Research Article

Starlikeness of General Integral Operator for Meromorphic Multivalent Functions

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We introduce a new integral operator $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)$ for meromorphic multivalent functions. The starlikeness condition of this integral operator is determined. Several special cases are also discussed in the form of corollaries.

1. Introduction

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, \dots\}), \quad (1)$$

which are analytic and p -valent in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}, \quad (2)$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $\Sigma_1 = \Sigma$.

A function $f \in \Sigma_p$ is said to be meromorphic p -valent starlike and belongs to the class $\mathcal{MS}_p^*$, if it satisfies the inequality:

$$-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0. \quad (3)$$

A function $f \in \Sigma_p$ is said to be meromorphic p -valent convex and belongs to the class $\mathcal{MC}_p$, if it satisfies the inequality:

$$-\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0. \quad (4)$$

We note that

$$f \in \mathcal{MC}_p \iff -\frac{zf'}{p} \in \mathcal{MS}_p^*. \quad (5)$$

In particular, we set

$$\mathcal{MS}_1^* = \mathcal{MS}^*, \quad \mathcal{MC}_1 = \mathcal{MC} \quad (6)$$

for the classes of meromorphic starlike and convex functions, respectively.

Analogous to the integral operator defined by Frasin [1] on normalized analytic functions, we now define the following general integral operator for the meromorphic p -valent functions in the class Σ_p .

Definition 1. Let $n \in \mathbb{N}$, $\gamma_j, \delta_j \in \mathbb{R}_+ \cup \{0\}$ for all $j = 1, 2, \dots, n$ and $\gamma = (\gamma_1, \dots, \gamma_n)$, $\delta = (\delta_1, \dots, \delta_n)$. We introduce a new general integral operator

$$\begin{aligned} \mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n) : \Sigma_p^n &\longrightarrow \Sigma_p, \\ \mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)(z) &= \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \left(-\frac{u^{p+1}}{p} f'_j(u) \right)^{\gamma_j} \\ &\quad \times (u^p f_j(u))^{\delta_j} du \quad (z \in \mathbb{U}^*). \end{aligned} \quad (7)$$

Remark 2. The integral operator introduced here generalizes the following integral operators.

(i) If $p = 1$, then we have the integral operator

$$\mathcal{J}_{\gamma, \delta}(f_1, \dots, f_n)(z) = \frac{1}{z^2} \int_0^z \prod_{j=1}^n (-u^2 f'_j(u))^{\gamma_j} \times (u f_j(u))^{\delta_j} du \quad (8)$$

introduced and studied by Bulut and Goswami [2].

(ii) If $\delta = (0, \dots, 0)$, then it reduces to the integral operator

$$\mathcal{J}_{\gamma}^p(f_1, \dots, f_n)(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \left(-\frac{u^{p+1}}{p} f'_j(u) \right)^{\gamma_j} du. \quad (9)$$

(iii) If $\gamma = (0, \dots, 0)$, then it reduces to the integral operator

$$\mathcal{E}_{\delta}^p(f_1, \dots, f_n)(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n (u^p f_j(u))^{\delta_j} du. \quad (10)$$

(iv) For $n = 1$, $\gamma_1 = \gamma$, $\delta_1 = \delta$, and $f_1 = f$, it reduces to the new integral operator

$$\mathcal{J}_{\gamma, \delta}^p(f)(z) = \frac{1}{z^{p+1}} \int_0^z \left(-\frac{u^{p+1}}{p} f'(u) \right)^{\gamma} (u^p f(u))^{\delta} du. \quad (11)$$

The integral operators defined in (ii) and (iii) are introduced and studied by Mohammed and Darus [3] (also the case $p = 1$ in (ii) and (iii) is defined in [4] and [12], resp.). Furthermore, the case $p = 1$ for the integral operator in (iv) is defined by Bulut and Goswami [2]. For the recent developments on meromorphic functions, refer to [5–10].

For the starlikeness of the integral operator $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)$ defined in Definition 1, we need to use the following lemma.

Lemma 3 (see [11]). *Let $\psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfy the following condition:*

$$\Re \{ \psi(is, t) \} \leq 0, \quad \left(s, t \in \mathbb{R}; t \leq -\frac{|a + is|^2}{2} \right). \quad (12)$$

If the function $h(z) = a + h_1 z + h_2 z^2 + \dots$, where $\Re(a) > 0$, is analytic in $\mathbb{U}$ and

$$\Re \left\{ \psi \left(h(z), zh'(z) \right) \right\} > 0 \quad (z \in \mathbb{U}), \quad (13)$$

then

$$\Re \{ h(z) \} > 0 \quad (z \in \mathbb{U}). \quad (14)$$

2. Starlikeness of the Integral Operator $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)$

In this section, we investigate sufficient conditions for the meromorphically starlikeness of the integral operator $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)$ which is defined in Definition 1.

