Hindawi
Journal of Advanced Transportation
Volume 2017, Article ID 5062984, 9 pages
https://doi.org/10.1155/2017/5062984

Research Article

Numerical Bounds on the Price of Anarchy

Louis de Grange,[1] Carlos Melo-Riquelme,[1] Cristóbal Burgos,[1] Felipe González,[1] and Sebastián Raveau[2]

[1] *Industrial Engineering Department, Diego Portales University, Santiago, Chile*
[2] *Department of Transport Engineering and Logistics, Pontificia Universidad Católica de Chile, Santiago, Chile*

Correspondence should be addressed to Louis de Grange; louis.degrange@udp.cl

Received 7 June 2017; Accepted 27 August 2017; Published 17 October 2017

Academic Editor: Sara Moridpour

Theoretical upper bounds for price of anarchy have been calculated in previous studies. We present an empirical analysis for the price of anarchy for congested transportation networks; different network sizes and demand levels are considered for each network. We obtain a maximum price of anarchy for the cases studied, which is notably lower than the theoretical bounds reported in the literature. This result should be carefully considered in the design and implementation of road pricing mechanisms for cities.

1. Introduction

The "price of anarchy" [1, 2], within the context of the traffic balance of transportation networks [3], is a concept related to the percentage or relative difference of the total costs in a network obtained from a traffic assignment equilibrium (i.e., Wardrop's first principle, [4]) with respect to the total costs in a network obtained from an optimal system assignment (i.e., Wardrop's second principle).

In this article, we present an empirical analysis of the price of anarchy for congested transportation networks: three networks of different types and sizes and different levels of demand for each network are considered. For the three cases analyzed, we find that the maximum price of anarchy never exceeds 9%, which is significantly lower than the theoretical bounds reported in the literature. The price of anarchy tends to zero even for hypercongested networks. This result suggests that the potential benefits from an optimal road pricing scheme can be quite small within the context of a traffic assignment with fixed demand: this result should be carefully considered in the design and implementation of road pricing mechanisms for cities.

We obtain a second interesting result that the price of anarchy (which represents the relative benefit of an optimal system assignment with respect to a balanced user assignment) depends strongly on the difference between the fixed costs and the variable costs of the arcs (or routes) of the network. The price of anarchy tends to zero for cases in which the fixed cost represents nearly 100% or 0% of the cost of the routes. That is, for high and low levels of congestion, the price of anarchy tends to take on increasingly smaller values. This second result may be relevant in the design of transportation networks and their optimal use.

The analyses were performed using the Bureau of Public Roads (BPR) volume-delay functions for strategic road networks in three cities of various sizes: Santiago de Chile, Chicago, and Anaheim. The equilibrium based on Wardrop's first and second principles was solved using the Origin-Based Assignment (OBA) traffic algorithm described by Bar-Gera [5] to ensure proper convergence of the equilibrium flows and trip times in the tests. This approach was considered also in Shi et al. [6] but including link-capacitated traffic assignment problem.

These results are important considerations for the design and implementation of road pricing mechanisms. In many cases of road pricing, a similar equilibrium is sought to that obtained based on Wardrop's second principle (i.e., first-best pricing). Thus, if an optimal assignment of the system produces very few benefits (i.e., a low price of anarchy), it follows that first-best pricing will also generate few benefits. However, if a change in demand results (instead of only a change in the routes used by travelers within the congested network), for example, by a transfer to other transportation modes or by alternative trip schedules, the upper bounds on the price of anarchy would be higher, indicating that a first-best policy has more potential benefits.

In Section 2, we present an analytical definition of the price of anarchy, a literature review on this subject, and the central hypothesis of our research. In Section 3, we present the methodology used for and the results obtained from this research, as well as two extensions for simple networks. Finally, in Section 4, we present the main findings and discuss future extensions of this work.

2. Definition of the Price of Anarchy, Literature Review, and Research Hypothesis

2.1. Price of Anarchy in Transportation Networks. The concept of the "price of anarchy" was first introduced by Koutsoupias and Papadimitriou [1] and was later formally defined by Papadimitriou [2]; both of these definitions were presented within the context of computer and telecommunications networks, which become congested with use.

The concept of the price of anarchy has also been applied to transportation issues [7–12], resource allocation to broadband internet [13, 14], the design of communication networks [15, 16], supply chain management [17], and divisible goods allocation [18].

The price of anarchy in congested transportation networks [3, 7, 19] refers to the loss of efficiency, which is measured as the increase in cost for travelers who are assigned to the arcs of a congested transportation network when the travelers minimize their individual costs (i.e., a behavior criterion based on Wardrop's first principle) with respect to the behavior that minimizes the sum of the costs for all of the individuals traveling in the network (i.e., a behavior criterion based on Wardrop's second principle).

For a network with linear arc cost functions, Roughgarden and Tardos [7] showed that the price of anarchy has an upper bound of 4/3. That is, there is a maximum increase of 33.33% in the welfare (which is measured as the total time spent in the network) of an optimal assignment of the system traffic with respect to an optimal assignment of users. Christodoulou et al. [20] found a bound below 4/3 using coordination mechanisms for linear cost functions. Chen and Zhang [21] also obtained a bound below 4/3 for the price of anarchy using linear costs in resource allocation in telecommunications by considering incentives for agents who released information to the network.

However, the loss of efficiency (i.e., the price of anarchy) may depend on several factors such as the size and topology of the network (i.e., the number of zones, nodes, and arcs), the level of demand, the capacity of the arcs, and the type of arc cost functions. Yang et al. [22] have discussed the price of anarchy for different families of cost functions for road network arcs (i.e., linear, BPR, quadratic, or cubic functions) and have extended the analysis to the case of variable demand. The authors stated that the bounds on the price of anarchy for a variable demand depend on the parameters of the demand function; that is, the price of anarchy can grow indefinitely. For a variable demand (e.g., changes in the total number of trips, the means of transport, or the trip schedule), the additional costs for the user equilibrium with respect to the optimum balance of the system could be much higher than that for a fixed trip matrix.

Gairing et al. [23] analyzed the bounds on the price of anarchy for general polynomial functions. Cole et al. [24] showed that, for linear cost functions (and some other specific network topologies), an optimal road pricing based on marginal costs reduces costs with respect to Wardrop's (Nash) equilibrium in the same way that eliminating some links from the network does.

