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Research Article

Theoretical Researches about $\mathfrak{u}$-Maximal Subgroups and Its Applications in Charactering $\mathrm{Int}_{\mathfrak{u}}(G)$

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Let G be a finite group and $\mathfrak{u}$ be the class of all finite supersoluble groups. A supersoluble subgroup U of G is called $\mathfrak{u}$-maximal in G if for any supersoluble subgroup V of G containing U, $V = U$. Moreover, $\mathrm{Int}_{\mathfrak{u}}(G)$ is the intersection of all $\mathfrak{u}$-maximal subgroups of G. This paper obtains some new criteria on $\mathrm{Int}_{\mathfrak{u}}(G)$, by assuming that some subgroups of G are either Φ-I-supplemented or Φ-I-embedded in G. Here, a subgroup H of G is called Φ-I-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G \leq \Phi(H/H_G)\mathrm{Int}_{\mathfrak{u}}(G)$ and Φ-I-embedded in G if there exists a S-quasinormal subgroup T of G such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq \Phi(H/H_G)\mathrm{Int}_{\mathfrak{u}}(G)$.

1. Introduction

As we know, the model of chemical substances, such as crystal, is a graph, whose change process can be represented by symmetric groups or others. Therefore, group theory plays an important role in chemistry and physics ([1, 2]). However, this paper focuses on a question in group theory, which will promote its development and, consequently, contribute chemistry and physics in many ways.

Throughout this paper, all groups are finite. G always denotes a group, p denotes a prime, and $\pi(G)$ is the set of all prime divisors of $|G|$. All unexplained notation and terminology are standard, as in [3, 4, 5].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be (1) saturated, if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ and (2) hereditary, if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$. Following ([3], Chap. III, Definition 3.1), a subgroup U of G is called $\mathfrak{F}$-maximal in G provided that (1) $U \in \mathfrak{F}$ and (2) if $U \leq V \leq G$ and $V \in \mathfrak{F}$, then $U = V$. Moreover, $\mathrm{Int}_{\mathfrak{F}}(G)$ [6] denotes the intersection of all $\mathfrak{F}$-maximal subgroups of G. As we know, the $\mathfrak{F}$-hypercentre $Z_{\mathfrak{F}}(G)$ of G is the largest normal subgroup of G such that each G-chief factor H/K below $Z_{\mathfrak{F}}(G)$ satisfies $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}$. Clearly, $Z_{\mathfrak{F}}(G) \leq \mathrm{Int}_{\mathfrak{F}}(G)$ for any group G (see ([6], Theorem C)).

Let $\mathfrak{F}$ be a hereditary saturated formation and N be a normal subgroup of G contained in $Z_{\mathfrak{F}}(G)$. Then, the following holds (1) $AN \in \mathfrak{F}$ for any subgroup A of G with $A \in \mathfrak{F}$ and (2) $T \in \mathfrak{F}$ for any subgroup T of G with $T/N \in \mathfrak{F}$. It is well known that the extensive applications of $\mathfrak{F}$-hypercentral subgroups are based on the above properties, and there are two main topics about $\mathfrak{F}$-hypercentre: (1) the influence of $\mathfrak{F}$-hypercentral subgroups on the structure of finite groups; (2) the criteria of $\mathfrak{F}$-hypercentral subgroups.

However, in [6], Theorem C shows that the above two properties still hold when $N \leq \mathrm{Int}_{\mathfrak{F}}(G)$, instead of the stronger condition $N \leq Z_{\mathfrak{F}}(G)$. Therefore, it would be rather natural and of great significance to study $\mathrm{Int}_{\mathfrak{F}}(G)$. In fact, some recent results in this topic can be found in, for example, [6, 7, 8–12]. Particularly, in [6, 10], the authors have shown that $Z_{\mathfrak{u}}(G) < \mathrm{Int}_{\mathfrak{u}}(G)$ in general, given the condition under which $Z_{\mathfrak{u}}(G) = \mathrm{Int}_{\mathfrak{u}}(G)$ holds for every group G.

