An Iterative Method for Finding Common Solution of the Fixed Point Problem of a Finite Family of Nonexpansive Mappings and a Finite Family of Variational Inequality Problems in Hilbert Space

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In this paper, a hybrid iterative algorithm is proposed for finding a common element of the set of common fixed points of finite family of nonexpansive mappings and the set of common solutions of the variational inequality for an inverse strongly monotone mapping on the real Hilbert space. We establish the strong convergence of the proposed method for approximating a common element of the above defined sets under some suitable conditions. The results presented in this paper extend and improve some well-known corresponding results in the earlier and recent literature.

1. Introduction

Throughout, let \( \mathcal{H} \) be a real Hilbert space whose inner product and norm are denoted as \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively. Let \( W \) be a nonempty closed convex subset of \( \mathcal{H} \) and \( P_W \) be the metric projection of \( \mathcal{H} \) onto \( W \). Let \( \mathcal{S} : W \to W \) be a nonexpansive mapping if
\[
\| \mathcal{S}s - \mathcal{S}t \| \leq \| s - t \| \quad \forall s, t \in W.
\]
The set of fixed point of \( \mathcal{S} \) is denoted by \( \text{Fix} \mathcal{S} \).

Let \( G : W \to \mathcal{H} \) be a nonlinear mapping. The classical variational inequality problem denoted by \( \text{VIP} \) associated with the set \( W \) is to find \( t \in W \) such that
\[
\langle Gt, s - t \rangle \geq 0, \quad \forall s \in W.
\]
The solution set of the \( \text{VIP} \) is denoted by \( \text{VI}(W,G) \). The \( \text{VIP} \) is a fundamental problem in variational analysis which has been extensively studied by many researchers in the past decades, see Yao and Chadli [1], Zeng, Schaible, and Yao [2], and the references therein.

The \( \text{VIP} \) was first discussed by Lions [3], further many different approaches are given to solve (2) in finite dimensional and infinite dimensional spaces, and the research in this direction is still continued.

Korpelevich [4] proposed a modification of an iterative algorithm for solving the \( \text{VIP} \) in Euclidean space \( \mathbb{R}^n \):
\[
t_p = P_W \left( s_p - \frac{\zeta}{2} Gs_p \right),
\]
\[
s_{p+1} = P_W \left( s_p - \frac{\zeta}{2} Gt_p \right), \quad \forall p \geq 0,
\]
with \( \zeta > 0 \), called the extragradient algorithm under certain assumptions, and it has received great attention. He proved that the sequences \( \{s_p\} \) and \( \{t_p\} \) converge strongly.

Takahashi and Toyoda [5] introduced an iterative algorithm for finding an element of \( \text{Fix}(\mathcal{S}) \cap \text{VI}(F,W) \) under the assumption that \( \mathcal{S} \) is nonexpansive and \( F \) is inverse strongly monotone as
\[
s_1 = s \in W \quad \text{chosen arbitrarily}
\]
\[
s_{p+1} = a_p s_p + \left( 1 - a_p \right) \mathcal{S} P_W \left( s_p - \eta_p F s_p \right), \quad p \geq 1
\]
where \( a_p \) is a sequence in \( (0, 1) \) and \( \eta_p \) is a sequence in \( (0, 2a) \).
Qin and Cho [8] introduced an extended composite iterative algorithm \( \{ s_p \} \) defined as follows:

\[
\begin{align*}
    s_1 &= s \in W, \text{ chosen arbitrarily } \\
    z_p &= \xi_p s_p + \left( 1 - \xi_p \right) \delta s_p, \\
    t_p &= b_y s_p + \left( 1 - b_y \right) \delta z_p, \\
    s_{p+1} &= a_y \xi g(s_p) + c_y s_p + \left[ \left( 1 - c_y \right) I - a_y G \right] t_p,
\end{align*}
\]

where \( g \) is a contraction, \( \delta \) is a nonexpansive mapping, and \( G \) is a strongly positive bounded linear self-adjoint operator. Under certain assumptions on the parameters, \( \{ s_p \} \) converges strongly to a fixed point of \( \delta \) provided certain conditions are satisfied.

The hybrid steepest descent method for solving the VIP over the set of fixed points of a nonexpansive mapping was introduced by Mann [9]. In this paper, we introduce and analyze an iterative algorithm by combining Korpelevich's extragradient method, viscosity approximation method, and hybrid steepest method. We prove that under certain conditions the proposed algorithm converges strongly to a common element of \( \{ s_p \} \).

In this paper, we introduce and analyze an iterative algorithm by combining Korpelevich's extragradient method, viscosity approximation method, and hybrid steepest method. We prove that under certain conditions the proposed algorithm converges strongly to a common element of \( \{ s_p \} \).

\section{Preliminaries}

Let \( \mathcal{Y} \) be a real Hilbert space and \( W \) be a nonempty closed convex subset of \( \mathcal{Y} \). For every point \( s \in \mathcal{Y} \), there exists a unique nearest point in \( W \), denoted by \( P_W s \), such that

\[ \| s - P_W s \| \leq \| s - t \|, \quad \forall t \in W. \]  

P \( W \) is called the metric projection of \( \mathcal{Y} \) onto \( W \). It is well known that \( P_W \) is a nonexpansive mapping of \( \mathcal{Y} \) onto \( W \) and satisfies

\[ \langle s - t, P_W s - P_W t \rangle \geq \| P_W s - P_W t \|^2. \]  

for every \( s, t \in \mathcal{Y} \). It is easy to see that

\[ u \in \text{VI}(F, W) \iff u = P_W (u - \lambda Fu), \quad \lambda > 0. \]  

\section*{Definition 1}

A mapping \( A : W \to \mathcal{Y} \) is said to be

(i) monotone, if

\[ \langle A s - A t, s - t \rangle \geq 0, \quad \forall s, t \in W; \]

(ii) \( L \)-Lipschitz, if there exists a constant \( L > 0 \) such that

\[ \| A s - A t \| \leq L \| s - t \|, \quad \forall s, t \in W; \]

(iii) \( \alpha \)-inverse strongly monotone, if there exists a positive real number \( \alpha \) such that

\[ \langle A s - A t, s - t \rangle \geq \alpha \| A s - A t \|^2, \quad \forall s, t \in W. \]

\section*{Remark 2}

Any \( \alpha \)-inverse strongly monotone mapping \( A \) is monotone and \( (1/\alpha) \)-Lipschitz continuous.