Nagurney and Qiang [25] used the price of anarchy to analyze the robustness of a road transport network when the capacity of the network roads decreases over time because of the absence of maintenance. Yin and Lawphongpanich [26] applied the price of anarchy concept to the costs of polluting emissions in transportation networks. Karakostas et al. [27] applied the price of anarchy concept to a traffic assignment model in which some of the travelers did not make decisions based on the trip time but on other information, such as the minimum trip distance obtained from maps. Han and Yang [28] consider the effect of the so-called second-best tolls on the price of anarchy of the traffic equilibrium problem, where there are multiple classes of users with a discrete set of values of time. Han et al. [29] analyze the efficiency of road space rationing schemes by establishing the bounds of the reduction in the system cost associated with the restricted flow pattern at user equilibrium in comparison with the system cost at the original user equilibrium. Liu et al. [30] study the existence and efficiency of oligopoly equilibrium under simultaneous toll and capacity competition in a parallel-link network subject to congestion. They show that the bounds are demand-function-free and are only dependent upon the number of competitive roads. Xu et al. [31] analyze how ridesharing impacts traffic congestion; the model is built by combining a ridesharing market model with a classic elastic demand Wardrop traffic equilibrium model. Their results show that the ridesharing base price influences the congestion level, and the utilization of ridesharing increases as the congestion increases.

Although the price of anarchy has spawned an entire line of theoretical research, few empirical studies have been conducted. Levinson [32] reported that, in the Twin Cities, Minneapolis-Saint Paul, the additional costs from congestion generated by the traffic equilibrium for the morning rush hour (7:30–8:30 hours) were only 1.7%, despite the fact that more than 630,000 vehicles were in circulation during that period. Previously, Youn et al. [33] estimated that the price of anarchy for Boston/Cambridge, London, and New York road networks ranged between 3% and 4% on average and decreased as the number of agents (travelers) increased.

Therefore, we undertake an empirical study of the price of anarchy attained for different scenarios. As we will see later, the price of anarchy is directly related to several exogenous factors associated with the topology of the road network and the demand levels.

2.2. Analytic Definition of the Network and the Price of Anarchy. A road network can be defined as a graph $G(N, A)$, where N represents the set of nodes (including areas or centroids) and A represents the set of arcs of the network. We also define W as the set of origin-destination pairs within the network. We use the following notations: P_w denotes the set of existing routes between the pair w, T_w denotes the demand (e.g., the number of trips per hour), and h_w^p denotes the flow of the route, where $p \in P_w$. We characterize the arcs by defining $c_a(f_a)$ as the cost of arc a, which depends on its flow f_a (i.e., the cost function increases monotonically with the flow). The flow f_a is related to h_w^p via $f_a = \sum_{p \in P_w} \delta_a^p h_w^p$, where δ_a^p is unity if route p passes through arc a and is zero otherwise.

For a traffic equilibrium based on Wardrop's first principle, which is also known as user equilibrium (UE), the flow of equilibrium on arc a can be defined as f_a^{UE}, such that the cost of equilibrium of users on arc a is $c_a(f_a^{\mathrm{UE}}) = c_a^{\mathrm{UE}}$. The total cost of the system under user equilibrium can be written as follows:

$$C_{\mathrm{TOT}}^{\mathrm{UE}} = \sum_{a \in A} \left(c_a^{\mathrm{UE}} \cdot f_a^{\mathrm{UE}} \right). \quad (1)$$

Similarly, for a traffic equilibrium based on Wardrop's second principle, which is also known as an optimum system (OS), the flow of equilibrium on arc a can be defined as f_a^{OS}, and the respective cost of arc a is $c_a(f_a^{\mathrm{SO}}) = c_a^{\mathrm{SO}}$. The total cost of the system under an optimum equilibrium of the system can be written as follows:

$$C_{\mathrm{TOT}}^{\mathrm{SO}} = \sum_{a \in A} \left(c_a^{\mathrm{SO}} \cdot f_a^{\mathrm{SO}} \right). \quad (2)$$

The ratio between the expressions given by (1) and (2) can be defined as follows:

$$\rho = \frac{C_{\mathrm{TOT}}^{\mathrm{UE}}}{C_{\mathrm{TOT}}^{\mathrm{SO}}} = \frac{\sum_{a \in A} \left(c_a^{\mathrm{UE}} \cdot f_a^{\mathrm{UE}} \right)}{\sum_{a \in A} \left(c_a^{\mathrm{SO}} \cdot f_a^{\mathrm{SO}} \right)}. \quad (3)$$

Within the context of transportation networks, the ρ term in (3) is known as the price of anarchy, where clearly and by construction $\rho \geq 1$. Let us define $\phi = \rho - 1$, where $0 \leq \phi < 1$ represents the percentage loss in efficiency of the optimum assignment of users with respect to the optimal system assignment. Both ρ and ϕ are used to refer to the price of anarchy.

2.3. Research Hypothesis. Considering (2) and (3), we note that $\rho = 1$ (and therefore $\phi = 0$) for the specific cases given as follows:

(i) The arc cost functions are constant; that is, they do not depend on their flow ($\partial c_a / \partial f_a = 0$); therefore, the assignments based on Waldrop's first and second principles are equal; that is, there is no congestion.

(ii) The arc cost functions are homogeneous with degree m; that is, $c_a(\lambda \cdot f_a) = \lambda^m c_a(f_a)$, $\forall m \geq 0$ is satisfied. The assignments based on Waldrop's first and second principles are also equal in this case [34, 35].

The previous result follows easily from changing variables ($x = \lambda f_a \rightarrow dx = \lambda d f_a$) in the objective function of Beckman et al.'s [36] classic traffic equilibrium problem, which yields the following result:

$$\begin{aligned} \sum_{a \in A} \int_0^{f_a} c_a(x)\, dx &= \sum_{a \in A} \int_0^1 f_a \cdot c_a(\lambda f_a)\, d\lambda \\ &= \sum_{a \in A} \int_0^1 f_a \cdot \lambda^m c_a(f_a)\, d\lambda, \end{aligned} \quad (4)$$

$$\begin{aligned} \sum_{a \in A} \int_0^1 f_a \cdot \lambda^m c_a(f_a)\, d\lambda &= \sum_{a \in A} f_a c_a(f_a) \int_0^1 \lambda^m d\lambda \\ &= \frac{1}{1+m} \sum_{a \in A} c_a(f_a) f_a. \end{aligned} \quad (5)$$

Therefore, when the arc cost functions are homogeneous with degree m, we can directly derive from (5) that the assignments based on Wardrop's first and second principle are the same.

