lim_{p\longrightarrow\infty}\left\|F_lt_{p,l}-F_ls^*\right\|=0\quad\forall\left\{l=1,2,\ldots,M\right\}.
\tag{46}
$$

Now,

$$
\begin{aligned}
s_p-w_{p,l}&=s_p-s_{p+1}+s_{p+1}-w_{p,l}\\
&=s_p-s_{p+1}+a_p\zeta g\left(s_p\right)+c_ps_p\\
&\quad+\left[\left(1-c_p\right)I-a_pG\right]w_{p,l}-w_{p,l}\\
&=s_p-s_{p+1}+a_p\left(\zeta g\left(s_p\right)-Gw_{p,l}\right)\\
&\quad+c_p\left(s_p-w_{p,l}\right).
\end{aligned}
\tag{47}
$$

Further,

$$
\begin{aligned}
\left(1-c_p\right)\left(s_p-w_{p,l}\right)&=s_p-s_{p+1}\\
&\quad+a_p\left(\zeta g\left(s_p\right)-Gw_{p,l}\right).
\end{aligned}
\tag{48}
$$

So, we get

$$
\begin{aligned}
\left(1-c_p\right)\left\|s_p-w_{p,l}\right\|&\le\left\|s_p-s_{p+1}\right\|\\
&\quad+a_p\left\|\zeta g\left(s_p\right)-Gw_{p,l}\right\|.
\end{aligned}
\tag{49}
$$

From condition (i) and (34), we have

$$
\lim_{p\longrightarrow\infty}\left\|s_p-w_{p,l}\right\|=0.
\tag{50}
$$

Now from (10), we have

$$
\begin{aligned}
&\left\|w_{p,l}-s^*\right\|^2=\left\|P_W\left(t_{p,l}-\theta_lF_lt_{p,l}\right)\right.\\
&\left.-P_W\left(s^*-\theta_lF_ls^*\right)\right\|^2\le\left\langle\left(t_{p,l}-\theta_lF_lt_{p,l}\right)\right.\\
&\left.-\left(s^*-\theta_lF_ls^*\right),w_{p,l}-s^*\right\rangle\\
&=\frac{1}{2}\left\{\left\|\left(t_{p,l}-\theta_lF_lt_{p,l}\right)-\left(s^*-\theta_lF_ls^*\right)\right\|^2\right.\\
&+\left\|w_{p,l}-s^*\right\|^2\\
&\left.-\left\|\left(t_{p,l}-\theta_lF_lt_{p,l}\right)-\left(s^*-\theta_lF_ls^*\right)-\left(w_{p,l}-s^*\right)\right\|^2\right\}\\
&\le\frac{1}{2}\left\{\left\|t_{p,l}-s^*\right\|^2+\left\|w_{p,l}-s^*\right\|^2\right.\\
&\left.-\left\|\left(t_{p,l}-w_{p,l}\right)-\theta_l\left(F_lt_{p,l}-F_ls^*\right)\right\|^2\right\}\\
&=\frac{1}{2}\left\{\left\|t_{p,l}-s^*\right\|^2+\left\|w_{p,l}-s^*\right\|^2-\left\|t_{p,l}-w_{p,l}\right\|^2\right.\\
&+2\theta_l\left\langle t_{p,l}-w_{p,l},F_lt_{p,l}-F_ls^*\right\rangle\\
&\left.-\theta_l^2\left\|F_lt_{p,l}-F_ls^*\right\|^2\right\}\le\left\|t_{p,l}-s^*\right\|^2-\left\|t_{p,l}-w_{p,l}\right\|^2\\
&+2\theta_l\left\langle t_{p,l}-w_{p,l},F_lt_{p,l}-F_ls^*\right\rangle-\theta_l^2\left\|F_lt_{p,l}-F_ls^*\right\|^2.
\end{aligned}
\tag{51}
$$

