

## Research Article

# On Eternal Domination of Generalized $J_{s,m}$

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An eternal dominating set of a graph  $G$  is a set of guards distributed on the vertices of a dominating set so that each vertex can be occupied by one guard only. These guards can defend any infinite series of attacks, an attack is defended by moving one guard along an edge from its position to the attacked vertex. We consider the “all guards move” of the eternal dominating set problem, in which one guard has to move to the attacked vertex, and all the remaining guards are allowed to move to an adjacent vertex or stay in their current positions after each attack in order to form a dominating set on the graph and at each step can be moved after each attack. The “all guards move model” is called the  $m$ -eternal domination model. The size of the smallest  $m$ -eternal dominating set is called the  $m$ -eternal domination number and is denoted by  $\gamma_m^{\infty}(G)$ . In this paper, we find the domination number of Jahangir graph  $J_{s,m}$  for  $s \equiv 1, 2 \pmod{3}$ , and the  $m$ -eternal domination numbers of  $J_{s,m}$  for  $s, m$  are arbitrary.

## 1. Introduction

The term graph protection refers to the process of placing guards or mobile agents in order to defend against a sequence of attacks on a network. Go to [1–5] for more background of the graph protection problem. In 2004, Burger et al. [2] introduced the concept of eternal domination. Goddard et al. [3] introduced the “all guards move model” and determined general bounds of  $\gamma(G) \leq \gamma_m^{\infty}(G) \leq \alpha(G)$ , where  $\alpha(G)$  denotes the independence number of  $G$  and  $\gamma(G)$  denotes the domination number of  $G$ . The  $m$ -eternal domination numbers for cycles  $C_n$  and paths  $P_n$  were found by Goddard et al. [3] as follows:  $\gamma_m^{\infty}(C_n) = \lceil n/3 \rceil$  and  $\gamma_m^{\infty}(P_n) = \lceil n/2 \rceil$ . For further information on eternal domination, see survey [1]. The  $k$ -dominating graph  $H(G, k)$  was defined by Goldwasser, Klostermeyer, and Mynhardt in [6] as follows: let  $G$  be a graph with a dominating set of cardinality  $k$ . The vertex set of the  $k$ -dominating graph  $H(G, k)$ , denoted  $V(H)$ , is the set of all subsets of  $V(G)$  of the size  $k$  which are dominating sets, and two vertices of  $H$  are adjacent if and only if the  $k$  guards occupying the vertices of  $G$  of one can move (at most distance one each) to the vertices of the other,  $\gamma_m^{\infty}(G) \leq k$  if and only if  $H(G, k)$  has an induced subgraph  $S(G, k)$  such that for each vertex  $x$  of  $S(G, k)$ , the union of the vertices in the closed neighborhood of  $x$  in  $S(G, k)$  is equal to  $V(G)$ .

A generalized Jahangir graph  $J_{s,m}$  for  $m \geq 2$  is a graph on  $sm + 1$  vertices, i.e., a graph consisting of a cycle  $C_{sm}$  with one additional vertex which is adjacent to  $m$  vertices of  $C_{sm}$  at the distance  $s$  from each other on  $C_{sm}$ , see [7] for more information on the Jahangir graph. Let  $v_{sm+1}$  be the label of the central vertex and  $v_1, v_2, \dots, v_{sm}$  be the labels of the vertices that incident clockwise on cycle  $C_{sm}$  so that  $\deg(v_1) = 3$ . We will use this labeling for the rest of the paper. The vertices that are adjacent to  $v_{sm+1}$  have the labels  $v_1, v_{1+s}, v_{1+2s}, \dots, v_{1+(m-1)s}$ . We denote the set  $\{v_1, v_{1+s}, \dots, v_{1+(m-1)s}\}$  by  $R$ . So,  $R = \{v_{1+is} : i = 0, 1, \dots, m-1\}$ .

Mtarneh, Hasni, Akhbari, and Movahedi claim to have found the domination number of  $J_{s,m}$  in [8] as follows:

**Theorem 1** [8]. For  $s, m \geq 2, \gamma(J_{s,m}) = \lceil sm/3 \rceil$ .

However, in this paper, we will prove that this theorem does not hold when  $s \equiv 1, 2 \pmod{3}$ .

