Research Article

Fekete-Szegő Inequalities for Starlike Functions with respect to k -Symmetric Points of Complex Order

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Received 5 May 2013; Accepted 27 November 2013; Published 19 March 2014

Academic Editor: J. Dziok

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Sharp upper bounds of $|a_3 - \mu a_2^2|$ for the function $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ belonging to certain subclass of starlike functions with respect to k -symmetric points of complex order are obtained. Also, applications of our results to certain functions defined through convolution with a normalized analytic function are given. In particular, Fekete-Szegő inequalities for certain classes of functions defined through fractional derivatives are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the following form:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad (z \in U = \{z : z \in \mathbb{C}, |z| < 1\}). \quad (1)$$

And let $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions.

Let $g(z) \in \mathcal{S}$ be given by the following:

$$g(z) = z + \sum_{m=2}^{\infty} g_m z^m. \quad (2)$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m = (g * f)(z). \quad (3)$$

If f and g are analytic functions in U , we say that f is subordinate to g , written $f < g$ if there exists a Schwarz function w , which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U , then we have the following equivalence (see [1, 2]):

$$f(z) < g(z) \iff f(0) = g(0), \quad f(U) \subset g(U). \quad (4)$$

Sakaguchi [3] introduced a class S_s^* of functions starlike with respect to symmetric points, which consists of functions $f(z) \in \mathcal{S}$ satisfying the inequality

$$\operatorname{Re} \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in U). \quad (5)$$

Chand and Singh [4] introduced a class $S_s^{(k)}$ of functions starlike with respect to k -symmetric points, which consists of functions $f(z) \in \mathcal{S}$ satisfying the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} > 0 \quad (z \in U), \quad (6)$$

where

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu} f(\epsilon^{\nu} z) \quad (k \in \mathbb{N}; \epsilon^k = 1). \quad (7)$$

Al-Shaqsi and Darus [5] defined the linear operator $D_\lambda^n f(z) : \mathcal{A} \rightarrow \mathcal{A}$ as follows:

$$\begin{aligned} D_\lambda^0 f(z) &= (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \\ D_\lambda^1 f(z) &= (1 - \lambda) z f'(z) + \lambda z (z f'(z))' \quad (\lambda \geq 0), \\ D_\lambda^2 f(z) &= \frac{z}{2!} \left[(1 - \lambda) (z f'(z))'' + \lambda (z (z f'(z)))'' \right] \\ &= \frac{z}{2!} [z (D_\lambda f(z))]' \quad (\lambda \geq 0), \\ &\vdots \\ D_\lambda^n f(z) &= \frac{z}{n!} [z^{n-1} (D_\lambda f(z))]^{(n)} \quad (\lambda \geq 0; n \in \mathbb{N}), \end{aligned} \quad (8)$$

and in general

$$\begin{aligned} D_\lambda^n f(z) &= z + \sum_{m=2}^{\infty} [1 + \lambda(m-1)] C(n, m) a_m z^m \\ &\quad (\lambda \geq 0; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, \dots\}), \end{aligned} \quad (9)$$

where

$$C(n, m) = \binom{m+n-1}{n}, \quad m = 2, 3, 4, \dots \quad (10)$$

In this paper, we define the following class $S_s^{(k)}(b, \lambda, n; \Phi)$ ($b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$) as follows.

Definition 1. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ be univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis. Let $b \neq 0$ be a complex number and let $B_1 > 0$. Then functions $f(z) \in \mathcal{A}$ are in the class $S_s^{(k)}(b, \lambda, n; \Phi)$ if

$$\begin{aligned} 1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1 \right) &< \Phi(z) \\ (b \in \mathbb{C}^*; n \in \mathbb{N}_0; \lambda \geq 0; z \in U), \end{aligned} \quad (11)$$

where $D_\lambda^n f(z)$ is defined by (9) and $f_k(z)$ is defined by (7).

We note that for suitable choices of b, k, n, λ , and $\Phi(z)$ we obtain the following subclasses:

- (i) $S_s^{(k)}(1, 0, 0; \Phi) = S_s^{(k)}(\Phi)$ (see Al-Shaqsi and Darus [6]),
- (ii) $S_s^{(k)}(1, 0, 0; (1 + Az)/(1 + Bz)) = S_s^{(k)}[A, B]$ ($-1 \leq B < A \leq 1$) (see Al-Shaqsi and Darus [6]),
- (iii) $S_s^{(k)}(1, 0, 0; (1 + (1 - 2\alpha)z)/(1 - z)) = S_s^{(k)}(\alpha)$ ($0 \leq \alpha < 1$) (see Al-Shaqsi and Darus [6]),
- (iv) $S_s^{(2)}(1, 0, 0; \Phi) = S_s^*(\Phi)$ (see Shanmugam et al. [7]),
- (v) $S_s^{(2)}(1, 0, 0; (1 + z)/(1 - z)) = S_s^*$ (see Sakaguchi [3]),