Theorem 4. *Let $f_j \in \Sigma_p$, $\gamma_j, \delta_j \in \mathbb{R}_+ \cup \{0\}$ for all $j = 1, 2, \dots, n$ and $\gamma = (\gamma_1, \dots, \gamma_n)$, $\delta = (\delta_1, \dots, \delta_n)$. If*

$$\Re \left\{ -\gamma_j \frac{z f_j''(z)}{f_j'(z)} - \delta_j \frac{z f_j'(z)}{f_j(z)} \right\} > -\frac{p}{n} + (p+1) \gamma_j + p \delta_j \quad (15)$$

for all $j = 1, 2, \dots, n$, then the general integral operator $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)$ defined in Definition 1 belongs to the meromorphic starlike function class $\mathcal{MS}_p^*$.

Proof. For the sake of simplicity, in the proof, we shall write $\mathcal{F}(z)$ instead of $\mathcal{J}_{\gamma, \delta}^p(f_1, \dots, f_n)(z)$. From (7), it is easy to see that

$$\begin{aligned} z^{p+1} \mathcal{F}'(z) + (p+1) z^p \mathcal{F}(z) \\ = \prod_{j=1}^n \left(-\frac{z^{p+1} f_j'(z)}{p} \right)^{\gamma_j} (z^p f_j(z))^{\delta_j}. \end{aligned} \quad (16)$$

Differentiating the above equality with respect to z , we obtain

$$\begin{aligned} z^{p+1} \mathcal{F}''(z) + 2(p+1) z^p \mathcal{F}'(z) + p(p+1) z^{p-1} \mathcal{F}(z) \\ = \sum_{j=1}^n \left\{ \gamma_j \left(-\frac{z^{p+1} f_j'(z)}{p} \right)^{\gamma_j} (z^p f_j(z))^{\delta_j} \right. \\ \times \left(\frac{p+1}{z} + \frac{f_j''(z)}{f_j'(z)} \right) + \delta_j \left(-\frac{z^{p+1} f_j'(z)}{p} \right)^{\gamma_j} \\ \times (z^p f_j(z))^{\delta_j} \left(\frac{p}{z} + \frac{f_j'(z)}{f_j(z)} \right) \left. \right\} \\ \times \prod_{\substack{s=1 \\ (s \neq j)}}^n \left(-\frac{z^{p+1} f_s'(z)}{p} \right)^{\gamma_s} (z^p f_s(z))^{\delta_s}. \end{aligned} \quad (17)$$

From (16) and (17), we get

$$\begin{aligned} \frac{z^{p+1} \mathcal{F}''(z) + 2(p+1) z^p \mathcal{F}'(z) + p(p+1) z^{p-1} \mathcal{F}(z)}{z^{p+1} \mathcal{F}'(z) + (p+1) z^p \mathcal{F}(z)} \\ = \sum_{j=1}^n \left\{ \gamma_j \left(\frac{p+1}{z} + \frac{f_j''(z)}{f_j'(z)} \right) + \delta_j \left(\frac{p}{z} + \frac{f_j'(z)}{f_j(z)} \right) \right\} \end{aligned} \quad (18)$$

or equivalently

$$\begin{aligned} & \frac{z^2 \mathcal{F}''(z) + 2(p+1)z\mathcal{F}'(z) + p(p+1)\mathcal{F}(z)}{z\mathcal{F}'(z) + (p+1)\mathcal{F}(z)} \\ &= \sum_{j=1}^n \left\{ \gamma_j \left(p+1 + \frac{zf_j''(z)}{f_j'(z)} \right) + \delta_j \left(p + \frac{zf_j'(z)}{f_j(z)} \right) \right\}. \end{aligned} \quad (19)$$

After some calculations, we have

$$\begin{aligned} & - \frac{z^2 \mathcal{F}''(z) + (p+2)z\mathcal{F}'(z)}{z\mathcal{F}'(z) + (p+1)\mathcal{F}(z)} \\ &= p + \sum_{j=1}^n \left\{ \gamma_j \left(-p-1 - \frac{zf_j''(z)}{f_j'(z)} \right) \right. \\ & \quad \left. + \delta_j \left(-p - \frac{zf_j'(z)}{f_j(z)} \right) \right\}. \end{aligned} \quad (20)$$