Further,

$$
\begin{aligned}
&\left\|s_{p+1}-s^*\right\|^2\le\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\left\|w_{p,l}-s^*\right\|^2\\
&+\left(1-a_p\bar{\zeta}\right)c_p\left\|s_p-s^*\right\|^2+d_p\le\left(1-a_p\bar{\zeta}\right)\left(1-c_p\right.\\
&\left.-a_p\bar{\gamma}\right)\left[\left\|t_{p,l}-s^*\right\|^2-\left\|t_{p,l}-w_{p,l}\right\|^2\right.\\
&+2\theta_l\left\langle t_{p,l}-w_{p,l},F_lt_{p,l}-F_ls^*\right\rangle\\
&\left.-\theta_l^2\left\|F_lt_{p,l}-F_ls^*\right\|^2\right]+\left(1-a_p\bar{\zeta}\right)c_p\left\|s_p-s^*\right\|^2\\
&+d_p\le\left(1-a_p\bar{\zeta}\right)\left\|s_p-s^*\right\|^2-\left(1-a_p\bar{\zeta}\right)\left(1-c_p\right.\\
&\left.-a_p\bar{\zeta}\right)\left\|t_{p,l}-w_{p,l}\right\|^2+2\theta_l\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\\
&\cdot\left\|t_{p,l}-w_{p,l}\right\|\left\|F_lt_{p,l}-F_ls^*\right\|-\theta_l^2\left(1-a_p\bar{\zeta}\right)\left(1-c_p\right.\\
&\left.-a_p\bar{\zeta}\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2+d_p.
\end{aligned}
\tag{52}
$$

Further, we have

$$\begin{aligned}
&\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\left\|t_{p,l}-w_{p,l}\right\|^2 \le \left\|s_p-s^*\right\|^2 \\
&\quad -\left\|s_{p+1}-s^*\right\|^2+2\theta_l\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right) \\
&\quad \cdot\left\|t_{p,l}-w_{p,l}\right\|\left\|F_lt_{p,l}-F_ls^*\right\|-\theta_l^2\left(1-a_p\bar{\zeta}\right) \\
&\quad \cdot\left(1-c_p-a_p\bar{\zeta}\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2+d_p \le \left\|s_p-s_{p+1}\right\| \\
&\quad \cdot\left(\left\|s_p-s^*\right\|+\left\|s_{p+1}-s^*\right\|\right)+2\theta_l\left(1-a_p\bar{\zeta}\right) \\
&\quad \cdot\left(1-c_p-a_p\bar{\zeta}\right)\left\|t_{p,l}-w_{p,l}\right\|\left\|F_lt_{p,l}-F_ls^*\right\| \\
&\quad -\theta_l^2\left(1-a_p\bar{\zeta}\right)\left(1-c_p-a_p\bar{\zeta}\right)\left\|F_lt_{p,l}-F_ls^*\right\|^2+d_p.
\end{aligned} \tag{53}$$

Using (34), (45), and (46), we get

$$\lim_{p\longrightarrow\infty}\left\|t_{p,l}-w_{p,l}\right\|=0. \tag{54}$$

From (50) and (54), we get

$$\left\|s_p-t_{p,l}\right\| \le \left\|s_p-w_{p,l}\right\|+\left\|w_{p,l}-t_{p,l}\right\| \longrightarrow 0 \quad \text{as } p\longrightarrow\infty. \tag{55}$$

Since $P_{\mathfrak{F}}(\zeta g+(I-G))$ is a contraction. So, by Banach Contraction Principle, $P_{\mathfrak{F}}(\zeta g+(I-G))$ has a unique fixed point, say $t\in V$ such that $t=P_{\mathfrak{F}}(\zeta g+(I-G))(t)$. Next, we have to show that

$$\lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, w_{p,l}-t\right\rangle \le 0. \tag{56}$$

So, we choose a subsequence $\{w_{p_k,l}\}$ of $\{w_{p,l}\}$ such that

$$\begin{aligned}
&\lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, w_{p,l}-t\right\rangle \\
&\quad =\lim_{k\longrightarrow\infty}\left\langle \zeta g(t)-Gt, w_{p_k,l}-t\right\rangle.
\end{aligned} \tag{57}$$