(vi) $S_s^{(1)}(1, \lambda, n; \Phi) = M_\lambda^n(\Phi)$ (see Shanthi et al. [8] and Al-Shaqsi and Darus [9]),

(vii) $S_s^{(1)}(1, 0, 0; \Phi) = S^*(\Phi)$ (see Ma and Minda [10]),

(viii) $S_s^{(1)}(1, 0, 0; (1 + Az)/(1 + Bz)) = S^*[A, B]$ ($-1 \leq B < A \leq 1$) (see Janowski [11]),

(ix) $S_s^{(1)}(b, 0, 0; \Phi) = S_b^*(\Phi)$ and $S_s^{(1)}(b, 0, 1; \Phi) = C_b(\Phi)$ (see Ravichandran et al. [12]),

(x) $S_s^{(1)}(b, 0, 0; (1 + z)/(1 - z)) = S(b)$ ($b \in \mathbb{C}^*$) (see Nasr and Aouf [13]),

(xi) $S_s^{(1)}(b, 0, 1; (1 + z)/(1 - z)) = C(b)$ ($b \in \mathbb{C}^*$) (see Nasr and Aouf [14] and Aouf et al. [15]),

(xii) $S_s^{(1)}((1 - \rho)e^{-i\lambda} \cos \lambda, 0, 0; (1 + z)/(1 - z)) = S^\lambda(\rho)$ ($|\lambda| < \pi/2; 0 \leq \rho < 1$) (see Libera [16]),

(xiii) $S_s^{(1)}((1 - \rho)e^{-i\lambda} \cos \lambda, 0, 1; (1 + z)/(1 - z)) = C^\lambda(\rho)$ ($|\lambda| < \pi/2; 0 \leq \rho < 1$) (see Chichra [17]),

(xiv) $S_s^{(1)}((1 - \rho)e^{-i\lambda} \cos \lambda, 0, 0; \Phi) = S_\rho^\lambda(\Phi)$ and $S_s^{(1)}((1 - \rho)e^{-i\lambda} \cos \lambda, 0, 1; \Phi) = C_\rho^\lambda(\Phi)$ ($|\lambda| < \pi/2; 0 \leq \rho < 1$) (see Aouf and Silverman [18]),

(xv) $S_s^{(1)}(e^{-i\gamma} \cos \gamma, 0, 0; (1 + (1 - 2\delta)z)/(1 - z)) = S^\gamma(\delta)$ ($|\gamma| < \pi/2, 0 \leq \delta < 1$) (see Keogh and Markes [19]).

Also, we note the following:

- (i) $S_s^{(1)}(b, \lambda, n; \Phi)$
 $= M_\lambda^n(b; \Phi)$
 $= \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1 \right) < \Phi(z) \right.$
 $\left. (b \in \mathbb{C}^*; n \in \mathbb{N}_0; \lambda \geq 0; z \in U) \right\};$
- (ii) $S_s^{(1)}((1 - \rho)e^{-i\alpha} \cos \alpha, \lambda, n; \Phi)$
 $= M_\lambda^n(\rho, \alpha; \Phi)$
 $= \left\{ f(z) \in \mathcal{A} : \left(e^{i\alpha} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} \right) - \rho \cos \alpha - i \sin \alpha \right) \right.$

$$\begin{aligned}
& \times ((1 - \rho) \cos \alpha)^{-1} < \Phi(z) \\
& \left(|\alpha| < \frac{\pi}{2}; \quad 0 \leq \rho < 1; \quad n \in \mathbb{N}_0; \right. \\
& \left. \lambda \geq 0; \quad z \in U \right) \Bigg\}; \\
\text{(iii)} \quad & S_s^{(2)}(b, \lambda, n; \Phi) \\
& = S_s^*(b, \lambda, n; \Phi) \\
& = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \left(\frac{2z(D_\lambda^n f(z))'}{D_\lambda^n f(z) - D_\lambda^n f(-z)} \right. \right. \\
& \quad \left. \left. - 1 \right) < \Phi(z) \right. \\
& \quad \left. (b \in \mathbb{C}^*; \quad n \in \mathbb{N}_0; \quad \lambda \geq 0; \quad z \in U) \right\}; \\
\text{(iv)} \quad & S_s^{(2)}((1 - \rho)e^{-i\alpha} \cos \alpha, \lambda, n; \Phi) \\
& = S_s^*(\rho, \alpha, \lambda, n; \Phi) \\
& = \left\{ f(z) \in \mathcal{A} : \left(\left(\frac{2e^{i\alpha} z(D_\lambda^n f(z))'}{D_\lambda^n f(z) - D_\lambda^n f(-z)} \right) \right. \right. \\
& \quad \left. \left. - \rho \cos \alpha - i \sin \alpha \right) \right. \\
& \quad \left. \times ((1 - \rho) \cos \alpha)^{-1} < \Phi(z) \right. \\
& \quad \left(|\alpha| < \frac{\pi}{2}; \quad 0 \leq \rho < 1; \quad n \in \mathbb{N}_0; \right. \\
& \quad \left. \lambda \geq 0; \quad z \in U \right) \Bigg\}; \\
\text{(v)} \quad & S_s^{(k)}(b, 0, 0; \Phi) \\
& = S_s^{(k)}(b; \Phi) \\
& = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b} \left(\frac{zf'(z)}{f_k(z)} - 1 \right) < \Phi(z) \right. \\
& \quad \left. (b \in \mathbb{C}; \quad z \in U) \right\}; \\
\text{(vi)} \quad & S_s^{(k)}((1 - \rho)e^{-i\alpha} \cos \alpha, 0, 0; \Phi) \\
& = S_s^{(k)}(\rho, \alpha; \Phi) \\
& = \left\{ f(z) \in \mathcal{A} : \left(e^{i\alpha} \left(\frac{zf'(z)}{f_k(z)} \right) - \rho \cos \alpha - i \sin \alpha \right) \right.
\end{aligned}$$