**Theorem 2** [3].

$$\gamma_m^{\infty}(C_n) = \left\lceil \frac{n}{3} \right\rceil. \quad (1)$$

In a previous paper, we found that

**Theorem 3** [9].

$$\gamma_m^\infty(J_{2,m}) = \begin{cases} 3::m = 3, \\ \gamma(J_{2,m}): m \in \{2, 4, 5, 6, 7, 9\}, \\ \left\lceil \frac{m}{2} \right\rceil + 2 \text{ for } m \geq 8 \text{ and } m \neq 9. \end{cases} \quad (2)$$

**Theorem 4** [9]. For  $m \geq 2$ ,  $\gamma_m^\infty(J_{3,m}) = m + 1$ .

**2. Main Results**

In this section, we prove that Theorem 1 does not hold if  $s \equiv 1, 2 \pmod{3}$ , and we find  $\gamma(J_{s,m})$  in these cases; then, we find  $\gamma_m^\infty(J_{s,m})$  for  $s \equiv 0, 1 \pmod{3}$  and arbitrary  $m$ . We also give an upper and a lower bound for the  $m$ -eternal domination number of  $J_{s,m}$  for  $s \equiv 2 \pmod{3}$  and arbitrary  $m$ .

**Proposition 5.** The  $sm/3$  – dominating set of  $J_{s,m}$  is unique for  $s \equiv 0 \pmod{3}$ .

*Proof.* Let  $D_0$  be a dominating set of  $J_{s,m}$  with cardinality  $sm/3$ . Since  $s \equiv 0 \pmod{3}$ , then  $sm \equiv 0 \pmod{3}$ ; therefore,  $\lceil sm/3 \rceil = sm/3$ . We imply that the vertices of  $R$  that are dominated by  $v_{sm+1}$  separate the cycle  $C_{sm}$  into  $m$  paths each of which has  $s - 1$  vertices and needs  $\lceil s - 1/3 \rceil$  vertices to dominate its vertices. If  $v_{sm+1} \in D_0$ , then we will need at least  $m \lceil s - 1/3 \rceil = sm/3$  vertices to dominate the  $m$  non-dominated paths of  $C_{sm}$  which means we will be needing  $(sm/3) + 1$  vertices, and that is a contradiction. This means  $D_0 \subset V(C_{sm})$ . We have three sets of cardinality  $sm/3$  that can dominate the cycle  $C_{sm}$  when  $sm \equiv 0 \pmod{3}$ . They are

$$\begin{aligned} S_1 &= \{v_1, v_4, \dots, v_{sm-2}\} = \{v_{1+3i}\} \text{ for } i = 0, 1, \dots, \frac{sm}{3} - 1, \\ S_2 &= \{v_2, v_5, \dots, v_{sm-1}\} = \{v_{2+3i}\} \text{ for } i = 0, 1, \dots, \frac{sm}{3} - 1, \\ S_3 &= \{v_3, v_6, \dots, v_{sm}\} = \{v_{3+3i}\} \text{ for } i = 0, 1, \dots, \frac{sm}{3} - 1. \end{aligned} \quad (3)$$

However,  $D_0$  must contain at least one vertex adjacent to vertex  $v_{sm+1}$  to dominate it. Since  $s \equiv 0 \pmod{3}$ , then  $s = 3k : k = 1, 2, \dots, (s/3)$ . Since  $s = 3k$ , then we can obviously see that  $R = \{v_{1+3ik} : i = 0, 1, \dots, m - 1 \text{ and } k = 1, 2, \dots, m - 1\}$ . However, considering  $ik > i$  for  $k > 1$ , then it is obvious that the set  $R \subset S_1$ , therefore  $S_1 \cap R = R, S_2 \cap R = \emptyset$ , and  $S_3 \cap R = \emptyset$ . Which means only  $S_1$  dominates both the cycle  $C_{sm}$  and the vertex  $v_{sm+1}$  at the same time. Therefore,  $D_0 = S_1$ , and this set is unique.

**Theorem 6.**  $\gamma(J_{s,m}) = m \lfloor s/3 \rfloor + 1$  if  $s \equiv 1 \pmod{3}$ .

*Proof.* Let  $S$  be a set of cardinality  $m(s - 1/3) + 1$ . Let  $v_{sm+1} \in S$ . We imply that the vertices of  $R$  that are dominated by  $v_{sm+1}$  separate the cycle  $C_{sm}$  into  $m$  paths each of which has  $s - 1$  vertices and needs  $\lceil s - 1/3 \rceil$  vertices to dominate its vertices. We denote these paths by  $T_i : i = 1, \dots, m$ . Since  $s \equiv 1 \pmod{3}$ ,

then  $s - 1 \equiv 0 \pmod{3}$ ; therefore,  $m \lceil s - 1/3 \rceil = m(s - 1/3)$  and  $|T_i| \equiv 0 \pmod{3}$ . Let  $S_i : i = 1, \dots, m$  be the family of dominating sets of paths  $T_i : i = 1, \dots, m$ , respectively. We know that for an arbitrary path  $T_k$ , the dominating set is defined as  $S_k = \{v_{s(k-1)+3}, v_{s(k-1)+6}, \dots, v_{ks-1}\}$ , which means  $S = \bigcup_{i=1}^m S_i \cup \{v_{sm+1}\}$  is a dominating set of  $J_{s,m}$  with cardinality  $m(s - 1/3) + 1$ . It is known that the  $k/3$  – dominating set of a path  $P_k : k \equiv 0 \pmod{3}$  is unique; therefore,  $S$  is unique because each of  $S_i : i = 1, \dots, m$  is unique. Let us prove that whatever set  $B$  which with cardinality  $m(s - 1/3)$  is not enough to dominate  $J_{s,m}$ . We consider the following cases, as illustrated in Figure 1 for  $J_{7,4}$ .