In connection with the topic of $Z_{\mathfrak{u}}(G)$, a question naturally arises as follows:

Question 1. Can we give a condition under which a normal subgroup of G is contained in $\mathrm{Int}_{\mathfrak{U}}(G)$?

In [9], Chen et al. gave some conditions under which a normal subgroup of G contained in $\mathrm{Int}_{\mathfrak{U}}(G)$. In this paper, we still pay attention to Question 1. Furthermore, we explore new criteria by the help of the following notion.

Definition 1. Let H be a subgroup of a group G. Then, H is called

(1) Φ-I-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H\cap T)H_G/H_G \le \Phi(H/H_G)\mathrm{Int}_{\mathfrak{U}}(G/H_G)$

(2) Φ-I-embedded in G if there exists a S-quasinormal subgroup T of G such that HT is S-quasinormal in G and $(H\cap T)H_G/H_G \le \Phi(H/H_G)\mathrm{Int}_{\mathfrak{U}}(G/H_G)$

Our main results are the following:

Theorem 1. *Let E be a normal subgroup of G. For every prime $p\in\pi(E)$ and every noncyclic Sylow p-subgroup P of E, assume that all maximal subgroups of P are either Φ-I-supplemented or Φ-I-embedded in G. Then $E \le \mathrm{Int}_{\mathfrak{U}}(G)$.*

Theorem 2. *Let E be a normal subgroup of G. For every prime $p\in\pi(E)$ and every noncyclic Sylow P-subgroup P of E, assume that all cyclic subgroups of P with order P and 4 (when P is a nonabelian 2-group) are either Φ-I-supplemented or Φ-I-embedded in G. Then, $E \le \mathrm{Int}_{\mathfrak{U}}(G)$.*

2. Preliminaries

Lemma 1 (see [13], Chapter 1 or [4], Chapter 1, Lemmas 5.34 and 5.35]). *Assume that H is a subgroup of G, $E\le G$, and $N \trianglelefteq G$.*

(1) If H is S-quasinormal in G, then $H\cap E$ is S-quasinormal in E

(2) If H is S-quasinormal in G, then HN/N is S-quasinormal in G/N

(3) Assume that H is a p-group, then H is S-quasinormal in G if and only if $O^p(G)\le N_G(H)$

(4) The set of S-quasinormal subgroups of G is a sublattice of the subnormal subgroup lattice of G

(5) If H is S-quasinormal in G, then H^G/H_G is nilpotent

(6) If H is a π-group and H is subnormal in G, then $H\le O_\pi(G)$

Lemma 2 ([6], Theorem C). *Let $\mathfrak{F}$ be a nonempty hereditary saturated formation. Assume that H, E, and N are subgroups of G with $N\trianglelefteq G$.*

(1) $\mathrm{Int}_{\mathfrak{F}}(H)N/N \le \mathrm{Int}_{\mathfrak{F}}(HN/N)$

(2) $\mathrm{Int}_{\mathfrak{F}}(H)\cap E \le \mathrm{Int}_{\mathfrak{F}}(H\cap E)$

(3) If $H/H\cap \mathrm{Int}_{\mathfrak{F}}(G)\in\mathfrak{F}$, then $H\in\mathfrak{F}$

(4) If $N\le \mathrm{Int}_{\mathfrak{F}}(G)$, then $\mathrm{Int}_{\mathfrak{F}}(G/N) = \mathrm{Int}_{\mathfrak{F}}(G)/N$

(5) $Z_{\mathfrak{F}}(G)\le \mathrm{Int}_{\mathfrak{F}}(G)$

Lemma 3. *Assume that H is a Φ-I-supplemented (resp., Φ-I-embedded) subgroup of G.*

(1) If N is a normal subgroup of G satisfying either $N\le H$ or $(|H|,|N|)=1$, then HN/N is Φ-I-supplemented (resp., Φ-I-embedded) in G/N

(2) If K is a subgroup of G containing H, then H is Φ-I-supplemented (resp., Φ-I-embedded) in K

Proof. As the proof for Φ-I-embedded subgroups is similar, we just assume that H is Φ-I-supplemented in G. Then, G has a subnormal subgroup T such that $G = HT$ and $(H\cap T)H_G/H_G \le \Phi(H/H_G)\mathrm{Int}_{\mathfrak{U}}(G/H_G)$.