$$\begin{aligned}
& \times ((1 - \rho) \cos \alpha)^{-1} < \Phi(z) \\
& \left(|\alpha| < \frac{\pi}{2}; \quad 0 \leq \rho < 1; \quad z \in U \right) \Bigg\}.
\end{aligned} \tag{12}$$

In this paper, we obtain the Fekete-Szegő inequalities for the functions in the class $S_s^{(k)}(b, \lambda, n; \Phi)$. We also give application of our results to certain functions defined through convolution and, in particular, we consider the class $S_s^{(k),\gamma}(b, \lambda, n; \Phi)$ defined by fractional derivatives.

2. Fekete-Szegő Problem

To prove our results, we need the following lemmas.

Lemma 2 (see [10]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function with positive real part in U and ν is a complex number, then*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}. \tag{13}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}. \tag{14}$$

Lemma 3 (see [10]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function with positive real part in U , then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases} \tag{15}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\vartheta \right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta \right) \frac{1-z}{1+z} \quad (0 \leq \vartheta \leq 1) \tag{16}$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{1}{2}\vartheta \right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta \right) \frac{1-z}{1+z} \quad (0 \leq \vartheta \leq 1). \tag{17}$$

Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \quad \left(0 < \nu < \frac{1}{2} \right), \tag{18}$$

$$|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad \left(\frac{1}{2} < \nu < 1 \right). \tag{19}$$

Theorem 4. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $S_s^{(k)}(b, \lambda, n; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{2|b|B_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{bB_1\psi_2}{(2-\psi_2)} \right. \right. \\ &\quad - \left((\mu b B_1 (1+2\lambda)(n+2) \right. \\ &\quad \times (3-\psi_3)) \\ &\quad \times (2(1+\lambda)^2 \\ &\quad \times (n+1)(2-\psi_2)^2)^{-1} \left. \right) \left. \right\}, \end{aligned} \quad (20)$$

where

$$\psi_m = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{(m-1)\nu}, \quad (k \in \mathbb{N}; m \geq 2; \epsilon^k = 1). \quad (21)$$

This result is sharp.

Proof. Let $f(z) \in S_s^{(k)}(b, \lambda, n; \Phi)$; then there is a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1 \right) = \Phi(w(z)). \quad (22)$$

If the function $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \dots \quad (23)$$

Since $w(z)$ is a Schwarz function, define

$$\begin{aligned} p(z) &= 1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1 \right) \\ &= 1 + d_1 z + d_2 z^2 + \dots \\ &\quad (z \in U). \end{aligned} \quad (24)$$

From (7), we obtain

$$\begin{aligned} f_k(z) &= \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu} f(\epsilon^\nu z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu} \left[\epsilon^\nu z + \sum_{m=2}^{\infty} a_m (\epsilon^\nu z)^m \right] \\ &= z + \sum_{m=2}^{\infty} a_m \psi_m z^m, \end{aligned} \quad (25)$$

where ψ_m is given by (21). In view of (22) and (23), we have

$$p(z) = \Phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right). \quad (26)$$

Since

$$\begin{aligned} \frac{p_1(z) - 1}{p_1(z) + 1} &= \frac{1}{2} \left[c_1 z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2 \right) z^3 + \dots \right], \end{aligned} \quad (27)$$

therefore, we have

$$\begin{aligned} \Phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) &= 1 + \frac{1}{2} B_1 c_1 z \\ &\quad + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \dots, \end{aligned} \quad (28)$$

and, from this equation and (28), we obtain

$$d_1 = \frac{1}{2} B_1 c_1, \quad d_2 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \quad (29)$$