*Case 1.*  $v_{sm+1} \in B$ .

In this case, we only have  $m(s - 1/3) - 1$  vertices to dominate  $T_i : i = 1, \dots, m$  which is impossible because it leaves one path  $T_k$  with  $(s - 1/3) - 1$  dominating vertices, which means three vertices of  $T_k$  will not be dominated, and that is a contradiction.

*Case 2.*  $v_{sm+1} \notin B$ .

In this case,  $\subset V(C_{sm})$ . We know that  $\gamma(C_{sm}) = \lceil sm/3 \rceil$ . However,  $\lceil sm/3 \rceil$  is one of the following:

$$\left\lceil \frac{sm}{3} \right\rceil = \begin{cases} \frac{sm}{3} ; sm \equiv 0 \pmod{3}, \\ \frac{sm + 2}{3} ; sm \equiv 1 \pmod{3}, \\ \frac{sm + 1}{3} ; sm \equiv 2 \pmod{3}. \end{cases} \quad (4)$$

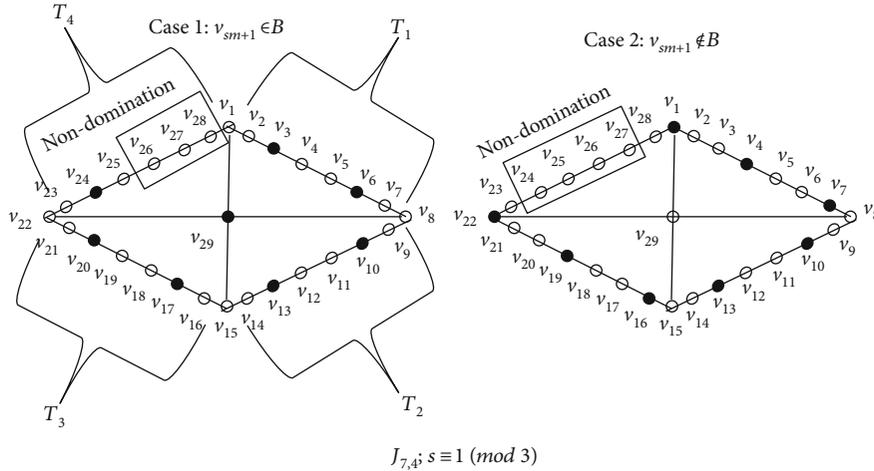
It is obvious that  $\lceil sm/3 \rceil > m(s - 1/3)$  for all cases of  $sm$ , which means  $B$  cannot dominate all the vertices of  $C_{sm}$ . Therefore,  $B$  cannot dominate  $J_{s,m}$  entirely. From Case 1 and Case 2, we imply that  $\gamma(J_{s,m}) > m(s - 1/3)$  for  $s \equiv 1 \pmod{3}$ . Hence,  $m(s - 1/3) < \gamma(J_{s,m}) \leq m(s - 1/3) + 1$  which means  $\gamma(J_{s,m}) = m(s - 1/3) + 1$ . However,  $(s - 1/3) = \lfloor s/3 \rfloor$  for  $s \equiv 1 \pmod{3}$ ; therefore,  $\gamma(J_{s,m}) = m \lfloor s/3 \rfloor + 1$  for  $s \equiv 1 \pmod{3}$ .

**Theorem 7.**  $\gamma(J_{s,m}) = \lceil sm - \lfloor m/2 \rfloor / 3 \rceil + 1$  if  $s \equiv 2 \pmod{3}$ .

*Proof.* We know that a set  $D_0$  of  $\lceil sm/3 \rceil$  vertices can dominate  $C_{sm}$ . Let us identify  $D_0$  as  $D_0 = A \cup \{v_{sm}\} = \{v_{3k} : 1 \leq k \leq sm - 2/3\} \cup \{v_{sm}\}$ . Since  $v_{1+s} = v_{1+3l+2} = v_{3(1+l)} \in D_0$ , therefore  $D_0$  dominates  $v_{sm+1}$  which means it dominates  $J_{s,m}$  entirely. However, let us consider another dominating set  $D_1$  of cardinality  $\lceil sm/3 \rceil + 1$ , where  $D_1 = D_0 \cup \{v_{sm+1}\}$ . We consider the two following cases:

*Case 1.*  $m$  is even.