Next, let us consider the typical BPR arc cost functions:

$$c_a(f_a) = c_a^0 \left(1 + \beta_a \left(\frac{f_a}{k_a} \right)^{n_a} \right) = c_a^0 + c_a^0 \beta_a \left(\frac{f_a}{k_a} \right)^{n_a}. \quad (6)$$

The term c_a^0 in (6) represents the time or cost at free flow in arc a, and the term $c_a^0 \beta_a (f_a/k_a)^{n_a}$, where $\beta_a > 0$ and $n_a > 1$, represents the cost increase in arc a caused by the congestion generated by the flow f_a. The term k_a is the fixed capacity of arc a.

Considering that β_a, n_a, and k_a are exogenous in (6), it is easy to see that if the parameter $n_a = n \ \forall a \in A$ is the same for all of the arcs in the network, the variable part of the cost of each arc a will be a homogeneous function of degree n.

Thus, we present the following hypothesis for cases (i) and (ii).

Hypothesis. If the variable component of the trip cost is very small compared to the fixed cost, the price of anarchy approaches unity. That is,

$$\begin{aligned} &\text{if } c_a^0 \gg c_a^0 \beta_a \left(\frac{f_a}{k_a} \right)^{n_a} \longrightarrow 1 \gg \beta_a \left(\frac{f_a}{k_a} \right)^{n_a}, \\ &\text{then } \rho \approx 1 \longrightarrow \phi \approx 0. \end{aligned} \quad (7)$$

Similarly, if the variable component of the trip cost of the arcs is very large with respect to the fixed cost, the price of anarchy approaches one. That is,

$$\begin{aligned} &\text{if } c_a^0 \ll c_a^0 \beta_a \left(\frac{f_a}{k_a} \right)^{n_a} \longrightarrow 1 \ll \beta_a \left(\frac{f_a}{k_a} \right)^{n_a}, \\ &\text{then } \rho \approx 1 \longrightarrow \phi \approx 0. \end{aligned} \quad (8)$$

In simple terms, the hypothesis represented by (7) and (8) is that if the congestion is too low or too high (hypercongestion), the price of anarchy approaches unity; therefore, the loss of relative efficiency between the assignments based on

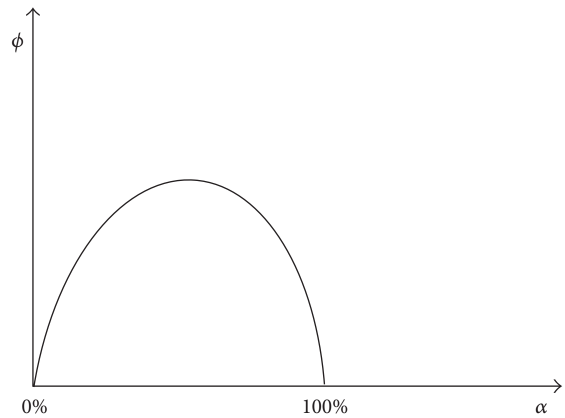

Figure 1: Relationship between the price of anarchy (ϕ) and the level of congestion (α) in a road network.

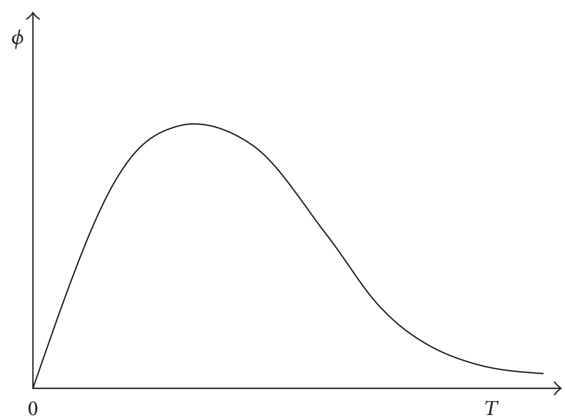

Figure 2: Relationship between the price of anarchy (ϕ) and the demand level (T) in a road network.

Wardrop's first and second principles approaches a vanishingly small value.

The hypothesis is represented graphically in Figure 1.

We previously defined $\phi = \rho - 1$. The value of α in the context of Figure 1 is defined as follows:

$$\alpha = \left(\frac{\mathrm{VC}}{\mathrm{VC} + \mathrm{FC}}\right) \times 100, \tag{9}$$

where VC represents the total variable cost and FC represents the total fixed cost of the network for any given equilibrium. Considering the BPR-type of cost functions in (6), FC and VC can be defined as follows:

$$\begin{aligned} \mathrm{FC} &= \sum_a \left(c_a^0\right), \\ \mathrm{VC} &= \sum_a \left\{c_a^0 \beta_a \left(\frac{f_a}{k_a}\right)^{n_a}\right\}. \end{aligned} \tag{10}$$

Usually, FC > 0; thus, it is always true that $\alpha < 100\%$. However, as the demand level in a network increases (i.e., there are more trips within the origin-destination matrix), the level of congestion and the value of α increase. Figure 2 shows the relationship between ϕ and the network demand level T.

Thus, Figures 1 and 2 are graphical representations of the hypothesis given by (7) and (8). In a real network, all of the arcs have some free flow cost ($c_a^0 > 0$, $\forall a$); therefore, the relationship shown in Figure 2 is more relevant than Figure 1 in our case. The hypothesis is presented empirically in the next section.