Then, from (24), we see that

$$\begin{aligned} d_1 &= \frac{(1+\lambda)(n+1)(2-\psi_2)a_2}{b}, \\ d_2 &= ((1+2\lambda)(n+1)(n+2)(3-\psi_3)a_3 \\ &\quad - 2(1+\lambda)^2(n+1)^2(2-\psi_2)\psi_2 a_2^2) \times (2b)^{-1}. \end{aligned} \quad (30)$$

Now from (24), (28), and (30), we have

$$\begin{aligned} a_2 &= \frac{bB_1 c_1}{2(1+\lambda)(n+1)(2-\psi_2)}, \\ a_3 &= \frac{bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ &\quad \times \left\{ c_2 - \frac{c_1^2}{2} + \frac{1}{2} \frac{B_2}{B_1} c_1^2 + \frac{bB_1 \psi_2 c_1^2}{2(2-\psi_2)} \right\}. \end{aligned} \quad (31)$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \{c_2 - \mu c_1^2\}, \quad (32)$$

where

$$\mu = \frac{1}{2} \left[1 - \frac{B_2}{B_1} - \frac{bB_1 \psi_2}{(2-\psi_2)} + \frac{\mu b B_1 (1+2\lambda)(n+2)(3-\psi_3)}{2(1+\lambda)^2(n+1)(2-\psi_2)^2} \right]. \quad (33)$$

Our result now follows by an application of Lemma 2. The result is sharp for the functions

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1 \right) = \Phi(z^2), \quad (34)$$

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f_k(z)} - 1 \right) = \Phi(z).$$

This completes the proof of Theorem 4. $\square$

Remark 5. (i) Putting $b = \cos \gamma e^{-i\gamma}$ ($|\gamma| < \pi/2$), $n = \lambda = 0$, $k = 2$, and $\Phi(z) = (1 + (1 - 2\delta)z)/(1 - z)$ ($0 \leq \delta < 1$) in Theorem 4, we obtain the result obtained by Keogh and Markes [19, Theorem 1].

(ii) Putting $b = 1$ and $k = 2$ in Theorem 4, we obtain the result obtained by Shanthi et al. [8, Theorem 2.6].

(iii) Putting $b = 1$, $n = \lambda = 0$, and $k = 1$ in Theorem 4, we obtain the result obtained by Ma and Minda [10].

Putting $b = 1$ and $k = 1$ in Theorem 4, we obtain the following corollary.

Corollary 6. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$. If $f(z)$ given by (1) belongs to the class $M_\lambda^n(\Phi)$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{B_1}{(1 + 2\lambda)(n + 1)(n + 2)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} - \left(\mu(1 + 2\lambda)(n + 2) \right. \right. \right. \\ &\quad \left. \left. \left. - (1 + \lambda)^2(n + 1) \right) \times ((1 + \lambda)^2(n + 1))^{-1} B_1 \right| \right\}. \end{aligned} \quad (35)$$

This result is sharp.

Putting $b = 1$, $n = \lambda = 0$, and $k = 2$ in Theorem 4, we obtain the following corollary.

Corollary 7. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $S_s^*(\Phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{\mu B_1}{2} \right| \right\}. \quad (36)$$

This result is sharp.

Putting $k = 1$ in Theorem 4, we obtain the following corollary.

Corollary 8. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $M_\lambda^n(b; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{2|b|B_1}{(1 + 2\lambda)(n + 1)(n + 2)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} - \left(\mu b(1 + 2\lambda)(n + 2) \right. \right. \right. \\ &\quad \left. \left. \left. - b(1 + \lambda)^2(n + 1) \right) \times ((1 + \lambda)^2(n + 1))^{-1} B_1 \right| \right\}. \end{aligned} \quad (37)$$

This result is sharp.

Putting $b = (1 - \rho)e^{-i\alpha}$ ($|\alpha| < \pi/2$; $0 \leq \rho < 1$) in Corollary 8, we obtain the following corollary.

Corollary 9. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $M_\lambda^n(\rho, \alpha; \Phi)$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{2B_1(1 - \rho)\cos \alpha}{(1 + 2\lambda)(n + 1)(n + 2)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} - \left(\mu(1 + 2\lambda)(n + 2) \right. \right. \right. \\ &\quad \left. \left. \left. - (1 + \lambda)^2(n + 1) \right) \times ((1 + \lambda)^2(n + 1))^{-1} B_1 \right. \right. \\ &\quad \left. \left. \times (1 - \rho)e^{-i\alpha}\cos \alpha \right| \right\}. \end{aligned} \quad (38)$$

This result is sharp.

Putting $n = \lambda = 0$ in Theorem 4, we obtain the following corollary.

Corollary 10. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $S_s^{(k)}(b; \Phi)$ and $b \in \mathbb{C}^*$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|b|B_1}{(3 - \psi_3)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{\mu(3 - \psi_3) - (2 - \psi_2)\psi_2}{(2 - \psi_2)^2} b B_1 \right| \right\}. \end{aligned} \quad (39)$$

This result is sharp.