\section*{Definition 3}

A mapping \( g : W \to W \) is a contraction if there exists a constant \( \tau \in (0, 1) \) such that

\[ \| g(s) - g(t) \| \leq \tau \| s - t \|, \quad \forall s, t \in W. \]

\section*{Lemma 4}

Assume that \( A \) is a strongly positive linear bounded self-adjoint operator on a Hilbert space \( \mathcal{Y} \) with coefficient \( \gamma > 0 \) and \( \rho \leq \| A \|^{-1} \). Then \( \| I - \rho A \| \leq 1 - \rho \gamma \).

\section*{Definition 5}

A mapping \( \delta : W \to \mathcal{Y} \) is said to be \( k \)-strictly pseudocontractive if there exists \( k \in (0, 1) \) such that

\[ \| \delta s - \delta t \|^2 \leq \| s - t \|^2 + k \| (I - \delta) s - (I - \delta) t \|^2, \quad \forall s, t \in W. \]

Let \( \mathcal{Y} \) be a Hilbert space and let \( W \) be a closed convex subset of \( \mathcal{Y} \). For any integer \( N \geq 1 \), assume that for each \( 1 \leq i \leq N \), \( \delta_i : W \to \mathcal{Y} \) is a \( k_i \)-strictly pseudocontractive mapping for some \( 0 \leq k_i < 1 \). Assume that \( \{ \delta_i \}_{i=1}^N \) is a positive sequence such that \( \sum_{i=1}^N \delta_i = 1 \). Then \( \delta_i \) is a \( k \)-strictly pseudocontractive mapping with \( k = \max\{k_i : 1 \leq i \leq N \} \).

\section*{Lemma 7}

Assume that \( \{ \delta_i \}_{i=1}^N \) is a sequence of nonnegative real numbers such that

\[ a_{n+1} = (1 - \delta_n) a_n + \delta_n, \]

where \( \{ \delta_n \} \) is a sequence in \( (0, 1) \) and \( \delta_n \) is a sequence such that

(i) \( \sum_{n=1}^\infty \delta_n = \infty \);

(ii) \( \limsup_{n \to \infty} (\delta_n/\delta_n) \leq 0 \) or \( \sum_{n=1}^\infty |\delta_n| < \infty \).

Then, \( \lim_{n \to \infty} a_n = 0 \).
Lemma 9 (see [13] demiclosed principle). Let $W$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{V}$. Let $\delta$ be a nonexpansive self-mapping on $W$ with $\text{Fix}(\delta) \neq \emptyset$. Then $I - \delta$ is demiclosed. That is, whenever $(s_n)$ is a sequence in $\mathcal{V}$ weakly converging to some $s \in \mathcal{V}$ and the sequence $(1 - \delta)s_n$ strongly converges to some $t$, it follows that $(I - \delta)s = t$. Here $I$ is the identity operator of $\mathcal{V}$.

Rockafellar [14] defined set-valued mapping $T : \mathcal{V} \to 2^\mathcal{V}$ to be called monotone if for all $s, t \in \mathcal{V}$, $f \in T_s$ and $g \in T_t$ imply

$$\langle s - t, f - g \rangle \geq 0.$$  

(18)

A set-valued mapping $T : \mathcal{V} \to 2^\mathcal{V}$ is maximal monotone if the graph of $H(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(s, f) \in \mathcal{V} \times \mathcal{V}$, $\langle s - t, f - g \rangle \geq 0$ for every $(t, g) \in H(T)$ implies $f \in T_s$. Let $F : W \to \mathcal{V}$ be a monotone mapping and let $N_W v$ be the normal cone to $W$ at $v \in W$, that is

$$N_W v = \{w \in \mathcal{V} : \langle u - v, w \rangle \geq 0, \forall u \in W\}.$$  

(19)

Define

$$T v = \begin{cases} F v + N_W v, & v \in W, \\ 0, & v \notin W. \end{cases}$$  

(20)

Then $T$ is the maximal monotone and

$$0 \in T v \quad \text{if and only if} \quad v \in VI(F, W).$$  

(21)

3. Main Results

Algorithm 10. Let $W$ be a closed convex subset of a real Hilbert space $\mathcal{V}$, let $g : W \to W$ be a $\xi$-contraction with coefficient $\xi \in [0, 1]$, $G$ be strongly positive linear bounded self-adjoint operator, $F_1 : W \to \mathcal{V}$ be a $\epsilon_i$-inverse strongly monotone mappings for each $1 \leq i \leq M$, where $M$ is some positive integer, and $\delta_1 : W \to W$ be a finite family of nonexpansive mappings for all $1 \leq i \leq M$. Let $(b_{p, l})$ for all $1 \leq l \leq M, (\{a_p\})$ and $\{\epsilon_p\}$ be the sequences in $[0, 1]$. For an arbitrarily given $s_1 = s \in W$, we propose the following hybrid iterative algorithm:

$$t_{p, 1} = b_{p, 1} \delta_1 s_1 + (1 - b_{p, 1}) s_p,$$

$$t_{p, l} = b_{p, l} \delta_1 s_{p, l - 1} + (1 - b_{p, l}) t_{p, l - 1}, \quad l = 1, 2, \ldots, M$$

$$s_{p, 1} = a_p g(s_p) + \epsilon_p s_p$$

$$+ \left(1 - \epsilon_p\right) I - a_p G P_W \left(t_{p, l} - \theta_l F_1 t_{p, l}\right),$$

$$\forall p \geq 1.$$  

(22)

The sequence $\{s_p\}$ defined by (22) converges strongly to a common element of the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of the variational inequalities for an inverse strongly monotone mapping which solves the variational inequality problem.