Since $\{w_{p_k,l}\}$ is bounded, so there exists a subsequence $\{w_{p_{k_m},l}\}$ of $\{w_{p_k,l}\}$ which converges weakly to $q\in W$. Now without loss of generality, we assume that $w_{p_k,l}\rightharpoonup q$. Therefore, we have

$$\begin{aligned}
&\lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, w_{p,l}-t\right\rangle \\
&\quad =\lim_{k\longrightarrow\infty}\left\langle \zeta g(t)-Gt, w_{p_k,l}-t\right\rangle \\
&\quad =\left\langle \zeta g(t)-Gt, q-t\right\rangle.
\end{aligned} \tag{58}$$

Now, we have to show that $q\in\mathfrak{F}=(\bigcap_{l=1}^{M}Fix(\mathcal{S}_l))\cap(\bigcap_{l=1}^{M}VI(F_l,W)))$. First, we show that

$$q\in\bigcap_{l=1}^{M}Fix(\mathcal{S}_l)=Fix(\mathcal{S}). \tag{59}$$

By using Lemma 9 (Demiclosed Principle) we have $q\in Fix(\mathcal{S}_l)$ for each $l=1,2,\ldots,M$, i.e., $q\in\bigcap_{l=1}^{M}Fix(\mathcal{S}_l)=Fix(\mathcal{S})$.

Next, we show that $q\in VI(F_l,W)$. Let T be the maximal monotone mapping defined as

$$Tv=\begin{cases}Fv+N_Wv, & v\in W;\\ \emptyset, & v\notin W.\end{cases} \tag{60}$$

For any given $(v,w)\in H(T)$, we have $w-F_lv\in N_W(v)$.

Since $w_{p,l}\in W$, we have

$$\left\langle v-w_{p,l}, w-F_lv\right\rangle \ge 0. \tag{61}$$

Since $w_{p,l}=P_W(t_{p,l}-\theta_lF_lt_{p,l})$, we have

$$\begin{aligned}
&\left\langle v-w_{p,l}, w_{p,l}-\left(t_{p,l}-\theta_lF_lt_{p,l}\right)\right\rangle \ge 0 \\
&\left\langle v-w_{p,l}, \frac{w_{p,l}-t_{p,l}}{\theta_l}+F_lt_{p,l}\right\rangle \ge 0.
\end{aligned} \tag{62}$$

So, we have

$$\begin{aligned}
\left\langle v-w_{p_k,l}, w\right\rangle &\ge \left\langle v-w_{p_k,l}, F_lv\right\rangle \\
&\ge \left\langle v-w_{p_k,l}, F_lv\right\rangle \\
&\quad -\left\langle v-w_{p_k,l}, \frac{w_{p_k,l}-t_{p_k,l}}{\theta_{l_k}}+F_lt_{p_k,l}\right\rangle \\
&=\left\langle v-w_{p_k,l}, F_lv-F_lt_{p_k,l}-\frac{w_{p_k,l}-t_{p_k,l}}{\theta_{l_k}}\right\rangle \\
&=\left\langle v-w_{p_k,l}, F_lv-F_lw_{p_k,l}\right\rangle \\
&\quad +\left\langle v-w_{p_k,l}, F_lw_{p_k,l}-F_lt_{p_k,l}\right\rangle \\
&\quad -\left\langle v-w_{p_k,l}-\frac{w_{p_k,l}-t_{p_k,l}}{\theta_{l_k}}\right\rangle \\
&\ge \left\langle v-w_{p_k,l}, F_lw_{p_k,l}\right\rangle \\
&\quad -\left\langle v-w_{p_k,l}, F_lt_{p_k,l}+\frac{w_{p_k,l}-t_{p_k,l}}{\theta_{l_k}}\right\rangle \\
&=\left\langle v-w_{p_k,l}, F_lw_{p_k,l}-F_lt_{p_k,l}\right\rangle \\
&\quad -\left\langle v-w_{p_k,l}, \frac{w_{p_k,l}-t_{p_k,l}}{\theta_{l_k}}\right\rangle.
\end{aligned} \tag{63}$$