Putting $b = (1 - \rho)e^{-i\alpha}$ ($|\alpha| < \pi/2$; $0 \leq \rho < 1$) in Corollary 10, we obtain the following corollary.

Corollary 11. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $S_s^{(k)}(\rho, \alpha; \Phi)$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{B_1(1 - \rho)\cos \alpha}{(3 - \psi_3)} \\ &\cdot \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{\mu(3 - \psi_3) - (2 - \psi_2)\psi_2}{(2 - \psi_2)^2} B_1 \right. \right. \\ &\quad \left. \left. \times (1 - \rho)e^{-i\alpha}\cos \alpha \right| \right\}. \end{aligned} \quad (40)$$

This result is sharp.

By using Lemma 3, we can obtain the following theorem.

Theorem 12. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $S_s^{(k)}(b, \lambda, n; \Phi)$ and $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2bB_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ + \frac{2b^2 B_1^2 \psi_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)} \\ - \frac{\mu b^2 B_1^2}{(1+\lambda)^2(n+1)^2(2-\psi_2)^2} \\ \text{if } \mu \leq \sigma_1, \\ \frac{2bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{2bB_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{2b^2 B_1^2 \psi_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)} \\ + \frac{\mu b^2 B_1^2}{(1+\lambda)^2(n+1)^2(2-\psi_2)^2} \\ \text{if } \mu \geq \sigma_2, \end{cases} \quad (41)$$

where

$$\sigma_1 = \frac{2(2-\psi_2)(1+\lambda)^2(n+1)[(B_2-B_1)(2-\psi_2)+bB_1^2\psi_2]}{bB_1^2(1+2\lambda)(n+2)(3-\psi_3)},$$

$$\sigma_2 = \frac{2(2-\psi_2)(1+\lambda)^2(n+1)[(B_2+B_1)(2-\psi_2)+bB_1^2\psi_2]}{bB_1^2(1+2\lambda)(n+2)(3-\psi_3)}, \quad (42)$$

and ψ_m is given by (21). The result is sharp.

Proof. To show that the bounds are sharp, we define the functions K_{φ_δ} ($\delta \geq 2$) by

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n K_{\varphi_\delta}(z))'}{D_\lambda^n K_{\varphi_\delta}(z)} - 1 \right) = \Phi(z^{\delta-1}), \quad (43)$$

$$K_{\varphi_\delta}(0) = 0 = K'_{\varphi_\delta}(0) - 1,$$

and the functions F_ρ and G_ρ ($0 \leq \rho \leq 1$) by

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n F_\rho(z))'}{D_\lambda^n F_\rho(z)} - 1 \right) = \phi\left(\frac{z(z+\rho)}{1+\rho z}\right),$$

$$F_\rho(0) = 0 = F'_\rho(0) - 1, \quad (44)$$

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n G_\rho(z))'}{D_\lambda^n G_\rho(z)} - 1 \right) = \Phi\left(-\frac{z(z+\rho)}{1+\rho z}\right),$$

$$G_\rho(0) = 0 = G'_\rho(0) - 1.$$

Clearly the functions K_{φ_δ} , F_ρ , and $G_\rho \in S_s^{(k)}(b, \lambda, n; \Phi)$. Also we write $K_\varphi = K_{\varphi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K_{φ_3} or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_ρ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_ρ or one of its rotations. If $\sigma_1 \leq \mu \leq \sigma_2$, in view of Lemma 3. This completes the proof of Theorem 12. $\square$

Remark 13. (i) Putting $b = 1$, $n = 0$, and $\lambda = 0$ in Theorem 12, we obtain the result obtained by Al-Shaqsi and Darus [6, Theorem 2.1].

(ii) Putting $b = 1$, $n = 0$, $\lambda = 0$, and $k = 2$ in Theorem 12, we obtain the result obtained by Shanmugam et al. [7, Theorem 2.1].

(iii) Putting $b = 1$ and $k = 2$ in Theorem 12, we obtain the result obtained by Al-Shaqsi and Darus [9, Theorem 2.1].

(iv) Putting $b = 1$, $n = \lambda = 0$, and $k = 1$ in Theorem 12, we obtain the result obtained by Ma and Minda [10].

Putting $k = 1$ in Theorem 12, we obtain the following corollary.