Theorem 12. Let $W$ be a closed convex subset of a real Hilbert space $\mathcal{V}$, let $g : W \to W$ be a $\xi$-contraction, let $F_1 : W \to \mathcal{V}$ be a $\epsilon_i$-inverse strongly monotone mappings, let $\delta_1 : W \to W$ be a finite family of nonexpansive mappings with $\mathcal{F}_l = (\bigcap_{i=1}^{M} \text{Fix}(\delta_1)) \cap (\bigcap_{i=1}^{M} V(I,F_1,W)) \neq \emptyset$, and let $G$ be a strongly positive linear bounded self-adjoint operator with the coefficient $\epsilon > 0$. Assume that $0 < \epsilon \leq \epsilon_0 < \infty$ and let $\{\epsilon_i\}$ be real numbers in $(0, \epsilon_0)$. Let $\{a_p\}, \{p_p\}$ and $\{\epsilon_p\}$ be sequences in $(0, 1)$ satisfying the following assumptions:

$$\xi \geq 0, \forall q \in \mathcal{F}_l.$$  

(23)

Proof. For any $s, t \in W$ and $\epsilon_i \in (0, \epsilon_0)$, we have

$$\|\left(I - \theta_l F_1\right) s - \left(I - \theta_l F_1\right) t\|^2$$

$$= \|(s - t) - \theta_l (F_1 s - F_1 t)\|^2$$

$$= \text{sup} \left\{\langle F_1 s - F_1 t, \epsilon \rangle : \epsilon \in [0, 1] \right\}^2$$

$$= \|s - t\|^2 - 2\theta_l \langle F_1 s - F_1 t, s - t \rangle + \theta_l^2 \|F_1 s - F_1 t\|^2$$

$$\leq \|s - t\|^2 - \theta_l (2\epsilon - \epsilon_0) \|F_1 s - F_1 t\|^2 \leq \|s - t\|^2.$$  

(24)

From (24), $(I - \theta_l F_1)$ is nonexpansive. Since $G$ is a linear bounded self-adjoint operator, so

$$\|G\| = \text{sup} \{|\langle G s, s \rangle| : s \in V, \|s\| = 1\}.$$  

(25)

We assume that

$$\|G\| a_p < 1 - c_p \quad \forall p \geq 0,$$  

(26)

and since $\lim_{p \to \infty} a_p = 0$, so without loss of generality, we have

$$\left\{\left(1 - c_p\right) I - a_p G, s, s\right\} = 1 - c_p - a_p \langle G s, s \rangle$$

$$\leq 1 - c_p - a_p \|G\| \leq 0,$$  

(27)

and

$$\left\{\left(1 - c_p\right) I - a_p G, s, s\right\} \leq 1 - c_p - a_p \|G\| \leq 0,$$  

(28)

so $(1 - c_p)I - a_p G$ is positive. Now, it follows that

$$\left\{\left(1 - c_p\right) I - a_p G, s, s\right\} \leq 1 - c_p - a_p \|G\| \leq 0,$$  

(28)
Next, we prove that the sequence \( \{ s_p \} \) is bounded. Fixing \( s^* \in \mathbb{R}_+ \), we have
\[
\| t_{p,1} - s^* \| = \| b_{p,1} \delta_1 s_p + (1 - b_{p,1}) s_p - s^* \|
\leq b_{p,1} \| s_p - s^* \| + (1 - b_{p,1}) \| s_p - s^* \|
\leq \| s_p - s^* \|. \tag{29}
\]
By induction from \( l = 2 \) to \( l = M \), we have
\[
\| t_{p,l} - s^* \| = \| b_{p,l} \delta_l s_p + (1 - b_{p,l}) t_{p,l-1} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p,l-1} - s^* \|
\leq \| s_p - s^* \|. \tag{30}
\]
Thus, for every \( l = 1, 2, \ldots, M \), we get
\[
\| t_{p,l} - s^* \| \leq \| s_p - s^* \|. \tag{31}
\]
From (22) and (31), we have
\[
\| s_{p+1} - s^* \| = \| a_p \zeta g(s_p) + c_p s_p + \left[ (1 - c_p) t_{p} - \theta_p s_p \right] P_W \left( t_{p,l} - \theta_p t_{p,l} \right) - s^* \|
\leq \| a_p \zeta g(s_p) - Gs^* \| + c_p \| s_p - s^* \| + (1 - c_p) \| t_{p,l} - s^* \|
+ a_p \| \zeta g(s^*) - Gs^* \| + c_p \| s_p - s^* \| + (1 - c_p) \| t_{p,l} - s^* \|
\leq \| a_p \zeta g(s_p) - Gs^* \| + c_p \| s_p - s^* \| + (1 - c_p) \| t_{p,l} - s^* \|
- Gs^* + c_p \| s_p - s^* \| + (1 - c_p - a_p) \| s_p - s^* \|
\leq (1 - a_p) \| \zeta - \zeta \| \| s_p - s^* \| + a_p \| \zeta g(s^*) - Gs^* \|
\leq \frac{a_p}{\zeta - \zeta} \| \zeta g(s^*) - Gs^* \|
\leq \max \left\{ \| s_p - s^* \|, \frac{1}{\zeta - \zeta} \| \zeta g(s^*) - Gs^* \| \right\}. \tag{32}
\]
By induction on \( p \), we have
\[
\| s_p - s^* \| \leq \max \left\{ \| s_0 - s^* \|, \frac{1}{\zeta - \zeta} \| \zeta g(s^*) - Gs^* \| \right\}, \tag{33}
\]
\[ \forall p \geq 0. \]

Hence, the given sequence \( \{ s_p \} \) is bounded and so is \( \{ t_{p,l} \} \) for all \( l = 1, 2, \ldots, M \).