Since, $\|w_{p_k,l}-t_{p_k,l}\|\longrightarrow 0$ as $p\longrightarrow\infty$ and F_l is Lipschitz continuous, therefore we get

$$\left\langle v-q, w\right\rangle \ge 0. \tag{64}$$

Since T is maximal monotone, we have $q\in T^{-1}0$ and hence $q\in VI(F_l,W)$. Therefore,

$$q\in\mathfrak{F}=\left[\bigcap_{l=1}^{M}Fix(\mathcal{S}_l)\right]\cap\left[\bigcap_{l=1}^{M}VI(F_l,W)\right]. \tag{65}$$

From (58), we get

$$\lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, w_{p,l}-t\right\rangle = \left\langle \zeta g(t)-Gt, q-t\right\rangle \le 0. \tag{66}$$

Since $t = P_{\mathfrak{F}}(\zeta g + (I-G))(t)$, from (50) and (66), we get

$$\begin{aligned}\lim_{p\longrightarrow\infty}\sup&\left\langle \zeta g(t)-Gt, s_p-t\right\rangle \\ &= \lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, \left(s_p-w_{p,l}\right)+\left(w_{p,l}-t\right)\right\rangle \\ &\le \lim_{p\longrightarrow\infty}\sup\left\langle \zeta g(t)-Gt, w_{p,l}-t\right\rangle \le 0.\end{aligned} \tag{67}$$