Corollary 14. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) belongs to the class $M_\lambda^n(b; \Phi)$ and if $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{bB_2}{(1+2\lambda)(n+1)(n+2)} \\ + \frac{b^2 B_1^2}{(1+2\lambda)(n+1)(n+2)} \\ - \frac{\mu b^2 B_1^2}{(1+\lambda)^2(n+1)^2} \\ \text{if } \mu \leq \sigma_1^*, \\ \frac{bB_1}{(1+2\lambda)(n+1)(n+2)} \\ - \frac{bB_2}{(1+2\lambda)(n+1)(n+2)} \\ - \frac{b^2 B_1^2}{(1+2\lambda)(n+1)(n+2)} \\ + \frac{\mu b^2 B_1^2}{(1+\lambda)^2(n+1)^2} \\ \text{if } \mu \geq \sigma_2^*, \end{cases} \quad (45)$$

where

$$\sigma_1^* = \frac{(1+\lambda)^2(n+1)[B_2 - B_1 + bB_1^2]}{bB_1^2(1+2\lambda)(n+2)},$$

$$\sigma_2^* = \frac{(1+\lambda)^2(n+1)[B_2 + B_1 + bB_1^2]}{bB_1^2(1+2\lambda)(n+2)}. \quad (46)$$

The result is sharp.

The proof of Theorem 15 is similar to the proof of Theorem 12, so the details are omitted.

Theorem 15. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ ($B_1 > 0$). Let $f(z)$ given by (1) belong to the class $S_s^{(k)}(b, \lambda, n; \Phi)$ and $b > 0$. Let σ_3 be given by

$$\sigma_3 = \frac{2(1+\lambda)^2(n+1)(2-\psi_2)^2}{bB_1(1+2\lambda)(n+2)(3-\psi_3)} \left[\frac{B_2}{B_1} + \frac{bB_1\psi_2}{(2-\psi_2)} \right], \quad (47)$$

where ψ_m is given by (21). If $\sigma_1 < \mu \leq \sigma_3$, then

$$\begin{aligned} & \left| a_3 - \mu a_2^2 \right| + \frac{2(1+\lambda)^2(n+1)(2-\psi_2)^2}{bB_1(1+2\lambda)(n+2)(3-\psi_3)} \\ & \times \left[1 - \frac{B_2}{B_1} - \frac{bB_1\psi_2}{(2-\psi_2)} \right. \\ & \quad \left. + \frac{\mu bB_1(1+2\lambda)(n+2)(3-\psi_3)}{2(1+\lambda)^2(n+1)(2-\psi_2)^2} \right] |a_2|^2 \\ & \leq \frac{2bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)}. \end{aligned} \quad (48)$$

If $\sigma_3 < \mu \leq \sigma_2$, then

$$\begin{aligned} & \left| a_3 - \mu a_2^2 \right| + \frac{2(1+\lambda)^2(n+1)(2-\psi_2)^2}{bB_1(1+2\lambda)(n+2)(3-\psi_3)} \\ & \times \left[1 + \frac{B_2}{B_1} + \frac{bB_1\psi_2}{(2-\psi_2)} \right. \\ & \quad \left. - \frac{\mu bB_1(1+2\lambda)(n+2)(3-\psi_3)}{2(1+\lambda)^2(n+1)(2-\psi_2)^2} \right] |a_2|^2 \\ & \leq \frac{2bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)}, \end{aligned} \quad (49)$$

where σ_1 and σ_2 are given in Theorem 12.

Remark 16. (i) Putting $b = 1$, $n = 0$, and $\lambda = 0$ in Theorem 15, we obtain the result obtained by Al-Shaqsi and Darus [6, Remark 2.1].

(ii) Putting $b = 1$, $n = \lambda = 0$, and $k = 2$ in Theorem 15, we obtain the result obtained by Shanmugam et al. [7, Theorem 2.2].

(iii) Putting $b = 1$ and $k = 2$ in Theorem 15, we obtain the result obtained by Al-Shaqsi and Darus [9, Remark 2.2].

Putting $k = 1$ in Theorem 15, we obtain the following corollary.

Corollary 17. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ ($B_1 > 0$). Let $f(z)$ given by (1) belong to the class $M_\lambda^n(b; \Phi)$ and $b > 0$. Let σ_3^* be given by

$$\sigma_3^* = \frac{(1+\lambda)^2(n+1)}{bB_1^2(1+2\lambda)(n+2)} [B_2 + bB_1^2]. \quad (50)$$

If $\sigma_1^* < \mu \leq \sigma_3^*$, then

$$\begin{aligned} & \left| a_3 - \mu a_2^2 \right| + \frac{(1+\lambda)^2(n+1)}{bB_1(1+2\lambda)(n+2)} \\ & \times \left[1 - \frac{B_2}{B_1} + \frac{\mu bB_1(1+2\lambda)(n+2) - bB_1(1+\lambda)^2(n+1)}{(1+\lambda)^2(n+1)} \right] \\ & \times |a_2|^2 \leq \frac{bB_1}{(1+2\lambda)(n+1)(n+2)}. \end{aligned} \quad (51)$$

If $\sigma_3^* < \mu \leq \sigma_2^*$, then

$$\begin{aligned} & \left| a_3 - \mu a_2^2 \right| + \frac{(1+\lambda)^2(n+1)}{bB_1(1+2\lambda)(n+2)} \\ & \times \left[1 + \frac{B_2}{B_1} - \frac{\mu bB_1(1+2\lambda)(n+2) - bB_1(1+\lambda)^2(n+1)}{(1+\lambda)^2(n+1)} \right] \\ & \times |a_2|^2 \leq \frac{bB_1}{(1+2\lambda)(n+1)(n+2)}, \end{aligned} \quad (52)$$

where σ_1^* and σ_2^* are given in Corollary 14.