Let us prove that we can derive a dominating set  $D_2$  of cardinality  $\lceil sm - m/2/3 \rceil + 1$  from  $D_1$ . Since  $v_{sm+1} \in D_1$ , it is obvious that  $v_{sm+1}$  dominates all the vertices of  $R$ ; therefore,  $v_{sm+1}$  dominates  $v_1, v_{1+s}, v_{1+2s}$ . Let us consider the first two paths  $T_1, T_2$  identified in Theorem 6. We notice that  $D_1 \cap$



In both cases  $m \lfloor s/3 \rfloor = 8$  vertices are not enough to dominate  $J_{7,4}$

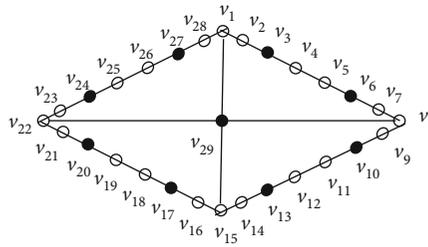


FIGURE 1:  $\gamma(J_{7,4}) = m \lfloor s/3 \rfloor + 1 = 9$ .

$V(T_1) = \{v_3, v_6, \dots, v_{s-2}\}$ , while  $D_1 \cap V(T_2) = \{v_{s+4}, v_{s+7}, \dots, v_{2s-1}\}$  and  $v_{1+s} \in D_1$ . We need  $v_{1+s}$  to dominate  $v_s$  and  $v_{2+s}$  so we include it in our new set  $D_2$ . However, since  $v_{2s}, v_{1+2s}$  are dominated by  $v_{2s-1}, v_{sm+1}$ , respectively, then we do not need to add  $v_{2+2s}$  to  $D_2$ ; therefore, we can start  $D_2 \cap V(T_3)$  with  $v_{3+2s}$  and recreate the same distribution of dominating sets we used on the unit  $\{T_1, v_{1+s}, T_2\}$ . This distribution spares one vertex ( $v_{1+2s}$ ) from the need of being protected by a dominating vertex from  $C_{sm}$ . Without loss of generality, we can use this distribution for every two consecutive paths of  $C_{sm}$ , since we have  $m$  paths, and  $m$  is even, and this distribution spares  $m/2$  vertices from the need of being dominated by a dominating vertex from  $C_{sm}$ ; therefore,  $|D_2| = \lceil sm - m/2/3 \rceil + 1$ .

Case 2.  $m$  is odd.

We use the same discussion in Case 1 on paths  $(T_1, \dots, T_{m-1})$ , but since we have one additional path  $(T_m)$ , we will have one less vertex to spare from the need of being dominated by a dominating vertex from  $C_{sm}$ ; therefore,  $|D_2| = \lceil sm - m - 1/2/3 \rceil + 1$  in this case.

From Cases 1 and 2, we imply that  $\lceil sm - \lfloor m/2 \rfloor / 3 \rceil + 1$  vertex can dominate  $J_{s,m}$  if  $s \equiv 2 \pmod{3}$ ; therefore,

$$\gamma(J_{s,m}) \leq \left\lceil \frac{sm - \lfloor m/2 \rfloor}{3} \right\rceil + 1 \text{ if } s \equiv 2 \pmod{3}. \quad (5)$$

Let  $X$  be a set of cardinality  $\lceil sm - \lfloor m/2 \rfloor / 3 \rceil$ , and we consider the following cases:

Case 1.  $v_{sm+1} \in X$ .

In this case, there will be at least two nondominated vertices of  $C_{sm}$  even after applying the strategy we showed earlier.

Case 2.  $v_{sm+1} \notin X$ .

In this case,  $X \cap C_{sm} = X$ , and since  $|X| = \lceil sm - \lfloor m/2 \rfloor / 3 \rceil$  and  $\gamma(C_{sm}) = \lceil sm/3 \rceil$ , we have a contradiction.

We conclude from both Case 1 and Case 2 that

$$\gamma(J_{s,m}) \geq \left\lceil \frac{sm - \lfloor m/2 \rfloor}{3} \right\rceil + 1 \text{ if } s \equiv 2 \pmod{3}. \quad (6)$$

From (5) and (6), we conclude that  $\gamma(J_{s,m}) = \lceil sm - \lfloor m/2 \rfloor / 3 \rceil + 1$  if  $s \equiv 2 \pmod{3}$ .