Next, we show that $s_p \longrightarrow t$. We have,

$$\begin{aligned}\left\|s_{p+1}-t\right\|^2 &= \left\|a_p\zeta g\left(s_p\right)+c_p s_p+\left[\left(1-c_p\right)I-a_pG\right]\right. \\ &\quad\cdot w_{p,l}-t\Big\|^2 = \Big\|a_p\left(\zeta g\left(s_p\right)-Gt\right)+c_p\left(s_p-t\right) \\ &\quad+\left[\left(1-c_p\right)I-a_pG\right]\left(w_{p,l}-t\right)\Big\|^2 = a_p^2\left\|\zeta g\left(s_p\right)\right. \\ &\quad-Gt\Big\|^2+\Big\|\left[\left(1-c_p\right)I-a_pG\right]\left(w_{p,l}-t\right)+c_p\left(s_p\right. \\ &\quad-t\Big)\Big\|^2+2c_pa_p\left\langle s_p-t,\zeta g\left(s_p\right)-Gt\right\rangle \\ &\quad+2a_p\left\langle\left[\left(1-c_p\right)I-a_pG\right)\left(w_{p,l}-t\right),\zeta g\left(s_p\right)\right. \\ &\quad-Gt\Big\rangle\le\left[\left(1-c_p-a_p\bar{\zeta}\right)\left\|w_{p,l}-t\right\|+c_p\left\|s_p-t\right\|\right]^2 \\ &\quad+a_p^2\left\|\zeta g\left(s_p\right)-Gt\right\|^2+2c_pa_p\zeta\left\langle s_p-t,g\left(s_p\right)\right. \\ &\quad-g(t)\Big\rangle+2c_pa_p\left\langle s_p-t,\zeta g(t)-Gt\right\rangle+2\left(1-c_p\right) \\ &\quad\cdot\zeta a_p\left\langle w_{p,l}-t,g\left(s_p\right)-g(t)\right\rangle+2\left(1-c_p\right) \\ &\quad\cdot a_p\left\langle w_{p,l}-t,\zeta g(t)-Gt\right\rangle-2a_p^2\left\langle G\left(w_{p,l}-t\right),\right. \\ &\quad\zeta g(t)-Gt\Big\rangle\le\left[\left(1-c_p-a_p\bar{\zeta}\right)\left\|s_p-t\right\|+c_p\left\|s_p\right.\right. \\ &\quad-t\Big\|\Big]^2+a_p^2\left\|\zeta g\left(s_p\right)-Gt\right\|^2+2c_pa_p\zeta\tau\left\|s_p-t\right\|^2 \\ &\quad+2c_pa_p\left\langle s_p-t,\zeta g(t)-Gt\right\rangle+2\left(1-c_p\right)\zeta a_p\tau\left\|s_p\right. \\ &\quad-t\Big\|^2+2\left(1-c_p\right)a_p\left\langle w_{p,l}-t,\zeta g(t)-Gt\right\rangle \\ &\quad-2a_p^2\left\langle G\left(w_{p,l}-t\right),\zeta g(t)-Gt\right\rangle=\left[\left(1-a_p\bar{\zeta}\right)^2\right. \\ &\quad+2c_pa_p\zeta\tau+2\left(1-c_p\right)\zeta a_p\tau\Big]\left\|s_p-t\right\|^2 \\ &\quad+a_p^2\left\|\zeta g\left(s_p\right)-Gt\right\|^2+2c_pa_p\left\langle s_p-t,\zeta g(t)-Gt\right\rangle \\ &\quad+2\left(1-c_p\right)a_p\left\langle w_{p,l}-t,\zeta g(t)-Gt\right\rangle \\ &\quad-2a_p^2\left\langle G\left(w_{p,l}-t\right),\zeta g(t)-Gt\right\rangle\le\Big[(1-2\left(\bar{\zeta}\right. \\ &\quad-a_p\zeta\Big)a_p\Big]\left\|s_p-t\right\|^2+a_p^2\bar{\zeta}^2\left\|s_p-t\right\|^2+a_p^2\left\|\zeta g\left(s_p\right)\right. \\ &\quad-Gt\Big\|^2+2c_pa_p\left\langle s_p-t,\zeta g(t)-Gt\right\rangle+2\left(1-c_p\right) \\ &\quad\cdot a_p\left\langle w_{p,l}-t,\zeta g(t)-Gt\right\rangle+2a_p^2\left\|G\left(w_{p,l}-t\right)\right\| \\ &\quad\cdot\left\|\zeta g(t)-Gt\right\|=\left[1-2\left(\bar{\zeta}-a_p\zeta\right)a_p\right]\left\|s_p-t\right\|^2 \\ &\quad+a_p\Big\{a_p\Big(\bar{\zeta}^2\left\|s_p-t\right\|^2+\left\|\zeta g\left(s_p\right)-Gt\right\|^2 \\ &\quad+2\left\|G\left(w_{p,l}-t\right)\right\|\left\|\zeta g(t)-Gt\right\|\Big)+2c_p\left\langle s_p\right. \\ &\quad-t,\zeta g(t)-Gt\Big\rangle+2\left(1-c_p\right)\left\langle w_{p,l}-t,\zeta g(t)\right. \\ &\quad-Gt\Big\rangle\Big\}.\end{aligned} \tag{68}$$

Since $\{s_p\}$, $\{g(s_p)\}$, and $w_{p,l}$ are bounded, we have

$$\begin{aligned}&\bar{\zeta}^2\left\|s_p-t\right\|^2+\left\|\zeta g\left(s_p\right)-Gt\right\|^2 \\ &\quad+2\left\|G\left(w_{p,l}-t\right)\right\|\left\|\zeta g(t)-Gt\right\|<M_2,\end{aligned} \tag{69}$$

where $M_2 > 0$. Further, we have

$$\left\|s_{p+1}-t\right\|^2\le\left[1-2\left(\bar{\zeta}-a_p\zeta\right)a_p\right]\left\|s_p-t\right\|^2+a_p\sigma_p, \tag{70}$$

where

$$\begin{aligned}\sigma_p &= 2c_p\left\langle s_p-t,\zeta g(t)-Gt\right\rangle \\ &\quad+2\left(1-c_p\right)\left\langle w_{p,l}-t,\zeta g(t)-Gt\right\rangle+a_pM_3.\end{aligned} \tag{71}$$

Using, condition (i), (66), and (67), we get $\limsup_{p\longrightarrow\infty}c_p \le 0$. Now applying Lemma 8 in (70), we get

$$s_p \longrightarrow t. \tag{72}$$

□

4. Numerical Example

In this section, we discuss the following example which shows the effectiveness and convergence of iterative sequence $\{s_p\}$ considered under Theorem 11.