We can write the left-hand side of (20) as the following:

$$\begin{aligned} & - \frac{(z\mathcal{F}'(z)/\mathcal{F}(z))((z\mathcal{F}''(z)/\mathcal{F}'(z)) + p+2)}{(z\mathcal{F}'(z)/\mathcal{F}(z)) + p+1} \\ &= p + \sum_{j=1}^n \left\{ \gamma_j \left(-p-1 - \frac{zf_j''(z)}{f_j'(z)} \right) \right. \\ & \quad \left. + \delta_j \left(-p - \frac{zf_j'(z)}{f_j(z)} \right) \right\}. \end{aligned} \quad (21)$$

Now we define a regular function $h(z)$ by

$$h(z) = -\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}, \quad (22)$$

and $h(0) = p$. Differentiating (22) logarithmically, we obtain

$$-h(z) + \frac{zh'(z)}{h(z)} = 1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}. \quad (23)$$

From (21), (22), and (23), we have

$$\begin{aligned} & h(z) + \frac{zh'(z)}{-h(z) + p+1} \\ &= p + \sum_{j=1}^n \left\{ \gamma_j \left(-p-1 - \frac{zf_j''(z)}{f_j'(z)} \right) \right. \\ & \quad \left. + \delta_j \left(-p - \frac{zf_j'(z)}{f_j(z)} \right) \right\}. \end{aligned} \quad (24)$$

Let us put

$$\psi(u, v) = u + \frac{v}{-u + p+1}. \quad (25)$$

From (15), (24) and (25), we get

$$\begin{aligned} & \Re \left\{ \psi(h(z), zh'(z)) \right\} \\ &= p + \sum_{j=1}^n \left\{ \gamma_j \left(-p-1 - \Re \left\{ \frac{zf_j''(z)}{f_j'(z)} \right\} \right) \right. \\ & \quad \left. + \delta_j \left(-p - \Re \left\{ \frac{zf_j'(z)}{f_j(z)} \right\} \right) \right\} \\ &= p - \sum_{j=1}^n ((p+1)\gamma_j + p\delta_j) \\ & \quad + \sum_{j=1}^n \Re \left\{ -\gamma_j \frac{zf_j''(z)}{f_j'(z)} - \delta_j \frac{zf_j'(z)}{f_j(z)} \right\} \\ &> p - \sum_{j=1}^n ((p+1)\gamma_j + p\delta_j) \\ & \quad + \sum_{j=1}^n \left(-\frac{p}{n} + (p+1)\gamma_j + p\delta_j \right) = 0. \end{aligned} \quad (26)$$

Now we have to show that

$$\Re \{ \psi(is, t) \} \leq 0, \quad \left(s, t \in \mathbb{R}; \quad t \leq -\frac{p^2 + s^2}{2} \right). \quad (27)$$

From (25), we have

$$\begin{aligned} \Re \{ \psi(is, t) \} &= \Re \left\{ is + \frac{t}{-is + p+1} \right\} \\ &= \frac{t(p+1)}{s^2 + (p+1)^2} \\ &\leq -\frac{(p^2 + s^2)(p+1)}{2(s^2 + (p+1)^2)} < 0. \end{aligned} \quad (28)$$

Thus using Lemma 3, and from (26) and (28), we conclude that $\Re\{h(z)\} > 0$, and so

$$-\Re \left\{ \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \right\} > 0, \quad (29)$$

that is, the integral operator $\mathcal{F} = \mathcal{F}_{\gamma, \delta}^p(f_1, \dots, f_n)$ is a meromorphic multivalent starlike function. $\square$

Putting $p = 1$ in Theorem 4, we get the following Corollary.

Corollary 5 (see [2, Theorem 2.1]). Let $f_j \in \Sigma, \gamma_j, \delta_j \in \mathbb{R}_+ \cup \{0\}$ for all $j = 1, 2, \dots, n$ and $\gamma = (\gamma_1, \dots, \gamma_n), \delta = (\delta_1, \dots, \delta_n)$. If

$$\Re \left\{ -\gamma_j \frac{zf_j''(z)}{f_j'(z)} - \delta_j \frac{zf_j'(z)}{f_j(z)} \right\} > -\frac{1}{n} + 2\gamma_j + \delta_j \quad (30)$$

for all $j = 1, 2, \dots, n$, then the general integral operator $\mathcal{J}_{\gamma, \delta}(f_1, \dots, f_n)$ defined in (8) belongs to the meromorphically starlike function class $\mathcal{MS}^*$.

