Research Article

Some Results of Univalent and Starlike Integral Operator

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The main objective of the present paper is to study the mapping properties of functions belonging to certain classes under a family of univalent and starlike integral operator. Relationships of these classes are also pointed out.

1. Introduction and Definitions

Let A denotes the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$U = \{z : |z| < 1\}. \quad (2)$$

Also let S denotes the class of all functions in A which are univalent in U . Then a function $f \in S$ is said to be starlike in U if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U). \quad (3)$$

We denote by S^* the class of all functions in S which are starlike in U . A function $f \in S$ is said to be starlike of order α in U if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U), \quad (4)$$

for some α ($0 \leq \alpha < 1$). We denote by $S^*(\alpha)$ the class of all functions in S which are starlike of order α in U . Clearly, we have $S^*(\alpha) \subseteq S^*(0) = S^*$ ($0 \leq \alpha < 1$).

With a view to introducing an interesting family of analytic functions, we should recall the concept of subordination between analytic functions. Given two functions f and g , which are analytic in U , the function f is said to be subordinate to g if there exists a function w , analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$, ($z \in U$), and symbolically written as the following:

$$f < g \quad (z \in U) \quad \text{or} \quad f(z) < g(z) \quad (z \in U). \quad (5)$$

It is known that $f(z) < g(z)$ ($z \in U$) $\Rightarrow$ $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1 (see [1]). For $-1 \leq B < A \leq 1$, a function $p(z)$, analytic in U with $p(0) = 1$, is said to belong to the class $\beta(A, B)$ if

$$p(z) < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1). \quad (6)$$

To prove our main result, we need the following.

Lemma 2 (see [2]). Let the functions N and D be analytic in U , and let D map U onto a starlike region. Suppose also that

$$N(0) = D(0) = 0, \quad \frac{N'(z)}{D'(z)} = k, \quad (7)$$

$$\frac{N'(z)}{kD'(z)} \in \beta(A, B) \quad (k \geq 1).$$

Then,

$$\frac{N(z)}{kD(z)} \in \beta(A, B), \quad (8)$$

for all $z \in U$.

Lemma 3 (see [3]). Let

$$p_j(z) \in \beta(A, B) \quad (j = 1, 2). \quad (9)$$

Then, for $\alpha > 0$ and $\beta > 0$,

$$\frac{\alpha p_1(z) + \beta p_2(z)}{\alpha + \beta} \in \beta(A, B), \quad (10)$$

for all $z \in U$.

Lemma 4 (see [4]). Let the functions M and N be analytic in U with

$$M(0) = N(0) = 0, \quad (11)$$

and let γ be a real number. Suppose also that N maps U onto a region which is starlike with respect to the origin. Then,

$$\begin{aligned} \operatorname{Re} \left\{ \frac{M'(z)}{N'(z)} \right\} &> \gamma \quad (z \in U) \\ \implies \operatorname{Re} \left\{ \frac{M(z)}{N(z)} \right\} &> \gamma \quad (z \in U). \end{aligned} \quad (12)$$

2. Main Result

We begin by introducing a new integral operator

$$I(f, g)(z) = \frac{\alpha + 1}{z^\alpha} \int_0^z (f(t) e^{g(t)})^\alpha dt = z + \sum_{n=2}^{\infty} c_n z^n, \quad (13)$$

where f and $g \in A$.

Bear in mind, there are various types of integral operators studied by many different authors such as [5–9], few to mention, that motivate us to come out with the abovementioned integral operators.

Now let us begin with our first result relating to the integral operator of (13).

Theorem 5. Let the functions f and h be in the class S^* . Then, the function $I(f, g)$ defined by (13) is also in the class S^* .

Proof. By logarithmic differentiation, we find from (13) that

$$\frac{zI'(f, g)(z)}{I(f, g)(z)} = \frac{N(z)}{D(z)}, \quad (14)$$

where

$$\begin{aligned} N(z) &= z(f(z) e^{g(z)})^\alpha - \alpha \int_0^z (f(t) e^{g(t)})^\alpha dt, \\ D(z) &= \int_0^z (f(t) e^{g(t)})^\alpha dt. \end{aligned} \quad (15)$$

Clearly, we have $N(0) = D(0) = 0$, and D satisfies the starlikeness condition of Lemma 4.

Next, let $zh(z) = e^{g(z)}$, where $h(z)$ is analytic function in $D = \{z : 0 < |z| < 1\}$, and $h(z) \neq 0, h(0) = 1$. From (15), it is easily seen that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{N'(z)}{D'(z)} \right\} &= \alpha \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} + \alpha \operatorname{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} + 1 \\ &= \gamma > 0 \quad (z \in U), \end{aligned} \quad (16)$$

hence by Lemma 4, we obtain

$$\operatorname{Re} \left\{ \frac{N(z)}{D(z)} \right\} = \operatorname{Re} \left\{ \frac{zI'(f, g)(z)}{I(f, g)(z)} \right\} > 0 \quad (z \in U), \quad (17)$$

that is

$$\operatorname{Re} \left\{ \frac{zI'(f, g)(z)}{I(f, g)(z)} \right\} > 0 \quad (z \in U), \quad (18)$$

which evidently proves Theorem 5. $\square$

Theorem 6. Let the functions f and g be in the class $S^*(A, B)$. Then, $(1/2\alpha)\{zI'(f, g)(z)/I(f, g)(z) - 1\}$ is in the class $\beta(A, B)$.

Proof. Since

$$f(z) \in S^*(A, B), \quad g(z) \in S^*(A, B), \quad (19)$$

we find from Definition 1 that

$$\frac{zf'(z)}{f(z)} \in \beta(A, B), \quad \frac{zg'(z)}{g(z)} \in \beta(A, B). \quad (20)$$

By logarithmic differentiation, we find from (13) that

$$\frac{zI'(f, g)(z)}{I(f, g)(z)} - 1 = \frac{N(z)}{D(z)}, \quad (21)$$

where

$$\begin{aligned} N(z) &= z(f(z) e^{g(z)})^\alpha - (\alpha + 1) \int_0^z (f(t) e^{g(t)})^\alpha dt, \\ D(z) &= \int_0^z (f(t) e^{g(t)})^\alpha dt. \end{aligned} \quad (22)$$

Next, let $h(z) = e^{g(z)}$, and from (15), it is easily seen that

$$\operatorname{Re} \left\{ \frac{N'(z)}{D'(z)} \right\} = \alpha \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} + \alpha \operatorname{Re} \left\{ \frac{zh'(z)}{h(z)} \right\}. \quad (23)$$

Now rewrite the equality in (23) in the form

$$\frac{1}{2\alpha} \frac{N'(z)}{D'(z)} = \frac{1}{2\alpha} \left\{ \alpha \frac{zf'(z)}{f(z)} + \alpha \frac{zh'(z)}{h(z)} \right\}, \quad (24)$$

so that by (20) and Lemma 3, we have

$$\frac{1}{2\alpha} \frac{N'(z)}{D'(z)} \in \beta(A, B) \quad (\alpha > 0). \quad (25)$$

It is easily seen that D and N satisfy conditions of Lemma 2. It follows from (21), (25), and Lemma 2 that

$$\frac{1}{2\alpha} \frac{N(z)}{D(z)} \in \beta(A, B), \quad (26)$$

which evidently proves Theorem 6. $\square$

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