Let $\mathscr{V} = \mathbb{R}$ and $W = [0,1]$. Let the mappings F_l, G, and g be defined by

$$\begin{aligned}F_l(s) &= 2s, \\ G(s) &= \frac{s}{5}, \\ g(s) &= \frac{s}{7}\end{aligned} \tag{73}$$

for all $s \in W$. Let $\mathcal{S}_l$ be the mapping defined by $\mathcal{S}_l(s) = s/l$ for all $s \in C$.

Define $\{a_p\}$, $\{b_{p,l}\}$, and $\{c_p\}$ in $(0,1)$ by $a_p = 1/2p$, $b_{p,l} = l/p^2$, and $c_p = 1/2p$. It is clear that $\mathcal{S}_l$ is a nonexpansive mapping and F_l is 2-inverse strongly monotone mapping. g is a (1/7)-contraction mapping.

Now we show that the sequences $\{a_p\}$, $\{b_{p,l}\}$, and $\{c_p\}$ satisfy the condition given.

$$\lim_{p\longrightarrow\infty} a_p = \frac{1}{2}\lim_{p\longrightarrow\infty}\frac{1}{p} = 0, \tag{74}$$

and

$$\sum_{p=1}^{\infty} a_p = \frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p} = \infty. \tag{75}$$

The sequence $\{a_p\}$ satisfies condition (i) of Algorithm 10.

Next we compute

$$\begin{aligned} a_p - a_{p-1} &= \frac{1}{2p} - \frac{1}{2(p-1)} = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p-1}\right) \\ &= \frac{-1}{2p(p-1)}. \end{aligned} \tag{76}$$

So,

$$\sum_{p=1}^{\infty}\left|a_p - a_{p-1}\right| < \infty. \tag{77}$$

Similarly, we show that

$$\sum_{p=1}^{\infty}\left|c_p - c_{p-1}\right| < \infty. \tag{78}$$

and

$$\sum_{p=1}^{\infty}\left|b_{p,l} - b_{p-1,l}\right| < \infty. \tag{79}$$

The sequences $\{a_p\}$, $\{b_{p,l}\}$, and $\{c_p\}$ satisfy conditions (i), (ii), and (iii). Clearly,

$$\mathfrak{F} := \left(\bigcap_{l=1}^{2} Fix(\mathcal{S}_l)\right) \cap \left(\bigcap_{l=1}^{2} VI(F_l, W)\right) = \{0\}. \tag{80}$$

The iterative algorithm (22) is written as follows.
When l=1

$$\begin{aligned} t_{p,l} &= s_p; \\ s_{p+1} &= \frac{11 s_p}{35p} + \frac{s_p}{3}. \end{aligned} \tag{81}$$

When l=2

$$\begin{aligned} t_{p,l} &= s_p - \frac{s_p}{p^2}; \\ s_{p+1} &= \frac{11 s_p}{35p} - \frac{3 s_p}{10p^2} + \frac{s_p}{6p^3} + \frac{s_p}{3}. \end{aligned} \tag{82}$$

Remark 12. Table 1 shows that the sequences $\{t_{p,l}\}$ and $\{s_p\}$ converge to 0. Also, $\{0\} \in \mathfrak{F}$.

Table 1: The values of $t_{p,l}$ and s_p with initial values $s_1 = 1$ when l=1,2.