3. Empirical Analysis

3.1. General Methodology. To empirically estimate the price of anarchy (i.e., ϕ) in congested transportation networks, we first use a model of traffic assignment based on Wardrop's first principle to solve the following optimization problem [36]:

$$\begin{aligned} \min_{\{h_p^w\}} \quad & Z_1 = \sum_a \int_0^{f_a} c_a(x)\,dx \\ \text{s.a.:} \quad & \sum_{p \in P_w} h_p^w = T_w(\lambda_w) \quad \forall w \in W \\ & f_a = \sum_{p \in P_w} \delta_{ap} h_p^w \quad \forall a \in A \\ & h_p^w \geq 0 \quad \forall p \in P. \end{aligned} \tag{11}$$

The variables in (11) have been previously defined. The conditions of optimality in this problem provide an equilibrium condition consistent with Wardrop's first principle. The user equilibrium flow vector $\mathbf{f}^{\mathrm{UE}} = [f_a^{\mathrm{UE}}]$, the arc costs $c_a^{\mathrm{UE}} = c_a(f_a^{\mathrm{UE}})$, and the total network costs $C_{\mathrm{TOT}}^{\mathrm{UE}} = \sum_{a\in A}(c_a^{\mathrm{UE}} \cdot f_a^{\mathrm{UE}})$ can be calculated using (11).

Next, we use a model of traffic assignment based on Wardrop's second principle to solve the following optimization problem:

$$\begin{aligned} \min_{\{h_p^w\}} \quad & Z_2 = \sum_a c_a(f_a) \cdot f_a \\ \text{s.a.:} \quad & \sum_{p \in P_w} h_p^w = T_w(\lambda_w) \quad \forall w \in W \\ & f_a = \sum_{p \in P_w} \delta_{ap} h_p^w \quad \forall a \in A \\ & h_p^w \geq 0 \quad \forall p \in P. \end{aligned} \tag{12}$$

The optimal system equilibrium flow vector $\mathbf{f}^{\mathrm{SO}} = [f_a^{\mathrm{SO}}]$, the arc costs $c_a^{\mathrm{SO}} = c_a(f_a^{\mathrm{SO}})$, and the total network costs $C_{\mathrm{TOT}}^{\mathrm{SO}} = \sum_{a\in A}(c_a^{\mathrm{SO}} \cdot f_a^{\mathrm{SO}})$ can be calculated using (12).

Then, the ratio between $C_{\mathrm{TOT}}^{\mathrm{UE}} = \sum_{a\in A}(c_a^{\mathrm{UE}} \cdot f_a^{\mathrm{UE}})$ and $C_{\mathrm{TOT}}^{\mathrm{SO}} = \sum_{a\in A}(c_a^{\mathrm{SO}} \cdot f_a^{\mathrm{SO}})$ can be calculated for different types of networks and demand levels to obtain different values for the price of anarchy ϕ.

Thus, the variations in the total demand for the network (i.e., the weighting factors of the original trip matrix for each of the three cities studied) are used to determine the relationship between the price of anarchy and the demand levels and network congestion.

To ensure that the solutions to the two previous problems converge properly, the OBA algorithm described by Bar-Gera [5] is used to solve both traffic assignment problems.

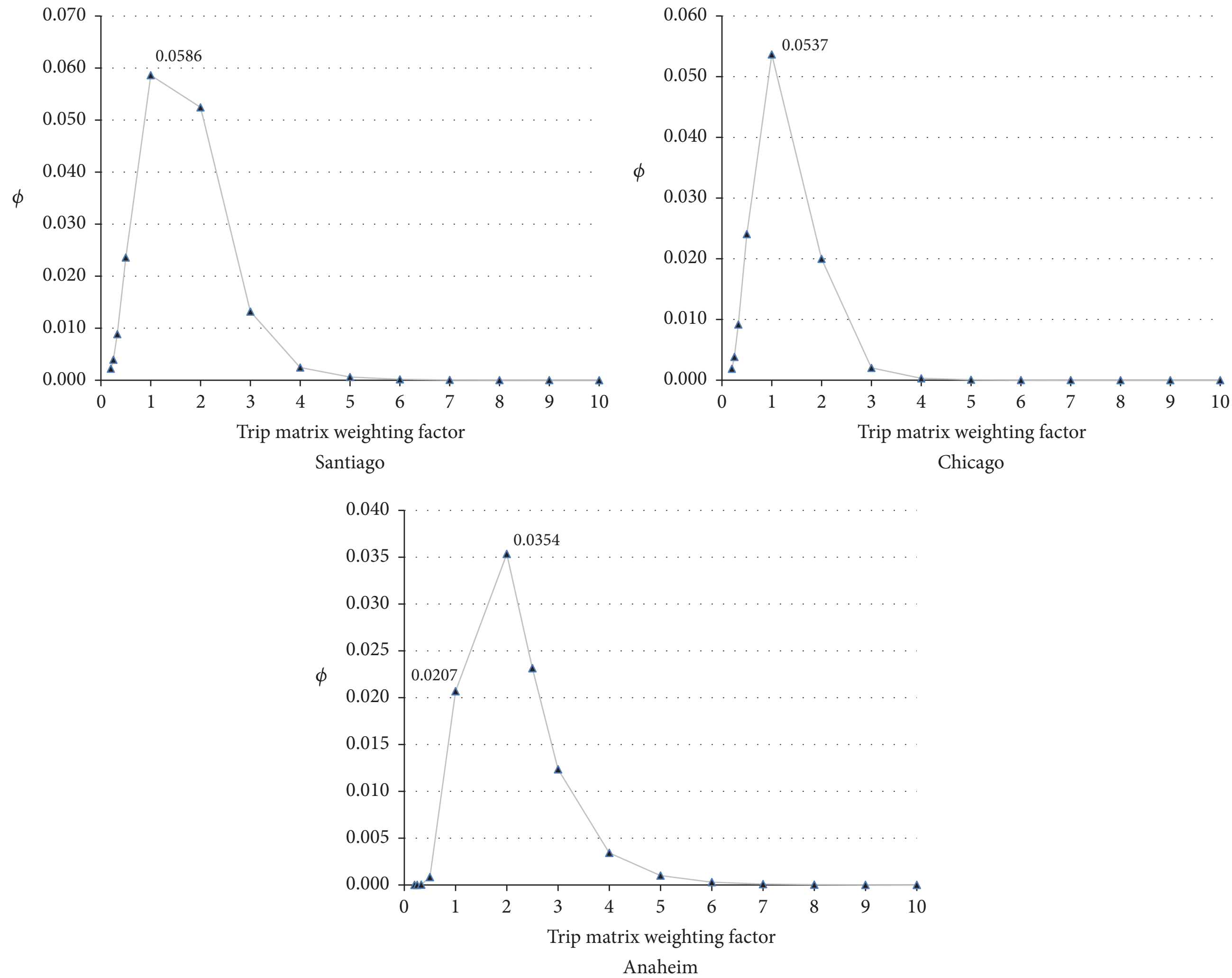

FIGURE 3: Relationship between ϕ and the demand level for each network (each curve specifies the price of anarchy using the trip matrix for the rush hour period estimated for each case. The price of anarchy for each of the Santiago and Chicago networks coincides with the maximum value. The maximum price of anarchy for the Anaheim network is obtained by doubling the number of trips (i.e., the matrix is weighted by a factor of 2)).