3. Applications to Functions Defined by Fractional Derivatives

Definition 18 (see [20]). Let $f(z)$ be analytic in a simply connected region of the z -plane containing origin. The fractional derivative of f of order γ is defined by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z (z-\zeta)^{-\gamma} f(\zeta) d\zeta \quad (0 \leq \gamma < 1), \quad (53)$$

where the multiplicity of $(z-\zeta)^{-\gamma}$ is removed by requiring that $\log(z-\zeta)$ be real for $(z-\zeta) > 0$.

Using Definition 18, Owa and Srivastava (see [21–23]) introduced a fractional derivative operator $\Omega^\gamma : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\Omega^\gamma f(z) = \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z), \quad (\gamma \neq 2, 3, 4, \dots). \quad (54)$$

Definition 19. A fixed $g(z)$ given by (2) and $f(z)$ given by (1) belong to the class $S_s^{(k),g}(b, \lambda, n; \Phi)$ if

$$1 + \frac{1}{b} \left(\frac{z(D_\lambda^n(f * g)(z))'}{D_\lambda^n(f_k * g)(z)} - 1 \right) \prec \Phi(z) \quad (55)$$

$$(b \in \mathbb{C}^*; n \in \mathbb{N}_0; \lambda \geq 0; z \in U),$$

where $\Phi(z)$ is defined in Definition 1 and D_λ^n is defined in (9).

The class $S_s^{(k),\gamma}(b, \lambda, n; \Phi)$ consists of the functions $f \in \mathcal{A}$ for which $\Omega^\gamma f \in S_s^{(k)}(b, \lambda, n; \Phi)$. The class $S_s^{(k),\gamma}(b, \lambda, n; \Phi)$ is a special case of the class $S_s^{(k),g}(b, \lambda, n; \Phi)$ when

$$g(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(2-\gamma)}{\Gamma(m+1-\gamma)} z^m \quad (z \in U). \quad (56)$$

Now applying Theorem 4 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots$, we get the following theorem.

Theorem 20. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $g(z)$ given by (2) and $f(z)$ given by (1) belong to the class $S_s^{(k),g}(b, \lambda, n; \Phi)$ and if $b \in \mathbb{C}^*$, then

$$|a_3 - \mu a_2^2| \leq \frac{2|b|B_1}{g_3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \cdot \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{bB_1\psi_2}{(2-\psi_2)} - \frac{\mu b B_1 g_3 (1+2\lambda)(n+2)(3-\psi_3)}{2g_2^2(1+\lambda)^2(n+1)(2-\psi_2)^2} \right| \right\}, \quad (57)$$

where ψ_m is given by (21). This result is sharp.

Proof. The proof of Theorem 20 is similar to the proof of Theorem 4, so the details are omitted. $\square$

Putting $b = 1$ and $n = \lambda = 0$ in Theorem 20, we obtain the following corollary.

Corollary 21. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $g(z)$ given by (2) and $f(z)$ given by (1) belong to the class $S_s^{(k),g}(\Phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{g_3(3-\psi_3)} \times \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{B_1\psi_2}{(2-\psi_2)} - \frac{\mu B_1 g_3 (3-\psi_3)}{g_2^2(2-\psi_2)^2} \right| \right\}, \quad (58)$$

where ψ_m is given by (21). This result is sharp.

Putting $b = 1$, $n = \lambda = 0$, and $k = 2$ in Theorem 20, we obtain the following corollary.

Corollary 22. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $g(z)$ given by (2) and $f(z)$ given by (1) belong to the class $S_s^{*,g}(\Phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2g_3} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{\mu g_3 B_1}{2g_2^2} \right| \right\}. \quad (59)$$

This result is sharp.

Now applying Theorem 12 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots$, we get the following theorem.