	$l = 1$, $s_1 = 1$		$l = 2$, $s_1 = 1$	
	$t_{p,l}$	s_p	$t_{p,l}$	s_p
n=1	1	1	0.5000	1
n=2	0.6476	0.6476	0.3855	0.5141
n=3	0.3176	0.3176	0.1993	0.2243
n=4	0.1391	0.1391	0.0863	0.0921
n=5	0.0572	0.0572	0.0349	0.0364
n=6	0.0226	0.0226	0.0136	0.0140
n=7	0.0087	0.0087	0.0050	0.0052
n=8	0.0032	0.0032	0.0018	0.0019
n=9	0.0011	0.0011	0.0005	0.0006
n=10	0.0004	0.0004	0.0001	0.0002
n=11	0.0001	0.0001	0.0000	0.0000
n=12	0.0000	0.0000	0.0000	0.0000

5. Application

In this section, by applying Theorem 11 we propose the following results.

Theorem 13. *Let W be a closed convex subset of a real Hilbert space V, let $g : W \longrightarrow W$ be a ζ-contraction, let $F_l : W \longrightarrow V$ be a ς_l-inverse strongly monotone mapping, let $\mathcal{S}_l : W \longrightarrow W$ be a finite family of nonexpansive mappings with $\mathfrak{F} := (\bigcap_{l=1}^{M} Fix(\mathcal{S}_l)) \cap (\bigcap_{l=1}^{M} F_l^{-1}(0)) \neq \emptyset$, and let G be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\zeta} > 0$. Assume that $0 < \zeta \leq \bar{\zeta}/\alpha$ and let $\{\theta_l\}$ be real numbers in $(0, 2\varsigma_l)$. Let $\{s_p\}$ and $\{t_{p,l}\}$ $\forall l = 1, 2, \ldots, M$, be the sequences given by*

$$\begin{aligned} t_{p,1} &= b_{p,1}\mathcal{S}_1 s_p + \left(1 - b_{p,1}\right) s_p \\ t_{p,l} &= b_{p,l}\mathcal{S}_l s_p + \left(1 - b_{p,l}\right) t_{p,l-1}, \quad l = 1, 2, \ldots, M \\ s_{p+1} &= a_p \zeta g\left(s_p\right) + c_p s_p \\ &\quad + \left[\left(1 - c_p\right) I - a_p G\right]\left(t_{p,l} - \theta_l F_l t_{p,l}\right), \\ &\qquad \forall p \geq 1. \end{aligned} \tag{83}$$

where $\{a_p\}$, $\{b_{p,l}\}$, and $\{c_p\}$ are sequences in (0,1) satisfying the following:

(i) $\lim_{p\longrightarrow\infty} a_p = 0$ *and* $\sum_{p=1}^{\infty} a_p = \infty$.

(ii) $\sum_{p=1}^{\infty} |a_p - a_{p-1}| < \infty$ *and* $\sum_{p=1}^{\infty} |c_p - c_{p-1}| < \infty$.

(iii) $\sum_{p=1}^{\infty} |b_{p,l} - b_{p-1,l}| < \infty$ *for each* $l = 1, 2, \ldots, M$.

(iv) $0 < \liminf_{p\longrightarrow\infty} b_{p,l} \leq \limsup_{p\longrightarrow\infty} b_{p,l} < 1$.

Then the sequence $\{s_p\}$ defined by (83) converges strongly to $t \in \mathfrak{F}$, where $t = P_{\mathfrak{F}}(\zeta g + (I - G))t$ which solves the following variational inequality:

$$\langle(\zeta g - G)t, q - t\rangle \leq 0, \quad \forall q \in \mathfrak{F}. \tag{84}$$

Again, we apply Theorem 11., the result to the problem for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings.