3.2. Description of Networks. A comparative analysis of the optimum user equilibrium and optimal system equilibrium was performed for three strategic road networks of different sizes, that is, the cities of Santiago de Chile, Chicago, and Anaheim, for different demand levels (and therefore different levels of congestion). Table 1 summarizes the main characteristics of these three networks.

We performed a comparative analysis of the results by using the following BPR arc cost function for the three aforementioned networks:

$$c_a\left(f_a\right) = c_a^0\left(1 + 0.15\left(\frac{f_a}{k_a}\right)^4\right). \tag{13}$$

The parameter values of 0.15 and 4 in (13) have been widely used in the specialized literature (see, e.g., [37–44], among many others). However, we used $n_a = 1$ (linear costs) and $n_a = 8$ in the calculations: the results are reported in Section 3.3.

3.3. Analysis of Results. Figure 3 shows the relationship between ϕ and the demand growth factor for each network. The growth factor varied between 0.25 and 10; that is, the trip matrix described in Table 1 was scaled by values between 0.25 and 10. Two assignments were performed for each case (i.e., the traffic equilibrium and optimal system equilibrium), and the corresponding values of ϕ were subsequently calculated.

The results shown in Figure 3 confirmed the hypothesis presented in Section 2.3 for the relationship between the price of anarchy and the demand and congestion levels of the network. The functional form of the price of anarchy shown in Figure 2 was also obtained for all of the cases.

Figure 3 shows that the maximum price of anarchy never exceeded 6%. We confirmed that the maximum price of anarchy was obtained for three out of the three networks when the reference matrix was used (i.e., a weighting factor of unity was used).

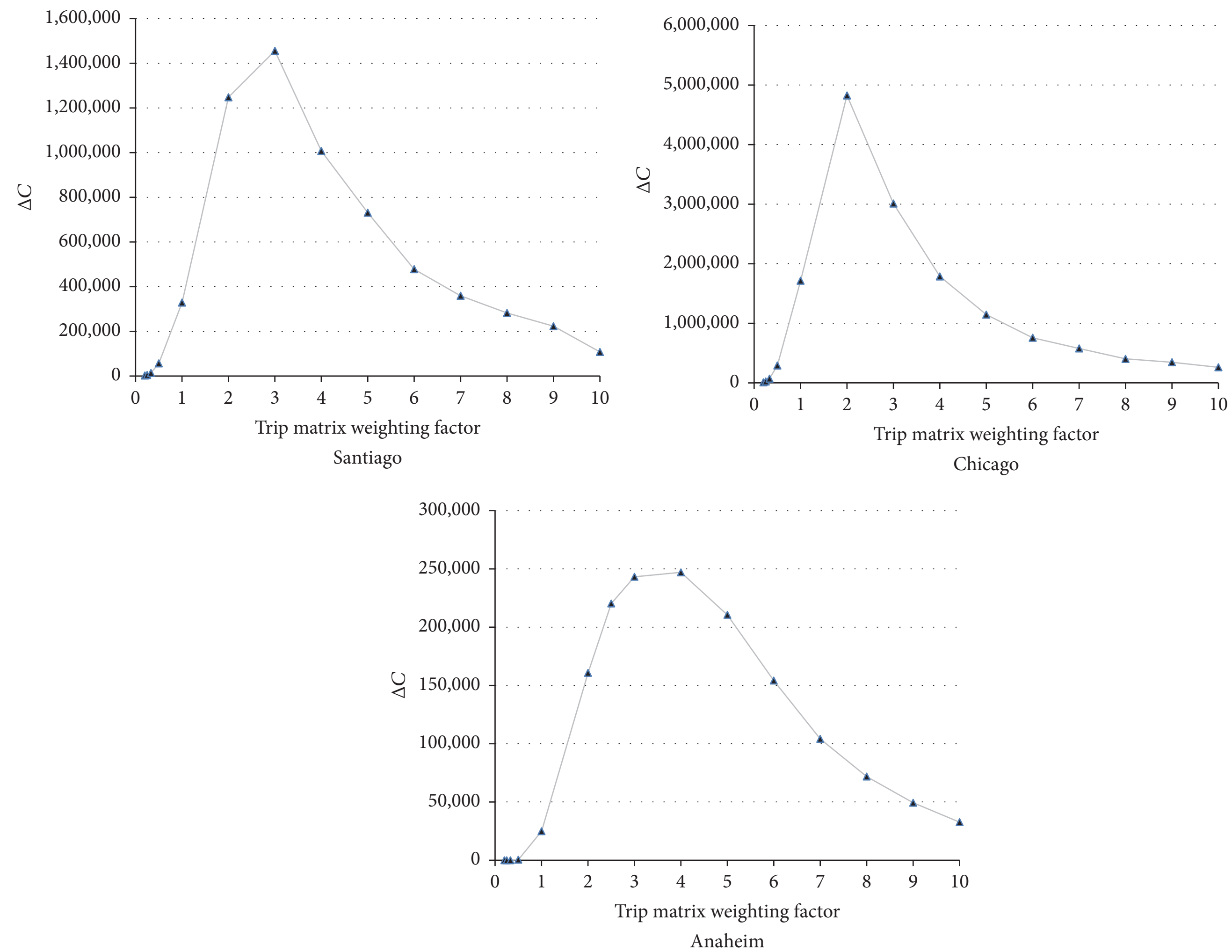

Figure 4: Relationship between ΔC and the demand level in each network.

Table 1: Strategic network indicators by city.