Theorem 23. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$). If $f(z)$ given by (1) and $g(z)$ given by (2), then $f(z)$ belongs to the class $S_s^{(k),g}(b, \lambda, n; \Phi)$ and $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2bB_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)g_3} + \frac{2b^2B_1^2\psi_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)g_3} - \frac{\mu b^2B_1^2}{(1+\lambda)^2(n+1)^2(2-\psi_2)^2g_2^2} & \text{if } \mu \leq \eta_1, \\ \frac{2bB_1}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)g_3} & \text{if } \eta_1 \leq \mu \leq \eta_2, \\ -\frac{2bB_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)g_3} - \frac{2b^2B_1^2\psi_2}{(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)g_3} + \frac{\mu b^2B_1^2}{(1+\lambda)^2(n+1)^2(2-\psi_2)^2g_2^2} & \text{if } \mu \geq \eta_2, \end{cases} \quad (60)$$

where

$$\begin{aligned} \eta_1 &= \left(2g_2^2(2-\psi_2)(1+\lambda)^2(n+1) \right. \\ &\quad \times \left[(B_2 - B_1)(2-\psi_2) + bB_1^2\psi_2 \right] \\ &\quad \times \left(bB_1^2g_3(1+2\lambda)(n+2)(3-\psi_3) \right)^{-1}, \\ \eta_2 &= \left(2g_2^2(2-\psi_2)(1+\lambda)^2(n+1) \right. \\ &\quad \times \left[(B_2 + B_1)(2-\psi_2) + bB_1^2\psi_2 \right] \\ &\quad \times \left(bB_1^2g_3(1+2\lambda)(n+2)(3-\psi_3) \right)^{-1}, \end{aligned} \quad (61)$$

and ψ_m is given by (21). The result is sharp.

Remark 24. (i) Putting $b = 1$, $n = 0$, and $\lambda = 0$ in Theorem 23, we obtain the result obtained by Al-Shaqsi and Darus [6, Theorem 3.1].

(ii) Putting $b = 1$, $n = \lambda = 0$, and $k = 2$ in Theorem 23, we obtain the result obtained by Shanmugam et al. [7, Theorem 3.2].

(iii) Putting $b = 1$ in Theorem 23, we obtain the result obtained by Al-Shaqsi and Darus [9, Theorem 3.2].

Since

$$\Omega^\nu f(z) = z + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(2-\nu)}{\Gamma(m+1-\nu)} a_m z^m, \quad (62)$$

we have

$$g_2 = \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}, \quad (63)$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}. \quad (64)$$

For g_2, g_3 given by (63) and (64), respectively, Theorem 23 reduces to the following theorem.

Theorem 25. Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ ($B_1 > 0$) and $\gamma < 2$. If $f(z)$ given by (1) belongs to the class $S_s^{(k),\gamma}(b, \lambda, n; \Phi)$ and $b > 0$, then

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \frac{bB_2(2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ + \frac{b^2 B_1^2 \psi_2 (2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)} \\ - \frac{\mu b^2 B_1^2 (2-\gamma)^2}{4(1+\lambda)^2(n+1)^2(2-\psi_2)^2} \\ \text{if } \mu \leq \eta_1^*, \\ \frac{bB_1(2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{bB_2(2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{b^2 B_1^2 \psi_2 (2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)} \\ + \frac{\mu b^2 B_1^2 (2-\gamma)^2}{4(1+\lambda)^2(n+1)^2(2-\psi_2)^2} \\ \text{if } \eta_1^* \leq \mu \leq \eta_2^*, \\ \frac{bB_1(2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{bB_2(2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)} \\ - \frac{b^2 B_1^2 \psi_2 (2-\gamma)(3-\gamma)}{3(1+2\lambda)(n+1)(n+2)(3-\psi_3)(2-\psi_2)} \\ + \frac{\mu b^2 B_1^2 (2-\gamma)^2}{4(1+\lambda)^2(n+1)^2(2-\psi_2)^2} \\ \text{if } \mu \geq \eta_2^*, \end{cases} \quad (65)$$

where

$$\begin{aligned} \eta_1^* &= \left(4(3-\gamma)(2-\psi_2)(1+\lambda)^2(n+1) \right. \\ &\quad \times \left[(B_2 - B_1)(2-\psi_2) + bB_1^2\psi_2 \right] \\ &\quad \times \left(3(2-\gamma)bB_1^2(1+2\lambda)(n+2)(3-\psi_3) \right)^{-1}, \\ \eta_2^* &= \left(4(3-\gamma)(2-\psi_2)(1+\lambda)^2(n+1) \right. \\ &\quad \times \left[(B_2 + B_1)(2-\psi_2) + bB_1^2\psi_2 \right] \\ &\quad \times \left(3(2-\gamma)bB_1^2(1+2\lambda)(n+2)(3-\psi_3) \right)^{-1}, \end{aligned} \quad (66)$$

and ψ_m is given by (21). The result is sharp.

Remark 26. (i) Putting $b = 1$, $n = 0$, and $\lambda = 0$ in Theorem 25, we obtain the result obtained by Al-Shaqsi and Darus [6, Theorem 3.3].

(ii) Putting $b = 1$, $n = \lambda = 0$, and $k = 2$ in Theorem 25, we obtain the result obtained by Shanmugam et al. [7, Theorem 3.4].

(iii) Putting $b = 1$ in Theorem 25, we obtain the result obtained by Al-Shaqsi and Darus [9, Theorem 3.3].

Conflict of Interests

A competing interest exists when professional judgment concerning the validity of research is influenced by a secondary interest, such as financial gain.

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