Next, we show that
\[
\lim_{p \to \infty} \| s_{p+1} - s_p \| = 0. \tag{34}
\]
From the definition of the sequence \( \{ t_{p,l} \} \) for each \( l = 1, 2, \ldots, M \) in Algorithm 10, we have
\[
\| t_{p,l} - t_{p-1,l} \| = \| b_{p,l} \delta_l s_p + (1 - b_{p,l}) t_{p-1,l} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p-1,l} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p-1,l} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p-1,l} - s^* \| \tag{35}
\]
When \( l = 1 \), we have
\[
\| t_{p,1} - t_{p-1,1} \| \leq b_{p,1} \| s_p - s^* \| + (1 - b_{p,1}) \| s_p - s^* \|
- b_{p-1,1} \| s_{p-1} - s^* \| + (1 - b_{p-1,1}) \| s_{p-1} - s^* \|
\leq b_{p,1} \| s_p - s^* \| + (1 - b_{p,1}) \| s_p - s^* \| + b_{p-1,1} \| s_{p-1} - s^* \| \tag{36}
\]
From (35) and (36), for \( l = 2, \ldots, M \), we get
\[
\| t_{p,l} - t_{p-1,l} \| \leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| s_p - s^* \|
+ \sum_{i=2}^{l} \| \delta_i s_{p-1} - t_{p-1,l-1} \| \| b_{p,l} - b_{p-1,l} \| \tag{37}
\]
Next,
\[
\| s_{p+1} - s_p \| = \| a_p \zeta g(s_p) + c_p s_p + \left[ (1 - c_p) t_{p} - \theta_p s_p \right] P_W \left( t_{p,l} - \theta_p t_{p,l} \right) - s^* \|
\leq \| a_p \zeta g(s_p) - Gs^* \| + c_p \| s_p - s^* \| + (1 - c_p) \| t_{p,l} - s^* \|
\leq \| a_p \zeta g(s_p) - Gs^* \| + (1 - c_p) \| t_{p,l} - s^* \|
\leq \| a_p \zeta g(s_p) - Gs^* \| + (1 - c_p) \| t_{p,l} - s^* \| \tag{38}
\]
Hence, the given sequence \( \{ s_p \} \) is bounded and so is \( \{ t_{p,l} \} \) for all \( l = 1, 2, \ldots, M \).

Next, we show that
\[
\lim_{p \to \infty} \| s_{p+1} - s_p \| = 0. \tag{34}
\]
From the definition of the sequence \( \{ t_{p,l} \} \) for each \( l = 1, 2, \ldots, M \) in Algorithm 10, we have
\[
\| t_{p,l} - t_{p-1,l} \| = \| b_{p,l} \delta_l s_p + (1 - b_{p,l}) t_{p-1,l} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p-1,l} - s^* \|
\leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| t_{p-1,l} - s^* \| \tag{35}
\]
When \( l = 1 \), we have
\[
\| t_{p,1} - t_{p-1,1} \| \leq b_{p,1} \| s_p - s^* \| + (1 - b_{p,1}) \| s_p - s^* \|
- b_{p-1,1} \| s_{p-1} - s^* \| + (1 - b_{p-1,1}) \| s_{p-1} - s^* \|
\leq b_{p,1} \| s_p - s^* \| + (1 - b_{p,1}) \| s_p - s^* \| + b_{p-1,1} \| s_{p-1} - s^* \| \tag{36}
\]
From (35) and (36), for \( l = 2, \ldots, M \), we get
\[
\| t_{p,l} - t_{p-1,l} \| \leq b_{p,l} \| s_p - s^* \| + (1 - b_{p,l}) \| s_p - s^* \|
+ \sum_{i=2}^{l} \| \delta_i s_{p-1} - t_{p-1,l-1} \| \| b_{p,l} - b_{p-1,l} \| \tag{37}
\]
Next,
\[
\| s_{p+1} - s_p \| = \| a_p \zeta g(s_p) + c_p s_p + \left[ (1 - c_p) t_{p} - \theta_p s_p \right] P_W \left( t_{p,l} - \theta_p t_{p,l} \right) - s^* \|
\leq a_p \| \zeta g(s_p) - Gs^* \| + c_p \| s_p - s^* \| + (1 - c_p) \| t_{p,l} - s^* \|
\leq a_p \| \zeta g(s_p) - Gs^* \| + (1 - c_p) \| t_{p,l} - s^* \| \tag{38}
\]
Hence, the given sequence \( \{ s_p \} \) is bounded and so is \( \{ t_{p,l} \} \) for all \( l = 1, 2, \ldots, M \).

Next, we show that
\[
\lim_{p \to \infty} \| s_{p+1} - s_p \| = 0. \tag{34}
\]
From conditions (i), (ii), and (iii), we have

$$s_{p+1} = a_p \zeta g(s_p) + c_p s_p + [(1 - c_p) I - a_p G] w_{pp}.$$  \hspace{1cm} (41)$$

Next for \( s^* \in \mathcal{F} \), we have

$$\|s_{p+1} - s^*\|^2 = \|a_p \zeta g(s_p) + c_p s_p$$

\[ + [(1 - c_p) I - a_p G] w_{pp} - s^*\|^2 \]

\[ = \|a_p \zeta g(s_p) - G s^*\|^2 + c_p (s_p - s^*)^2 \]

\[ + [(1 - c_p) I - a_p G] (w_{pp} - s^*)^2 = a_p^2 \|\zeta g(s_p) - G s^*\|^2 \]

\[ + c_p (s_p - s^*)^2 \zeta g(s_p) - G s^*\|^2 \]

\[ + c_p (s_p - s^*)^2 \zeta g(s_p) - G s^*\|^2 \]

\[ + [(1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]

\[ + 2a_p \zeta g(s_p) - G s^*\|^2 \]

\[ + (1 - c_p) I - a_p G] (w_{pp} - s^*)^2 + 2a_p c_p (s_p - s^*) \zeta g(s_p) - G s^*\|^2 \]
\[-a_p \xi \| w_{p,1} - s^* \|^2 + (1 - a_p \xi) c_p \| s_p - s^* \|^2 + d_p \leq (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
\cdot \left[ \| t_{p,1} - \theta_t F_t t_{p,1} \| - (s^* - \theta_t F_t s^*) \right]^2 \right] + (1 - a_p \xi) \]
\[
\cdot c_p \| s_p - s^* \|^2 + d_p \leq (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
\cdot \left[ \| t_{p,1} - s^* \|^2 + \theta_t (\theta_t - 2 \xi) \| F_t t_{p,1} - F_t s^* \| \right]^2 + (1 - a_p \xi) \]
\[
- 2 \xi c_p \| s_p - s^* \|^2 + d_p \leq \| s_p - s^* \|^2 + 2 \xi (\theta_t - 2 \xi) + 2 \xi \| w_{p,1} - s^* \|^2 + d_p, \tag{42} \]

where
\[
d_p = a_p^2 \| \xi g (s_p) - G s^* \|^2 + 2 c_p a_p \langle s_p - s^*, \xi g (s_p) - G s^* \rangle \]
\[
+ 2 \xi a_p \left( (1 - c_p) I - a_p G \right) (w_{p,1} - s^*) \xi g (s_p) - G s^* \rangle. \tag{43} \]