City	Zones	Vertices	Arcs	Trips 7:30–8:30 (*)
Santiago	630	6,770	14,731	465,457
Chicago	1,790	12,982	39,018	1,360,428
Anaheim	38	416	914	104,694

(*) These trips are in the reference origin-destination matrix. This matrix is then weighted by a scalar to vary the total demand on the network and calculate the price of anarchy.

Table 2: Maximum value of ϕ for each network and n_a value.

City	$n_a = 1$	$n_a = 4$	$n_a = 8$
Santiago	0.0082	0.0586	0.08378
Chicago	0.0056	0.0537	0.08762
Anaheim	0.0063	0.0354	0.02730
Maximum theoretical value (*)	*0.3333*	*1.1505*	*2.0808*
Ratio (**)	*40.7*	*19.6*	*23.2*

(*)See details in Roughgarden [3]. (**)Corresponds to the ratio of the maximum theoretical value and the largest numerical value calculated among the three networks analyzed.

The maximum price of anarchy was obtained using a weighting factor of two only for the Anaheim network. The maximum prices of anarchy for the two largest networks (Santiago and Chicago) were larger than those for the two smaller networks.

An interesting result that follows from the previous observation was obtained by plotting the difference in the costs ($\Delta C = C_{\mathrm{TOT}}^{\mathrm{UE}} - C_{\mathrm{TOT}}^{\mathrm{SO}}$) against the demand level (see Figure 4).

Figure 4 shows that as the level of demand and congestion in the network increased, the difference between the total costs obtained for the user equilibrium and the total costs of an optimal system assignment first grew and then decreased. The plots in Figure 4 confirmed the hypothesis presented in (7) and (8).

Unlike Figure 3, Figure 4 shows that, for three out of the three networks, the maximum ΔC value was obtained using weighting factors greater than unity, which corresponded to a larger number of trips than that in the reference matrix.

The above procedure is repeated using $n_a = 1$ (linear costs) and $n_a = 8$. Table 2 summarizes the maximum

Table 3: Trip matrix weighting factor (*) that generates the maximum value of ϕ as a function of the n_a value.

City	$n_a = 1$	$n_a = 4$	$n_a = 8$
Santiago	7	1	0.9
Chicago	3	1	0.8
Anaheim	9	2	1

(*) A weighting factor of unity indicates that the reference trip matrix used for the rush hour of each network was unchanged.

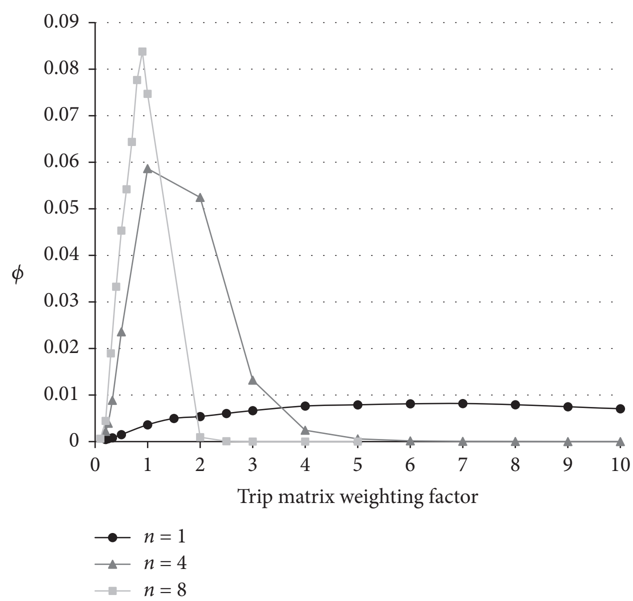

Figure 5: Relationship between ϕ and the demand level for the Santiago network for $n_a = 1, 4$, and 8.

calculated values of ϕ for each of the three networks, for three different n_a values. The last row shows the maximum theoretical value for ϕ for the respective cost function [3].

The maximum calculated value of ϕ for the linear case was 0.00819. Thus, the traffic equilibrium costs with respect to the optimum system equilibrium would increase by 0.82%. This value is significantly lower than the theoretical bound reported in the literature (for $\rho = 4/3$ or $\phi = 1/3$) of 33.33%. For $n_a = 4$ or $n_a = 8$, the maximum theoretical bounds reported in Roughgarden [3] are 19.6 and 23.2 times higher than our calculated results.

Table 3 shows the weighting factor for the trip matrix which produced the maximum price of anarchy. For the linear case, the original matrices constructed for the respective rush hour must be sufficiently amplified to produce the maximum price of anarchy. However, for $n_a = 4$, the maximum price of anarchy was obtained for a weighting factor of unity, that is, for the same matrix as the rush hour matrix. For $n_a = 8$, the maximum price of anarchy was obtained by decreasing the matrix weighting factor.

In Figure 5, the curves for ϕ versus the demand level are compared for $n_a = 1$, $n_a = 4$, and $n_a = 8$ for the Santiago network.

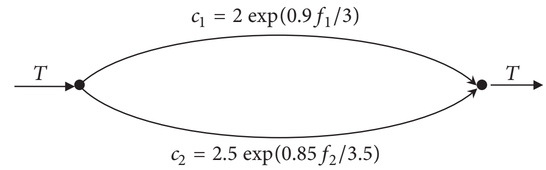

Figure 6: Simple network with exponential cost functions.

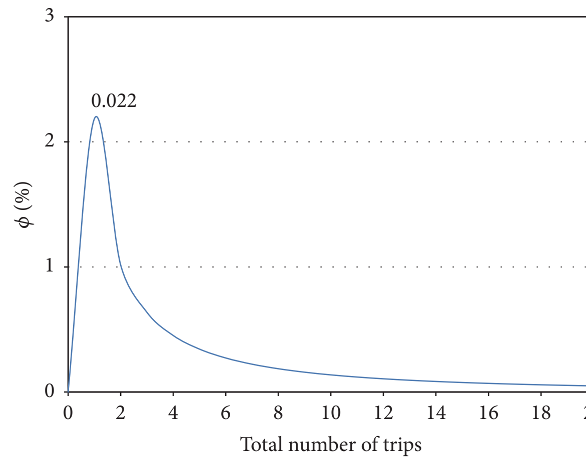

Figure 7: Relationship between ϕ and T in a simple network with exponential cost functions.