Theorem 14. *Let W be a closed convex subset of a real Hilbert space V. For any integer $M > 1$, assume that for each $1 \leq l \leq M$, $\mathcal{T}_l : W \longrightarrow V$ is a k_l-strictly pseudocontractive mapping for some $0 \leq k_l < 1$. Let $g : W \longrightarrow W$ be a ζ-contraction, let $F_l : W \longrightarrow V$ be a ς_l-inverse strongly monotone mapping, let $\mathcal{S}_l : W \longrightarrow W$ be a finite family of nonexpansive mappings with $\mathfrak{F} := (\bigcap_{l=1}^{M} Fix(\mathcal{S}_l)) \cap (\bigcap_{l=1}^{M} F(\mathcal{T}_l)) \neq \emptyset$, and let G be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\zeta} > 0$. Assume that $0 < \zeta \leq \overline{\zeta}/\alpha$. Let $\{s_p\}$ and $\{t_{p,l}\} \forall l = 1, 2, \ldots, M$, be the sequences given by*

$$\begin{aligned} t_{p,1} &= b_{p,1}\mathcal{S}_1 s_p + \left(1 - b_{p,1}\right) s_p \\ t_{p,l} &= b_{p,l}\mathcal{S}_l s_p + \left(1 - b_{p,l}\right) t_{p,l-1}, \quad l = 1, 2, \ldots, M \\ s_{p+1} &= a_p \zeta g\left(s_p\right) + c_p s_p + \left[\left(1 - c_p\right) I - a_p G\right] \\ &\quad \cdot P_W\left(\left(1 - \lambda_p\right) t_{p,l} - \lambda_p \sum_{l=1}^{M} \varphi_l \mathcal{T}_l t_{p,l}\right), \end{aligned} \tag{85}$$

where $\{a_p\}, \{b_{p,l}\}$, and $\{c_p\}$ are sequences in (0,1) satisfying the following:

(i) $\lim_{p \longrightarrow \infty} a_p = 0$ *and* $\sum_{p=1}^{\infty} a_p = \infty$.

(ii) $\sum_{p=1}^{\infty} |a_p - a_{p-1}| < \infty$ *and* $\sum_{p=1}^{\infty} |c_p - c_{p-1}| < \infty$.

(iii) $\sum_{p=1}^{\infty} |b_{p,l} - b_{p-1,l}| < \infty$ *for each* $l = 1, 2, \ldots, M$.

(iv) $0 < \liminf_{p \longrightarrow \infty} b_{p,l} \leq \limsup_{p \longrightarrow \infty} b_{p,l} < 1$.

Then the sequence $\{s_p\}$ defined by (85) converges strongly to $t \in \mathfrak{F}$, where $t = P_{\mathfrak{F}}(\zeta g + (I - G))t$ which solves the following variational inequality:

$$\langle (\zeta g - G) t, q - t \rangle \leq 0, \quad \forall q \in \mathfrak{F}. \tag{86}$$

Proof. Taking $F_l = I - \sum_{l=1}^{M} \varphi_l \mathcal{T}_l : W \longrightarrow \mathcal{V}$, we know that $F_l : W \longrightarrow \mathcal{V}$ is ς-inverse strongly monotone with $\varsigma = (1-k)/2$. Hence, F_l is a monotone L-Lipschitz continuous mapping with $L = 2/(1-k)$. From Lemma 7, we know that $\sum_{l=1}^{M} \varphi_l \mathcal{T}_l$ is a k-strictly pseudocontractive mapping with $k = \max\{k_l : 1 \leq l \leq M\}$ and then $F(\sum_{l=1}^{M} \varphi_l \mathcal{T}_l) = VI(F_l, W)$. So, we have

$$\begin{aligned} &P_W\left(t_{p,l} - \theta_l F_l t_{p,l}\right) \\ &\quad = P_W\left(\left(1 - \lambda_p\right) t_{p,l} - \lambda_p \sum_{l=1}^{M} \varphi_l \mathcal{T}_l t_{p,l}\right). \end{aligned} \tag{87}$$

Using Theorem 11 we get the desired result. □

Remark 15. Theorem 14 is a generalization of the theorems given by Iiduka and Takahashi [6] and Takahashi and Toyoda [5].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors contributed equally and approved the final manuscript.

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