From (42), we get
\[
- \theta_t (\theta_t - 2 \xi) \| F_t t_{p,1} - F_t s^* \|^2 \]
\[
\leq \| s_p - s^* \|^2 - \| s_{p+1} - s^* \|^2 + d_p \leq \| s_p - s_{p+1} \| \left( \| s_p - s^* \| + \| s_{p+1} - s^* \| \right) + d_p. \tag{44} \]

From condition (i), we get
\[
\lim_{p \to \infty} d_p = 0. \tag{45} \]

From (34), (44), and (45), we get
\[
\lim_{p \to \infty} \| F_t t_{p,1} - F_t s^* \|^2 = 0 \quad \forall \{ l = 1, 2, \ldots, M \}. \tag{46} \]

Now,
\[
s_p - w_{p,1} = s_p - s_{p+1} + s_{p+1} - w_{p,1} \]
\[
= s_p - s_{p+1} + a_p \xi g (s_p) + c_p s_p \]
\[
+ \left( (1 - c_p) I - a_p G \right) w_{p,1} - w_{p,1} \]
\[
= s_p - s_{p+1} + a_p \xi g (s_p) - G w_{p,1} \]
\[
+ c_p \xi g (s_p) - G w_{p,1}. \tag{47} \]

Further,
\[
(1 - c_p) (s_p - w_{p,1}) = s_p - s_{p+1} + a_p \xi g (s_p) - G w_{p,1}. \tag{48} \]

So, we get
\[
(1 - c_p) \| s_p - w_{p,1} \| \leq \| s_p - s_{p+1} \| + a_p \xi g (s_p) - G w_{p,1}. \tag{49} \]

From condition (i) and (34), we have
\[
\lim_{p \to \infty} \| s_p - w_{p,1} \| = 0. \tag{50} \]

Now from (10), we have
\[
\| w_{p,1} - s^* \|^2 = \| P_W (t_{p,1} - \theta_t F_t t_{p,1}) \|
\]
\[
- P_W (s^* - \theta_t F_t s^*) \| \leq \langle t_{p,1} - \theta_t F_t t_{p,1}, s^* - \theta_t F_t s^* \rangle \]
\[
= \frac{1}{2} \left\{ \| t_{p,1} - \theta_t F_t t_{p,1} \| \right. \} \]
\[
\| s^* - \theta_t F_t s^* \| + \underbrace{\| w_{p,1} - s^* \|^2}_{= \| t_{p,1} - s^* \|^2} - \| t_{p,1} - w_{p,1} \| \]
\[
\leq \frac{1}{2} \left\{ \| t_{p,1} - s^* \|^2 + \| w_{p,1} - s^* \|^2 \]
\[
- \| t_{p,1} - w_{p,1} \| \right\} \leq \| t_{p,1} - s^* \|^2 - \| t_{p,1} - w_{p,1} \|^2 \]
\[
+ 2 \theta_t \langle t_{p,1} - w_{p,1}, F_t t_{p,1} - F_t s^* \rangle - \theta_t^2 \| F_t t_{p,1} - F_t s^* \|^2. \tag{51} \]

Further,
\[
\| s_{p+1} - s^* \|^2 \leq (1 - a_p \xi) (1 - c_p - a_p \xi) \| w_{p,1} - s^* \|^2 \]
\[
+ (1 - a_p \xi) c_p \| s_p - s^* \|^2 + d_p \leq (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
- a_p \xi \left[ \| t_{p,1} - s^* \|^2 - \| t_{p,1} - w_{p,1} \|^2 \right] \]
\[
+ 2 \theta_t \langle t_{p,1} - w_{p,1}, F_t t_{p,1} - F_t s^* \rangle \]
\[
- \theta_t^2 \| F_t t_{p,1} - F_t s^* \|^2 \] + (1 - a_p \xi) \| s_p - s^* \|^2 \]
\[
+ d_p \leq (1 - a_p \xi) \| s_p - s^* \|^2 - (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
- a_p \xi \| t_{p,1} - s^* \|^2 + 2 \theta_t (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
+ 2 \theta_t \| t_{p,1} - w_{p,1} \|^2 \]
\[
- \theta_t^2 (1 - a_p \xi) (1 - c_p - a_p \xi) \]
\[
- a_p \xi \| F_t t_{p,1} - F_t s^* \|^2 + d_p. \tag{52} \]
Further, we have
\[
\begin{align*}
(1-a_p \tilde{\xi}) (1-c_p - a_p \tilde{\xi}) \| t_{p_j} - w_{p_j} \|^2 & \leq \| s_p - s^* \|^2 \\
- \| s_{p+1} - s^* \|^2 + 2 \theta (1-a_p \tilde{\xi}) (1-c_p - a_p \tilde{\xi}) & - \| t_{p_j} - w_{p_j} \| F_t t_{p_j} - F(s^*) - \theta \| (1-a_p \tilde{\xi}) \\
\cdot (1-c_p - a_p \tilde{\xi}) \| F_t t_{p_j} - F(s^*) - \theta \| (1-a_p \tilde{\xi}) & + \| s_{p+1} - s^* \|^2 + d_p \leq \| s_p - s_{p+1} \| \\
\cdot (1-c_p - a_p \tilde{\xi}) \| F_t t_{p_j} - F(s^*) - \theta \| (1-a_p \tilde{\xi}) & + \| s_{p+1} - s^* \|^2 + 2 \theta (1-a_p \tilde{\xi}) \\
\cdot \left( (1-c_p - a_p \tilde{\xi}) (1-c_p - a_p \tilde{\xi}) F_t t_{p_j} - F(s^*) \right)^2 & + d_p.
\end{align*}
\]
Using (34), (45), and (46), we get
\[
\lim_{p \to \infty} \| t_{p_j} - w_{p_j} \| = 0. \tag{54}
\]
From (50) and (54), we get
\[
\| s_p - t_{p_j} \| \leq \| s_p - w_{p_j} \| + \| w_{p_j} - t_{p_j} \| \to 0 \tag{55}
\]
as \(p \to \infty\).