**Theorem 8.**  $\gamma_m^\infty(J_{s,m}) \leq \lceil sm/3 \rceil + 1$  for

$$rs \geq 3, m \geq 2. \quad (7)$$

*Proof.* From Theorem 2, we have  $\gamma_m^\infty(C_{sm}) = \lceil sm/3 \rceil$ . With the right distribution of the required  $\lceil sm/3 \rceil$  guards on the vertices of the outer cycle  $C_{sm}$  and placing one additional

guard on the central vertex  $v_{sm+1}$ , this additional guard would not have to move to defend any attack on the vertices of  $C_{sm}$  since  $\lceil sm/3 \rceil$  guards are enough to eternally dominate the vertices of  $C_{sm}$ . The vertex  $v_{sm+1}$  cannot be attacked because it is occupied which means  $\gamma_m^{\infty}(J_{s,m}) \leq \lceil sm/3 \rceil + 1$ .

**Theorem 9.**  $\gamma_m^{\infty}(J_{s,m}) = (sm/3) + 1$  if  $s \equiv 0 \pmod{3}$ .

*Proof.* From Theorem 1, we know that  $\gamma(J_{s,m}) = (sm/3)$  for  $s \equiv 0 \pmod{3}$ . We imply that Theorem 7 that  $S_0 = \{v_1, v_4, \dots, v_{sm-2}\}$  is the unique  $sm/3$  – dominating set of  $J_{s,m}$ . We consider an attack on an arbitrary vertex  $v_2 \in V(C_{sm}) - S_0$ . The only guard that can move to  $v_2$  to defend the attack is the guard located on  $v_1$ . We can only consider the two following strategies:

Strategy 1: we move the guard from  $v_1$  to  $v_2$  without moving any other guard and that would leave  $v_{sm}$  unprotected.

Strategy 2: we make a full rotation (clockwise) from the vertices of set  $S_0$  to the vertices of the only set that contains  $v_2$  and can dominate  $C_{sm}$  as well. This required set is the set  $S_2 = \{v_2, v_5, \dots, v_{sm-1}\} = \{v_{2+3i} : i = 0, 1, \dots, sm/3 - 1\}$ . But from Theorem 7, we imply that  $S_2 \cap R = \emptyset$ ; therefore, the central vertex  $v_{sm+1}$  will not be protected anymore. Without loss of generality, both strategies fail for any vertex  $v_i \in V(C_{sm}) - S_0$ , and we conclude that  $\gamma_m^{\infty}(J_{s,m}) > (sm/3)$  for  $s \equiv 0 \pmod{3}$ . From Theorem 8, we know the result that  $\gamma_m^{\infty}(J_{s,m}) \leq (sm/3) + 1$  for  $s \equiv 0 \pmod{3}$ . Then,  $\gamma_m^{\infty}(J_{s,m}) = (sm/3) + 1$  for  $s \equiv 0 \pmod{3}$ , see Figures 2 and 3.

**Lemma 10.** For any path,  $P_k : k \equiv 0 \pmod{3}$  with the  $k/3$  – dominating set  $S = \{v_2, v_5, \dots, v_{k-1}\}$  and for any vertex  $v_i \in S$ , if  $v_{i-1}, v_{i+1}$  is attacked, then there is a way to move the guards to keep all vertices of  $P_k$  protected with the exception of  $v_k, v_1$ , respectively.

*Proof.* It is obvious that any path  $P_k : k \equiv 0 \pmod{3}$  can be divided into  $k/3$  subsets each of which contains three vertices. To dominate each subset, we need only one guard located on the middle vertex, so each vertex  $v_i \in S$  dominates itself,  $v_{i-1}$ ,  $v_{i+1}$ .

If  $v_{i-1}$  is attacked, then the guard on  $v_i$  moves to  $v_{i-1}$  which would leave  $v_{i+1}$  unprotected. However, the guard on  $v_{i+3}$  moves to  $v_{i+2}$ , and so all the guards need to move one step to the left which means the guard on  $v_{k-1}$  moves to  $v_{k-2}$  leaving  $v_k$  unprotected.

If  $v_{i+1}$  is attacked, then the guard on  $v_i$  moves to  $v_{i+1}$  which would leave  $v_{i-1}$  unprotected. However, the guard on  $v_{i-3}$  moves to  $v_{i-2}$ , and so all the guards need to move one step to the right which means the guard on  $v_2$  moves to  $v_3$  leaving  $v_1$  unprotected.