3.4. Extension: Simple Network with Exponential Cost Functions. In this section, we present the results for a simple network of two parallel arcs for which we used exponential cost functions instead of BPR-type cost functions. This extension was used to test our hypothesis for the case where the cost functions were not homogeneous and of degree n.

We considered a simple network of two parallel arcs with exponential cost functions, as shown in Figure 6.

Figure 7 shows the price of anarchy for different values of T (which is the sum of f_1 and f_2). The same functional was obtained as in the hypothesis (see Figure 2) and for the cases analyzed in Section 3. An upper bound of 2.2% was attained for the price of anarchy in this simple example.

4. Conclusions

In this paper, we present an empirical analysis to calculate the social loss (measured as the increase in trip costs) that occurs in congested transportation networks, between an optimal user equilibrium (Wardrop's first principle) and an optimal system equilibrium (Wardrop's second principle). This social loss, in percentage terms, is known as the price of anarchy. The analysis was carried out in three transportation networks of three different cities around the world and with different demand levels (trip matrices).

The price of anarchy is defined as the percentage loss of efficiency associated with the user equilibrium with respect to the optimal system equilibrium. The first important conclusion obtained in this study was that the price of anarchy was always small and much lower than the theoretical upper bounds reported in the specialized literature. The maximum price of anarchy never exceeded 9%. Even for cases with linear cost functions, the maximum calculated values were always lower than 1%, which is significantly below the theoretical bound of 33.33% reported for linear costs.

A second important conclusion was that the social loss that produced an optimum user equilibrium tended to zero in two cases: when the fixed costs of the network arcs were dominant and when the variable costs of the network arcs were dominant. These two cases correspond to very low congestion (in the limit where there are only fixed costs and there is no congestion) and very high congestion. This correspondence can be explained by assuming that the variable part of the arc cost function is homogeneous. However, this conclusion is also likely to hold when the variable part of the cost is not homogeneous, as we found for a simple example with exponential cost functions.

Finally, we concluded that as the marginal effect of congestion increased (e.g., as powers of polynomial functions), the price of anarchy tended to increase but remained well below the theoretical bounds reported in the literature for polynomial cost functions such as the BPR.

We submit that these findings are relevant considerations in the designing and implementation of road pricing mechanisms because our results showed that the maximum social benefit from first-best road pricing was small. This result was obtained because first-best pricing encourages an equilibrium such as Wardrop's second principle.

Thus, second-best pricing policies, which can be feasibly implemented in practice, could generate even lower profits.

However, higher bounds on the price of anarchy could be obtained when there is a change in the demand (instead of only a change in the routes used by the travelers in the congested network), such as in changing transportation modes or trip schedules, showing the potential benefits of a first-best pricing policy.

To develop this research further, variable demand should be included in the analysis by incorporating the surplus of consumers and the total cost of the travelers in the network into the price of anarchy. An interesting relationship may be uncovered between the price of anarchy and the demand elasticity with respect to the generalized trip cost.

It would also be interesting to perform analyses with dynamic assignment models or microsimulations to better capture the behavior of vehicles in very congested networks because static models cannot be used to model phenomena such as road queues, blocked intersections, and the eventual cessation of vehicle flow at very high vehicle densities.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by FONDECYT (Grant no. 1160251).

References

[1] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," in *Proceedings of the 16th Symposium on Theoretical Aspects of Computer Science*, vol. 1563 of *Lecture Notes in Computer Science*, pp. 404–413, Springer.

[2] C. H. Papadimitriou, "Algorithms, games, and the internet," in *Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing*, pp. 749–753, 2001.

[3] T. Roughgarden, "The price of anarchy is independent of the network topology," *Journal of Computer and System Sciences*, vol. 67, no. 2, pp. 341–364, 2003.

[4] J. G. Wardrop, "Some theoretical aspects of road traffic research. Part II," in *Proceedings of the Institute of Civil Engineers*, vol. 1, pp. 325–378, 1952.

[5] H. Bar-Gera, "Origin-based algorithm for the traffic assignment problem," *Transportation Science*, vol. 36, no. 4, pp. 398–417, 2002.

[6] F. Shi, G.-M. Xu, and H. Huang, "An augmented Lagrangian origin-based algorithm for link-capacitated traffic assignment problem," *Journal of Advanced Transportation*, vol. 49, no. 4, pp. 553–567, 2015.

[7] T. Roughgarden and E. Tardos, "How bad is selfish routing?" *Journal of the ACM*, vol. 49, no. 2, pp. 236–259, 2002.

[8] J. R. Correa, A. S. Schulz, and N. E. Stier-Moses, "Selfish routing in capacitated networks," *Mathematics of Operations Research*, vol. 29, no. 4, pp. 961–976, 2004.

[9] L. Fleischer, K. Jain, and M. Mahdian, "Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games," in *Proceedings of the 45th Annual Symposium of the Foundations of Computer Science (FOCS)*, pp. 277–285, October 2004.

[10] G. Karakostas and S. G. Kolliopoulos, "Edge pricing of multicommodity networks for heterogeneous selfish users," in *Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2004*, pp. 268–276, October 2004.

[11] T. Roughgarden, *Selfish Routing and the Price of Anarchy*, MIT Press, 2005.

[12] V. Zverovich and E. Avineri, "Braess' paradox in a generalised traffic network," *Journal of Advanced Transportation*, vol. 49, no. 1, pp. 114–138, 2015.

[13] R. Johari and J. N. Tsitsiklis, "Efficiency loss in a network resource allocation game," *Mathematics of Operations Research*, vol. 29, no. 3, pp. 407–435, 2004.

[14] T. Harks and K. Miller, "The worst-case efficiency of cost sharing methods in resource allocation games," *Operations Research*, vol. 59, no. 6, pp. 1491–1503, 2011.

[15] D. Acemoglu and A. Ozdaglar, "Competition and efficiency in congested markets," *Mathematics of Operations Research*, vol. 32, no. 1, pp. 1–31, 2007.

[16] E. Anshelevich, A. Dasgupta, E. Tardos, and T. Wexler, "Near-optimal network design with selfish agents," in *Proceedings of the 35th Annual ACM Symposium on the Theory of Computing*, pp. 511–520, ACM, New York, NY, USA.