Since \(P_R(\zeta g + (I - G))\) is a contraction. So, by Banach Contraction Principle, \(P_R(\zeta g + (I - G))\) has a unique fixed point, say \(t \in V\) such that \(t = P_R(\zeta g + (I - G))(t)\). Next, we have to show that
\[
\lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, w_{p_j} - t \rangle \leq 0. \tag{56}
\]
So, we choose a subsequence \(\{w_{p_k}\}\) of \(\{w_{p_j}\}\) such that
\[
\lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, w_{p_j} - t \rangle = \lim_{k \to \infty} \langle \zeta g(t) - Gt, w_{p_k} - t \rangle. \tag{57}
\]
Since \(\{w_{p_j}\}\) is bounded, so there exists a subsequence \(\{w_{p_{k_l}}\}\) of \(\{w_{p_j}\}\) which converges weakly to \(q \in W\). Now without loss of generality, we assume that \(w_{p_{k_l}} \to q\). Therefore, we have
\[
\lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, w_{p_j} - t \rangle = \lim_{k \to \infty} \langle \zeta g(t) - Gt, w_{p_{k_l}} - t \rangle \tag{58}
\]
Now, we have to show that \(q \in F = \bigcap_{l=1}^M \text{Fix}(\delta_l) \cap (\bigcap_{l=1}^M \text{VI}(F_l, W))\). First, we show that \(q \in \bigcap_{l=1}^M \text{Fix}(\delta_l) = \text{Fix}(\delta)\).

Next, we show that \(q \in \text{VI}(F_l, W)\). Let \(T\) be the maximal monotone mapping defined as
\[
T_{v} = \begin{cases} 
F_{v} + N_{W_{v}}, & v \in W; \\
\emptyset, & v \notin W.
\end{cases} \tag{60}
\]
For any given \((v, w) \in H(T)\), we have \(w - F_{v} \in \text{N}_{W_{v}}(v)\).

Since \(w_{p_j} \in W\), we have
\[
\langle v - w_{p_j}, w - F_{v} \rangle \geq 0. \tag{61}
\]
Since \(w_{p_j} = P_{W}(t_{p_j} - \theta F_{t_{p_j}})\), we have
\[
\begin{align*}
\langle v - w_{p_j}, w_{p_j} - (t_{p_j} - \theta F_{t_{p_j}}) \rangle & \geq 0 \\
\langle v - w_{p_j}, w_{p_j} - t_{p_j} \rangle + F_{t_{p_j}} & \geq 0.
\end{align*} \tag{62}
\]
So, we have
\[
\langle v - w_{p_{k_l}}, w \rangle \geq \langle v - w_{p_{k_l}}, F_{v} \rangle \geq \langle v - w_{p_{k_l}}, \theta F_{t_{p_j}} \rangle \tag{63}
\]
\[
\langle v - w_{p_{k_l}}, F_{v} - F_{t_{p_j}} \rangle \geq 0.
\]
Since \(\|w_{p_j} - t_{p_j}\| \to 0\) as \(p \to \infty\) and \(F_{t}\) is Lipschitz continuous, therefore we get
\[
\langle v - q, w \rangle \geq 0. \tag{64}
\]
Since \(T\) is maximal monotone, we have \(q \in T^{-1} 0\) and hence \(q \in \text{VI}(F_{l}, W)\). Therefore,
\[
q \in F = \bigcap_{l=1}^M \text{Fix}(\delta_l) \cap (\bigcap_{l=1}^M \text{VI}(F_l, W)). \tag{65}
\]
From (58), we get
\[
\lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, w_{p,t} - t \rangle = \langle \zeta g(t) - Gt, q - t \rangle \leq 0. \tag{66}
\]
Since \( t = P_G(\zeta g + (I - G))(t) \), from (50) and (66), we get
\[
\lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, s_p - t \rangle = \lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, (s_p - w_{p,t}) + (w_{p,t} - t) \rangle \leq \lim_{p \to \infty} \sup \langle \zeta g(t) - Gt, w_{p,t} - t \rangle \leq 0. \tag{67}
\]
Next, we show that \( s_p \to t \). We have,
\[
\| s_{p+1} - t \|^2 = \| a_p \zeta g(s_p) + c_p s_p + [(1 - c_p) I - a_p G] w_{p,t} - t \|^2 = \| a_p \zeta g(s_p) + c_p s_p - t \|^2 + \| (1 - c_p) I - a_p G \| (w_{p,t} - t) \|^2 + c_p (s_p - t) - t \|^2 + 2 c_p a_p \langle s_p - t, \zeta g(s_p) - Gt \rangle + 2 (1 - c_p) \langle w_{p,t} - t, \zeta g(t) - Gt \rangle \tag{68}
\]
Since \( \{ s_p \}, \{ g(s_p) \} \), and \( w_{p,t} \) are bounded, we have
\[
\| s_{p+1} - t \|^2 \leq \| (1 - c_p) I - a_p G \| (w_{p,t} - t) \|^2 + 2 (1 - c_p) \| \zeta g(t) - Gt \| + 2 \langle w_{p,t} - t, \zeta g(t) - Gt \rangle \tag{69}
\]
where \( M_2 > 0 \). Further, we have
\[
\| s_{p+1} - t \|^2 \leq \| (1 - c_p) I - a_p G \| (w_{p,t} - t) \|^2 + 2 (1 - c_p) \| g(t) - Gt \| + 2 \langle w_{p,t} - t, \zeta g(t) - Gt \rangle \tag{70}
\]
where
\[
\sigma_p = 2c_p \langle s_p - t, \zeta g(t) - Gt \rangle \tag{71}
\]
Using, condition \((i),(66),(67)\), we get \( \lim_{p \to \infty} \langle s_p - t, \zeta g(t) - Gt \rangle = 0 \). Now applying Lemma 8 in (70), we get
\[
s_p \to t. \tag{72}
\]

4. Numerical Example

In this section, we discuss the following example which shows the effectiveness and convergence of iterative sequence \( \{ s_p \} \) considered under Theorem 2.