(1) Clearly, TN/N is a subnormal subgroup of G such that $G/N = HN/N\cdot TN/N$. Consider $HN\cap TN$. If $N\le H$, then $H\cap TN = (H\cap T)N$ by the modular law. Assume that $(|H|,|N|)=1$. Then $(|HN\cap T : H\cap T|, |HN\cap T : N\cap T|) = (|N\cap HT|, |H\cap NT|) = 1$, which implies that $HN\cap T = (H\cap T)(N\cap T)$. Thus, $HN\cap TN = (H\cap T)N$ in both cases. Note that $H_GN/N \le (HN/N)_{G/N} = (HN)_G/N$, and there exists the isomorphism:

$$\frac{(HN\cap TN)(HN)_G}{(HN)_G} = \frac{(H\cap T)(HN)_G}{(HN)_G} \cong \frac{(H\cap T)H_G/H_G\cdot (HN)_G/H_G}{(HN)_G/H_G}. \quad (1)$$

So from ([3], Chap. A, Theorem 9.2(e)) and Lemma 2(1), it follows that $(HN\cap TN)(HN)_G/(HN)_G \le \Phi(HN/(HN)_G)\mathrm{Int}_{\mathfrak{U}}(G/(HN)_G)$. By the definition, HN/N is Φ-I-supplemented in G/N.

(2) Assume that $T_0 = T\cap K$. Clearly, T_0 is subnormal in G and $HT_0 = HT\cap K = K$. Note that $H_G\le H_K$ and the isomorphism

$$\frac{(H\cap T)H_K}{H_K} \cong \frac{(H\cap T)H_G/H_G\cdot H_K/H_G}{H_K/H_G}. \quad (2)$$

Therefore, by ([3], Chap. A, Theorem 9.2(e)) and Lemma 2(2), $(H\cap T)H_K/H_K \le \Phi(H/H_K)\mathrm{Int}_{\mathfrak{U}}(K/H_K)$. So H is Φ-I-supplemented in K. □

Lemma 4 ([9], Lemmas 2.3 and 2.8).

(1) Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then, $\mathrm{Int}_{\mathfrak{U}}(G) \le \mathrm{Int}_{\mathfrak{N}^p}(G)$, where $\mathfrak{N}^p$ denotes the class of all p-nilpotent groups.

(2) Assume that $\mathfrak{F}$ is a nonempty hereditary saturated formation. Let N_1, N_2 be normal subgroups of G, $I_1/N_1 = \mathrm{Int}_{\mathfrak{F}}(G/N_1)$ and $I_2/N_2 = \mathrm{Int}_{\mathfrak{F}}(G/N_2)$. Then, $I_1\cap I_2/(N_1\cap N_2) = \mathrm{Int}_{\mathfrak{F}}(G/(N_1\cap N_2))$.

Lemma 5.

(1) If T is a subnormal subgroup of G such that $|G:T|$ is a power of p, then $O^p(G)\le T$([13], Lemma 1.1.11).

(2) Assume that N is a minimal normal subgroup of G. Then, $N \leq O^p(G)$ or $|N| = p$.