**Theorem 11.**  $\gamma_m^{\infty}(J_{s,m}) = m\lfloor s/3 \rfloor + 2$  for  $s \equiv 1 \pmod{3}$ .

*Proof.* Let  $J_{s,m}$  be a Jahangir graph with  $s \equiv 1 \pmod{3}$ , and let  $S$  be the unique dominating set of cardinality  $m\lfloor s/3 \rfloor + 1$  for  $J_{s,m}$  that was identified in Theorem 9. We position a guard on each vertex of  $S$ . Since  $S$  is unique, then it will fail at defending  $J_{s,m}$  against the first attack. We study a series of

arbitrary attacks on  $J_{s,m}$  denoted by  $(A_1, A_2, \dots, A_i, \dots)$ , and we assume that  $A_1$  occurs on  $v_l \in R$ ; then, the only guard capable of defending the attack is located on  $v_{sm+1}$  but moving this guard to  $v_l$  would leave the remaining vertices of  $R$  unprotected. And since we cannot bring any other guard to  $v_{sm+1}$  to protect them which means  $\gamma_m^{\infty}(J_{s,m}) \neq m\lfloor s/3 \rfloor + 1$ , but from the definition of Jahangir graph and also from Theorem 8, we can easily find that  $\gamma_m^{\infty}(J_{s,m}) \geq m\lfloor s/3 \rfloor + 1$  which means

$$\gamma_m^{\infty}(J_{s,m}) > m\left\lfloor \frac{s}{3} \right\rfloor + 1. \quad (8)$$

We add a guard on an arbitrary vertex  $y \in R$  so the new set of guards consisting of  $m\lfloor s/3 \rfloor + 2$  guards is  $S \cup \{y\}$ . We will denote this distribution of guards by  $M_0$ .

We notice that the vertices of  $R$  partition  $J_{s,m}$  into  $m$  subgraph of  $s + 2$  vertices so that  $v_{sm+1}$  belongs to every subgraph. We denote these subgraphs by  $SJ_i : i = 1, \dots, m$ . We have

$$\begin{aligned} SJ_1 &= \{v_1, \dots, v_{s+1}, v_{sm+1}\}, \\ SJ_2 &= \{v_{s+1}, \dots, v_{2s+1}, v_{sm+1}\}, \\ SJ_m &= \{v_{(m-1)s+1}, \dots, v_{sm}, v_1, v_{sm+1}\}. \end{aligned} \quad (9)$$

We assume  $A_1$  occurs on a vertex  $v_i \in SJ_1$ , and that  $y \notin SJ_1$ ; then, the guard set on  $SJ_1$  is  $\{v_3, v_6, \dots, v_{s-1}\}$ , and we consider the following cases:

*Case I.*  $v_{i-1}$  is occupied.

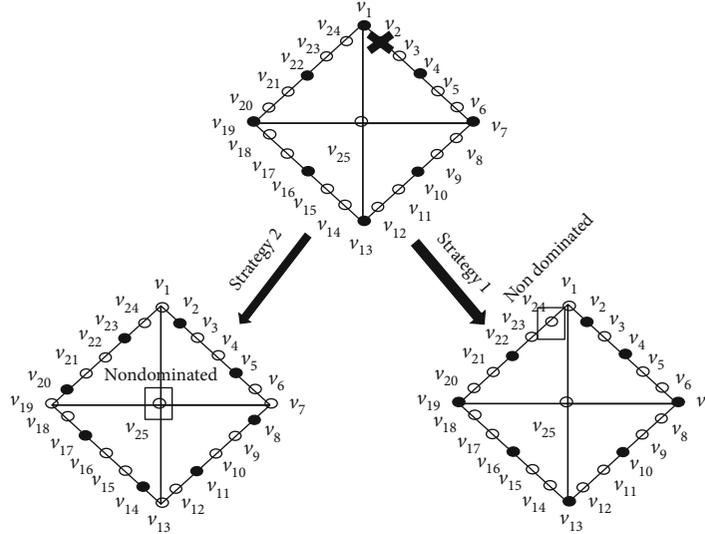
In this case, the only guard capable of moving to  $v_i$  is located on  $v_{i-1}$  which leaves  $v_{i-2}$  unprotected and in order to protect it, the guard on  $v_{i-4}$  moves to  $v_{i-3}$ , as in Lemma 10 which all the guards located on vertices of  $T_1$  move one step towards the higher index which leaves the end vertex with the lowest index  $v_2$  unprotected, and the guard on  $v_{sm+1}$  is the only one capable of moving directly to  $v_1$  to protect  $v_2$ . This is possible because of  $v_1 \in R$ , and this movement will make  $R - \{v_1, y\}$  unprotected. To avoid that, we move the guard on  $y$  to  $v_{sm+1}$  to protect all the vertices of  $R$ . So, the new set of guards located on  $SJ_1$  is

$$W_1 = \{v_1, v_4, v_7, \dots, v_{s-3}, v_s, v_{sm+1}\}. \quad (10)$$

We assume the attacks  $A_2, A_3, \dots$  occur on  $SJ_1$  as well. We notice that the previous distribution  $W_1$  with the two following distributions:

$$\begin{aligned} W_2 &= \{v_2, v_5, v_8, \dots, v_{s-2}, v_{s+1}, v_{sm+1}\}, \\ W_3 &= \{v_3, v_6, v_9, \dots, v_{s-1}, v_{s+1}, v_{sm+1}\}. \end{aligned} \quad (11)$$