Let \( F = \mathbb{R} \) and \( W = [0,1] \). Let the mappings \( F_1, G \), and \( g \) be defined by
\[
F_1(s) = 2s, \tag{73}
\]
\[
G(s) = \frac{s}{5}, \tag{74}
\]
\[
g(s) = \frac{s}{7}
\]
for all \( s \in W \). Let \( \delta_1 \) be the mapping defined by \( \delta_1(s) = s/l \) for all \( s \in C \).
Define \( \{a_p\}, \{b_p\}, \) and \( \{c_p\} \) in (0, 1) by \( a_p = 1/2p, b_p = 1/p^2, \) and \( c_p = 1/2p. \) It is clear that \( \delta_{ij} \) is a nonexpansive mapping and \( F_j \) is 2-inverse strongly monotone mapping.g is a \((1/7)\)-contraction mapping.

Now we show that the sequences \( \{a_p\}, \{b_p\}, \) and \( \{c_p\} \) satisfy the condition given.

\[
\lim_{p \to \infty} a_p = \frac{1}{2} \lim_{p \to \infty} \frac{1}{p} = 0, \tag{74}
\]

and

\[
\sum_{p=1}^{\infty} a_p = \frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} = \infty. \tag{75}
\]

The sequence \( \{a_p\} \) satisfies condition (i) of Algorithm 10.

Next we compute

\[
a_p - a_{p-1} = \frac{1}{2p} - \frac{1}{2(p-1)} = \frac{1}{2} \left( \frac{1}{p} - \frac{1}{p-1} \right), \tag{76}
\]

So,

\[
\sum_{p=1}^{\infty} \left| a_p - a_{p-1} \right| < \infty. \tag{77}
\]

Similarly, we show that

\[
\sum_{p=1}^{\infty} \left| c_p - c_{p-1} \right| < \infty. \tag{78}
\]

and

\[
\sum_{p=1}^{\infty} \left| b_p - b_{p-1} \right| < \infty. \tag{79}
\]

The sequences \( \{a_p\}, \{b_p\}, \) and \( \{c_p\} \) satisfy conditions (i), (ii), and (iii). Clearly,

\[
\mathcal{F} = \left( \bigcap_{l=1}^{M} \text{Fix}(\delta_l) \right) \cap \left( \bigcap_{l=1}^{M} \text{VI}(F_p, W) \right) = \{0\}. \tag{80}
\]

The iterative algorithm (22) is written as follows. When \( l=1 \)

\[
t_{p,1} = s_p; \tag{81}
\]

\[
s_{p+1} = \frac{11s_p}{35p} + \frac{s_p}{3}.
\]

When \( l=2 \)

\[
t_{p,1} = s_p - \frac{s_p}{p^2}; \tag{82}
\]

\[
s_{p+1} = \frac{11s_p}{35p} - \frac{3s_p}{10p^2} + \frac{s_p}{6p^3} + \frac{s_p}{3}.
\]

Remark 12. Table 1 shows that the sequences \( \{t_{p,l}\} \) and \( \{s_p\} \) converge to 0. Also, \( \{0\} \in \mathcal{F}. \)

<table>
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<tr>
<th>( l )</th>
<th>( t_{p,l} )</th>
<th>( s_p )</th>
<th>( t_{p,l} )</th>
<th>( s_p )</th>
</tr>
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<td>1</td>
<td>0.5000</td>
<td>1</td>
</tr>
<tr>
<td>( n=2 )</td>
<td>0.6476</td>
<td>0.6476</td>
<td>0.3855</td>
<td>0.5341</td>
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<tr>
<td>( n=3 )</td>
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<td>0.3176</td>
<td>0.1993</td>
<td>0.2243</td>
</tr>
<tr>
<td>( n=4 )</td>
<td>0.1391</td>
<td>0.1391</td>
<td>0.0863</td>
<td>0.0921</td>
</tr>
<tr>
<td>( n=5 )</td>
<td>0.0572</td>
<td>0.0572</td>
<td>0.0349</td>
<td>0.0364</td>
</tr>
<tr>
<td>( n=6 )</td>
<td>0.0226</td>
<td>0.0226</td>
<td>0.0136</td>
<td>0.0140</td>
</tr>
<tr>
<td>( n=7 )</td>
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<td>0.0087</td>
<td>0.0050</td>
<td>0.0052</td>
</tr>
<tr>
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<td>0.0018</td>
<td>0.0019</td>
</tr>
<tr>
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<td>0.0011</td>
<td>0.0005</td>
<td>0.0006</td>
</tr>
<tr>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( n=12 )</td>
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<td>0.0000</td>
<td>0.0000</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( l = 1 )</th>
<th>( s_{p,1} )</th>
<th>( s_{p,2} )</th>
<th>( s_{p,3} )</th>
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5. Application

In this section, by applying Theorem 11 we propose the following results.