Taking $\delta = (0, \dots, 0)$ in Theorem 4, we get the following.

Corollary 6. Let $f_j \in \Sigma_p, \gamma_j > 0$ for all $j = 1, 2, \dots, n$ and $\gamma = (\gamma_1, \dots, \gamma_n)$. If

$$\Re \left\{ -\gamma_j \frac{zf_j''(z)}{f_j'(z)} \right\} > -\frac{p}{n} + (p+1)\gamma_j \quad (31)$$

for all $j = 1, 2, \dots, n$, then the general integral operator $\mathcal{J}_\gamma^p(f_1, \dots, f_n)$ defined in (9) belongs to the meromorphically starlike function class $\mathcal{MS}_p^*$.

Remark 7. If we set $p = 1$ in Corollary 6, then we have [12, Theorem 2.1].

Next, putting $\gamma = (0, \dots, 0)$ in Theorem 4, we get the following.

Corollary 8. Let $f_j \in \Sigma_p, \delta_j > 0$ for all $j = 1, 2, \dots, n$ and $\delta = (\delta_1, \dots, \delta_n)$. If

$$\Re \left\{ -\delta_j \frac{zf_j'(z)}{f_j(z)} \right\} > -\frac{p}{n} + p\delta_j \quad (32)$$

for all $j = 1, 2, \dots, n$, then the general integral operator $\mathcal{E}_\delta^p(f_1, \dots, f_n)$ defined in (10) belongs to the meromorphically starlike function class $\mathcal{MS}_p^*$.

Further taking $n = 1, \gamma_1 = \gamma, \delta_1 = \delta$, and $f_1 = f$ in Theorem 4, we have the following.

Corollary 9. Let $f \in \Sigma_p$ and $\gamma, \delta > 0$. If

$$\Re \left\{ -\gamma \frac{zf''(z)}{f'(z)} - \delta \frac{zf'(z)}{f(z)} \right\} > -p + (p+1)\gamma + p\delta, \quad (33)$$

then the general integral operator $\mathcal{J}_{\gamma, \delta}^p(f)$ defined in (11) belongs to the meromorphically starlike function class $\mathcal{MS}_p^*$.

References

- [1] B. A. Frasin, "Order of convexity and univalence of general integral operator," *Journal of the Franklin Institute*, vol. 348, no. 6, pp. 1013–1019, 2011.
- [2] S. Bulut and P. Goswami, "Starlikeness properties of general integral operator for meromorphic univalent functions," *Southeast Asian Bulletin of Mathematics*, vol. 37, 2013.
- [3] A. Mohammed and M. Darus, "The order of starlikeness of new p -valent meromorphic functions," *International Journal of Mathematical Analysis*, vol. 6, no. 25–28, pp. 1329–1340, 2012.
- [4] A. Mohammed and M. Darus, "A new integral operator for meromorphic functions," *Acta Universitatis Apulensis. Mathematics. Informatics*, no. 24, pp. 231–238, 2010.
- [5] A. Mohammed and M. Darus, "New criteria for meromorphic functions," *International Journal of Applied Mathematics & Statistics*, vol. 33, no. 3, 2013.
- [6] Y. Sun, W.-P. Kuang, and Z.-G. Wang, "On meromorphic starlike functions of reciprocal order α ," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, no. 2, pp. 469–477, 2012.
- [7] Z.-G. Wang, Z.-H. Liu, and A. Catas, "On neighborhoods and partial sums of certain meromorphic multivalent functions," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 864–868, 2011.
- [8] Z.-G. Wang, Z.-H. Liu, and R.-G. Xiang, "Some criteria for meromorphic multivalent starlike functions," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 1107–1111, 2011.
- [9] Z.-G. Wang, Y. Sun, and N. Xu, "Some properties of certain meromorphic close-to-convex functions," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 454–460, 2012.
- [10] Z.-G. Wang, Y. Sun, and Z.-H. Zhang, "Certain classes of meromorphic multivalent functions," *Computers and Mathematics with Applications*, vol. 58, no. 7, pp. 1408–1417, 2009.
- [11] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.
- [12] A. Mohammed and M. Darus, "Starlikeness properties of a new integral operator for meromorphic functions," *Journal of Applied Mathematics*, vol. 2011, Article ID 804150, 8 pages, 2011.