[17] B. Golany and U. G. Rothblum, "Inducing coordination in supply chains through linear reward schemes," *Naval Research Logistics (NRL)*, vol. 53, no. 1, pp. 1–15, 2006.

[18] S. Yang and H. Hajek, *Revenue and stability of a mechanism for efficient allocation of a divisible good. Working paper*, University of Illinois at Urbana-Champaign, 2005.

[19] G. Perakis, "The price of anarchy when costs are non-separable and asymmetric," in *Integer programming and combinatorial optimization*, vol. 3064 of *Lecture Notes in Comput. Sci.*, pp. 46–58, Springer, Berlin, Germany, 2004.

[20] G. Christodoulou, K. Mehlhorn, and E. Pyrga, "Improving the price of anarchy for Selfish routing via coordination mechanisms," *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics): Preface*, vol. 6942, pp. 119–130, 2011.

[21] Y.-J. Chen and J. Zhang, "Design of price mechanisms for network resource allocation via price of anarchy," *Mathematical Programming*, vol. 131, no. 1-2, Ser. A, pp. 333–364, 2012.

[22] H. Yang, W. Xu, and B. Heydecker, "Bounding the efficiency of road pricing," *Transportation Research Part E: Logistics and Transportation Review*, vol. 46, no. 1, pp. 90–108, 2010.

[23] M. Gairing, T. Lücking, M. Mavronicolas, and B. Monien, "The price of anarchy for polynomial social cost," in *Proceedings of the 29th Int. Symposium on Mathematical Foundations of Computer Science (MFCS'04)*, vol. 3153 of *Lecture Notes in Computer Science*, pp. 574–585, Springer, Berlin, Germany, 2004.

[24] R. Cole, Y. Dodis, and T. Roughgarden, "How much can taxes help selfish routing?" *Journal of Computer and System Sciences*, vol. 72, no. 3, pp. 444–467, 2006.

[25] A. Nagurney and Q. Qiang, "A relative total cost index for the evaluation of transportation network robustness in the presence of degradable links and alternative travel behavior," *International Transactions in Operational Research*, vol. 16, no. 1, pp. 49–67, 2009.

[26] Y. Yin and S. Lawphongpanich, "Internalizing emission externality on road networks," *Transportation Research Part D: Transport and Environment*, vol. 11, no. 4, pp. 292–301, 2006.

[27] G. Karakostas, T. Kim, A. Viglas, and H. Xia, "On the degradation of performance for traffic networks with oblivious users," *Transportation Research Part B: Methodological*, vol. 45, no. 2, pp. 364–371, 2011.

[28] D. Han and H. Yang, "The multi-class, multi-criterion traffic equilibrium and the efficiency of congestion pricing," *Transportation Research Part E: Logistics and Transportation Review*, vol. 44, no. 5, pp. 753–773, 2008.

[29] D. Han, H. Yang, and X. Wang, "Efficiency of the plate-number-based traffic rationing in general networks," *Transportation Research Part E: Logistics and Transportation Review*, vol. 46, no. 6, pp. 1095–1110, 2010.

[30] T.-L. Liu, J. Chen, and H.-J. Huang, "Existence and efficiency of oligopoly equilibrium under toll and capacity competition," *Transportation Research Part E: Logistics and Transportation Review*, vol. 47, no. 6, pp. 908–919, 2011.

[31] H. Xu, F. Ordõñez, and M. Dessouky, "A traffic assignment model for a ridesharing transportation market," *Journal of Advanced Transportation*, vol. 49, no. 7, pp. 793–816, 2015.

[32] D. Levinson, "The Physics of Traffic Congestion and Road Pricing in Transportation Planning," http://nexus.umn.edu/Presentations/Physics.pdf. Meeting of The American Physical Society, 2010.

[33] H. Youn, M. T. Gastner, and H. Jeong, "Erratum: Price of anarchy in transportation networks: Efficiency and optimality control (Physical Review Letters (2008) 101 (128701))," *Physical Review Letters*, vol. 102, no. 4, Article ID 049905, 2009.

[34] A. De Palma and Y. Nesterov, "Optimization formulations and static equilibrium in congested transportation networks," in *CORE discussion paper 9861*, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, 1998.

[35] H. Yang and H. J. Huang, *Mathematical and Economic Theory of Road Pricing*, Elsevier, Oxford, UK, 2005.

[36] M. Beckmann, C. B. McGuire, and C. B. Winsten, *Studies in the Economics of Transportation*, Yale University Press, New Haven, CT, USA, 1956.

[37] R. Dowling and A. Skabardonis, "Improving the average travel speeds estimated by planning models," in *Transportation Research Record, 1360*, pp. 68–74, National Research Council, Washington, DC, USA, 1993.

[38] R. G. Dowling, W. Kittelson, A. Skabardonis, and J. Zegeer, "Techniques for Estimating Speed and Service Volumes for Planning Applications," TRB NCHRP Report 387, National Research Council, Washington, DC, USA, 1997.

[39] R. Singh, "Improved speed-flow relationships: application to transportation planning models," in *Proceedings of the 7th TRB Conference on Application of Transportation Planning Methods*, Boston, Mass, USA, 1999.

[40] R. B. Dial, "Minimal-revenue congestion pricing part I: A fast algorithm for the single-origin case," *Transportation Research Part B: Methodological*, vol. 33, no. 3, pp. 189–202, 1999.

[41] R. B. Dial, "Minimal-revenue congestion pricing Part II: An efficient algorithm for the general case," *Transportation Research Part B: Methodological*, vol. 34, no. 8, pp. 645–665, 2000.

[42] H. Ham, T. J. Kim, and D. Boyce, "Implementation and estimation of a combined model of interregional, multimodal commodity shipments and transportation network flows," *Transportation Research Part B: Methodological*, vol. 39, no. 1, pp. 65–79, 2005.

[43] A. Nagurney and Q. Qiang, "Robustness of transportation networks subject to degradable links," *Europhysics Letters. EPL*, vol. 80, no. 6, Article ID 68001, Art. 68001, 6 pages, 2007.

[44] E. T. Verhoef, A. Koh, and S. Shepherd, "Pricing, capacity and long-run cost functions for first-best and second-best network problems," *Transportation Research Part B: Methodological*, vol. 44, no. 7, pp. 870–885, 2010.