**Theorem 13.** Let \( W \) be a closed convex subset of a real Hilbert space \( V, \) let \( g : W \to W \) be a \( \zeta \)-contraction, let \( F_l : W \to V \) be a \( g_l \)-inverse strongly monotone mapping, \( l \geq 1 \). Let \( \mathcal{F} = \{0\} \) and let \( \theta_l \) be real numbers in \((0, 2\zeta)\). Let \( \{s_p\} \) and \( \{t_{p,l}\} \) \( \forall l = 1, 2, \ldots, M, \) be the sequences given by

\[
t_{p,1} = b_{p,1} \delta_{l=1} s_p + (1 - b_{p,1}) s_p, \tag{83}
\]

\[
t_{p,l} = b_{p,l} \delta_{l=1} s_p + (1 - b_{p,l}) t_{p,l-1}, \quad l = 1, 2, \ldots, M, \tag{83a}
\]

\[
s_{p+1} = a_p \zeta g(s_p) + c_p p, \tag{83b}
\]

\[+ \left[ (1 - c_p) I - a_p G \right] (t_{p,l} - \theta_l F_l t_{p,l}), \tag{83c}
\]

\[\forall p \geq 1.\]

where \( \{a_p\}, \{b_p\}, \) and \( \{c_p\} \) are sequences in \((0,1)\) satisfying the following:

(i) \( \lim_{p \to \infty} a_p = 0 \) and \( \sum_{p=1}^{\infty} a_p = \infty. \)

(ii) \( \sum_{p=1}^{\infty} |a_p - a_{p-1}| < \infty \) and \( \sum_{p=1}^{\infty} |c_p - c_{p-1}| < \infty. \)

(iii) \( \sum_{p=1}^{\infty} |b_p - b_{p-1}| < \infty \) for each \( l = 1, 2, \ldots, M. \)

(iv) \( 0 < \lim \inf_{p \to \infty} b_p \leq \lim \sup_{p \to \infty} b_p < 1. \)

Then the sequence \( \{s_p\} \) defined by (83) converges strongly to \( t \in \mathcal{F}, \) where \( t = P_\mathcal{F} (\zeta g + (I - G)t) \) which solves the following variational inequality:

\[
\langle (\zeta g - G) t, q - t \rangle \leq 0, \quad \forall q \in \mathcal{F}. \tag{84}
\]
Again, we apply Theorem II., the result to the problem for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings.

**Theorem 14.** Let W be a closed convex subset of a real Hilbert space V. For any integer M > 1, assume that for each 1 ≤ l ≤ M, \( \mathcal{S}_l : W \rightarrow V \) is a k_l-strictly pseudocontractive mapping for some 0 ≤ k_l < 1. Let \( g : W \rightarrow W \) be a ζ-contraction, let \( F_l : W \rightarrow V \) be a \( \zeta_l \)-inverse strongly monotone mapping, let \( \mathcal{S}_l : W \rightarrow W \) be a finite family of nonexpansive mappings with \( \mathcal{F} = (\bigcap_{l=1}^{M} \text{Fix}(\mathcal{S}_l)) \cap (\bigcap_{l=1}^{M} \text{Fix}(\mathcal{F}_l)) \neq \emptyset \), and let \( G \) be a strongly positive linear bounded self-adjoint operator with the coefficient \( \zeta > 0 \). Assume that 0 < ζ ≤ \( \zeta_l^0 \). Let \{s_p\} and \{t_p\}\( \forall l = 1, 2, \ldots, M \) be the sequences given by

\[
\begin{align*}
    t_{p,l} &= b_{p,1} s_{p} + \left( 1 - b_{p,1} \right) s_{p} \\
    t_{p,l} &= b_{p,1} s_{p} + \left( 1 - b_{p,1} \right) t_{p,l-1}, \quad l = 1, 2, \ldots, M \\
    s_{p,l} &= a_p \zeta g (s_p) + c_p s_p + \left( 1 - c_p \right) I - a_p G \\
        &+ M \sum_{l=1}^{M} \mathcal{S}_l t_{p,l} \quad (85)
\end{align*}
\]

where \{a_p\}, \{b_p\}, and \{c_p\} are sequences in (0,1) satisfying the following:

(i) \( \lim_{p \to \infty} a_p = 0 \) and \( \sum_{p=1}^{\infty} a_p = \infty \).

(ii) \( \sum_{p=1}^{\infty} |a_p - a_{p-1}| < \infty \) and \( \sum_{p=1}^{\infty} |c_p - c_{p-1}| < \infty \).

(iii) \( \sum_{p=1}^{\infty} |b_p - b_{p-1}| < \infty \) for each \( l = 1, 2, \ldots, M \).

(iv) \( 0 < \liminf_{p \to \infty} b_{p,l} \leq \limsup_{p \to \infty} b_{p,l} < 1 \).

Then the sequence \{s_p\} defined by (85) converges strongly to \( t \in \mathcal{F} \), where \( t = P_{\mathcal{F}}(G + (I - G))t \) which solves the following variational inequality:

\[
\langle (G - G) t, q - t \rangle \leq 0, \quad \forall q \in \mathcal{F}. \quad (86)
\]

**Proof.** Taking \( F_l = I - \sum_{i=1}^{M} \mathcal{S}_i \mathcal{F}_i : W \rightarrow \mathcal{Y} \), we know that \( F_l : W \rightarrow \mathcal{Y} \) is \( \zeta \)-inverse strongly monotone with \( \zeta = (1-k)/2 \). Hence, \( F_l \) is a monotone L-Lipschitz continuous mapping with \( L = 2/(1-k) \). From Lemma 7, we know that \( \sum_{i=1}^{M} \mathcal{S}_i \mathcal{F}_i \) is a k-strictly pseudocontractive mapping with \( k = \max\{k_l : 1 \leq l \leq M \} \) and then \( F(\sum_{i=1}^{M} \mathcal{S}_i \mathcal{F}_i) = V(I(F_l, W) \).

So, we have

\[
\begin{align*}
    P_{\mathcal{F}}(t_{p,l} - \zeta_l F_l t_{p,l}) &= P_{\mathcal{F}}\left( 1 - \lambda_p \right) t_{p,l} - \lambda_p \sum_{l=1}^{M} \mathcal{S}_l t_{p,l} \quad (87)
\end{align*}
\]

Using Theorem II we get the desired result. \( \square \)

**Remark 15.** Theorem 14 is a generalization of the theorems given by Iiduka and Takahashi [6] and Takahashi and Toyoda [5].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

Both authors contributed equally and approved the final manuscript.

**References**