Form an eternal dominating family on  $SJ_1$  for the following reasons:



In both strategies  $sm/3 = 8$  guards are not enough to eternally dominate  $J_{6,4}$

FIGURE 2: For  $J_{6,4}; s \equiv 0 \pmod{3}$ .

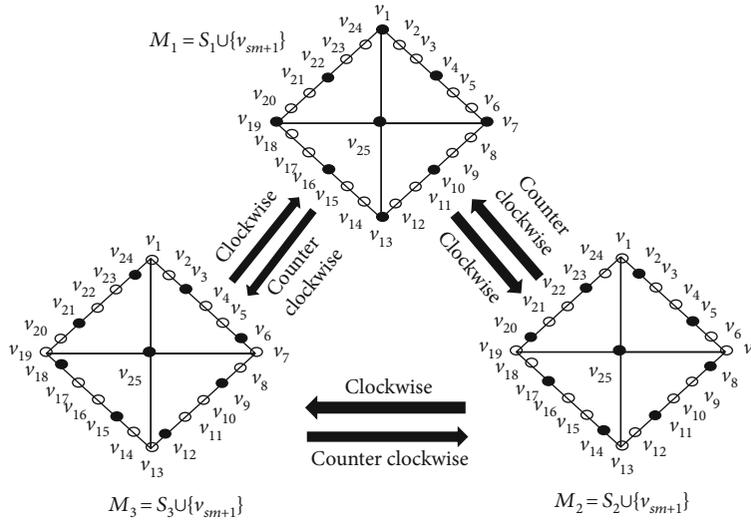


FIGURE 3: For  $J_{6,4}; s \equiv 0 \pmod{3}$ .

- (i) Each two sets of  $\{W_1, W_2, W_3\}$  are reachable from each other in one step, see Figure 4

$$V(W_1) \cup V(W_2) \cup V(W_3) = V(SJ_1). \quad (12)$$

It is obvious that  $W_3$  is a special case of  $M_0$ , because  $y \in \{v_1, v_{s+1}\}$  in  $W_3$ , while  $y \in R$  in  $M_0$ .

After a series of attacks on  $SJ_1$ , we assume an attack  $A_l$  occurs on  $v_k \in SJ_k : k \neq 1$ , because each one of  $W_1, W_2, W_3$  contains  $v_{sm+1}$  and a vertex of  $R(v_1 \text{ or } v_{s+1})$ ; then, it is possible to repeat the same previous strategy to defend the attack  $A_l$  and apply it to  $SJ_k$ . When trying to defend  $A_l$ , the guard of  $R$  that moves to  $v_{1+s(k-1)}$  to protect  $v_{2+s(k-1)}$  is the vertex  $y \in \{v_1, v_{s+1}\}$  taking into consideration that the distribu-

tions  $W_1 - y, W_2 - y$  can move back to  $M_0$  in one step while  $W_3$  is already a special case of  $M_0$ . Without loss of generality, this strategy applies to every subgraph  $SJ_k$ , and any series of attacks taking into consideration that the proof does not change if  $y \in SJ_1$  when  $A_1$  occurs (in this case, we start at  $W_3$  instead of  $M_0$ ).

Case 2.  $v_{i+1}$  is occupied.

By following the same process in Case 1 with one difference which is moving the guards of  $T_1$  towards the lower index when defending  $A_1$ ,  $v_s$  becomes unprotected, as in Case 1, and the guard on  $v_{sm+1}$  moves to  $v_{s+1}$  to protect  $v_s$ , and  $y$  moves to  $v_{sm+1}$  forming  $W_2$ , in a similar way to Case 1.  $W_1$ ,

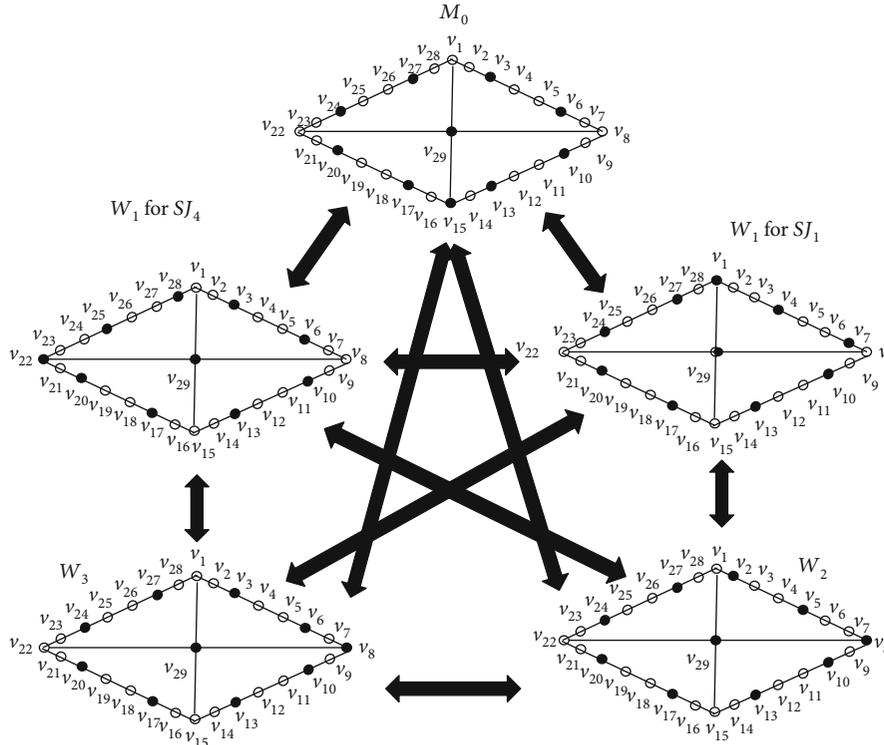


FIGURE 4:  $\gamma_m^\infty(J_{7,A}) = m \lfloor s/3 \rfloor + 2 = 10$ .

$W_2, W_3$  form an eternal dominating family on  $SJ_1$ , and the same argument about  $A_l$  can be followed here.

Case 3.  $v_i \in R$ .

When  $A_1$  occurs, we consider the following subcases:

Case 3(a). The current distribution is  $M_0$ , and then the guard set is  $S \cup \{y\}$ . The guard on  $v_{sm+1}$  moves to  $v_i$ , the guard on  $y$  moves to  $v_{sm+1}$ , and the new set of guards is  $S \cup \{v_i\}$ ; therefore, we remain in  $M_0$ .

Case 3(b). The current distribution is  $W_1$  in the previously attacked subgraph, and let us assume this subgraph is  $SJ_1$  while the remaining subgraphs fall under  $M_0$ ; then, the guard on  $v_{sm+1}$  moves to  $v_i$ , the guard on  $v_1$  moves to  $v_{sm+1}$  to defend the attack, and the guards  $W_1 - y$  move back to  $M_0$  so the new set of guards is  $S \cup \{v_i\}$ ; therefore, we remain in  $M_0$ . Without loss of generality, this strategy applies no matter which subgraph was attacked in the previous attack.

Case 3(c). The current distribution is  $W_2$  in the previously attacked subgraph, and we use the same argument in Case 3(b).

From all the previous cases, we can see that  $m \lfloor s/3 \rfloor + 2$  guards can eternally dominate  $J_{s,m}$ . Therefore,

$$\gamma_m^\infty(J_{s,m}) \leq m \left\lfloor \frac{s}{3} \right\rfloor + 2. \tag{13}$$

From (8) and (13), we conclude that  $\gamma_m^\infty(J_{s,m}) = m \lfloor s/3 \rfloor + 2$  for  $s \equiv 1 \pmod{3}$ . Figure 4 also demonstrates how 10 guards can move directly from any guard distribution of

$SJ_1$  to the distribution  $W_1$  of  $SJ_4$ , and this direct transportation can occur from every guard distribution of any subgraph  $SJ_k$  to all other distributions of every subgraph  $SJ_k : 2 \leq k \leq m$ .

**Corollary 12.**

$$\gamma_m^\infty(J_{s,m}) = \begin{cases} \left\lceil \frac{sm}{3} \right\rceil + 1 & \text{for } s \equiv 0 \pmod{3}, \\ m \left\lfloor \frac{s}{3} \right\rfloor + 2 & \text{for } s \equiv 1 \pmod{3}, \end{cases} \tag{14}$$

$$\left\lceil \frac{sm - \lfloor m/2 \rfloor}{3} \right\rceil + 1 \leq \gamma_m^\infty(J_{s,m}) \leq \left\lceil \frac{sm}{3} \right\rceil + 1.$$

*Proof.* We conclude 1 by combining Theorems 9 and 11.

We know that  $\gamma_m^\infty(J_{s,m}) \geq \gamma(J_{s,m})$ ; therefore, from Theorems 7 and 8, we conclude 2.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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