Proof. If $N \nleq O^p(G)$, then $N \cap O^p(G) = 1$. Note that $NO^p(G)/O^p(G)$ is a minimal normal subgroup of the p-group $G/O^p(G)$. So the G-isomorphism $N \cong NO^p(G)/O^p(G)$ shows that $|N| = p$. □

Lemma 6 ([3], Chap. A, Lemma 8.4). *Let N be a nilpotent normal subgroup of G and M a maximal subgroup of G satisfying $N \nleq M$. Then, $N \cap M$ is a normal subgroup of G.*

Lemma 7 ([5], Chap. VI, Theorem 4.7). *Let P be a Sylow p-subgroup of G and N a normal subgroup of G. If $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

3. Proofs of Main Theorems

The following two propositions are main steps in the proof of Theorems 1 and 2, which also have independent meanings (see Corollaries 1 and 2).

Proposition 1. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Assume that all maximal subgroups of P are either Φ-I-supplemented or Φ-I-embedded in G. Then, G is p-nilpotent.*

Proof. Suppose that the assertion is false and let G be a counterexample for which $|G|$ is minimal. We proceed via the following steps:

(1) G has the unique minimal normal subgroup.

Let N be a minimal normal subgroup of G. Assume that M/N is an arbitrary maximal subgroup of PN/N, which is a Sylow p-subgroup of G/N. Then $M = M \cap PN = (M \cap P)N$. Denote $P_1 = M \cap P$. Since $|P : P_1| = |PN : M| = p$, P_1 is a maximal subgroup of P. By the hypothesis, G has a subnormal (resp., S-quasinormal) subgroup T such that P_1 is Φ-I-supplemented (resp., Φ-I-embedded) in G. Note that $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N, so $|P_1N \cap T : P_1 \cap T| = |N \cap P_1T : P_1 \cap N|$ is a p'-number. However, $|P_1N \cap T : N \cap T| = |P_1 \cap TN : P_1 \cap N|$ is a p-number. So we have $P_1N \cap T = (P_1 \cap T)(N \cap T)$ and then $M/N \cap TN/N = (P_1N \cap TN)/N = (P_1 \cap T)N/N$. Similarly as Lemma 3, it is easy to show that TN/N is a subnormal (resp., S-quasinormal) subgroup of G/N such that M/N is Φ-I-supplemented (resp., Φ-I-embedded) in G/N. Therefore, G/N satisfies the hypothesis. So the choice of G implies that G/N is p-nilpotent. Consequently, N is the unique minimal normal subgroup of G.

(2) $O_{p'}(G) = 1$ and $\mathrm{Int}_{\mathfrak{u}}(G) = 1$.

If $O_{p'}(G) \neq 1$, then the uniqueness of N implies that $N \leq O_{p'}(G)$. In the case, $G/O_{p'}(G)$ is p-nilpotent and so is G, a contradiction. Keep Lemmas 2(3) and 5(1) in mind. It is easy to obtain that $\mathrm{Int}_{\mathfrak{u}}(G) = 1$.

(3) $O_p(G) = 1$.

If $O_p(G) > 1$, then $N \leq O_p(G)$ and $|N| > p$. Clearly, $N \nleq \Phi(G)$. Then, there exists a maximal subgroup M of G such that $N \nleq M$. Together with the uniqueness of N, $G = N \rtimes M$. Note that $O_p(G) \cap M \trianglelefteq G$ by Lemma 6, so $O_p(G) \cap M = 1$ and, consequently, $O_p(G) = N(O_p(G) \cap M) = N$. Here, $P = N \rtimes (P \cap M)$ and then $|P : P \cap M| = |N| > p$. Thus, P has a maximal subgroup P_1 such that $P \cap M < P_1 < P$. Clearly, $(P_1)_G = 1$ and by the hypothesis, P_1 is either Φ-I-supplemented or Φ-I-embedded in G.

First assume that P_1 is Φ-I-supplemented in G. Combining with (2), there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)$. According to Lemma 5, we have $N \leq O^p(G) \leq T$, which deduces that $P_1 \cap N \leq P_1 \cap T \leq \Phi(P_1)$. In this case, $P_1 = P_1 \cap N(P \cap M) = (P_1 \cap N)(P \cap M) = P \cap M$, a contradiction.

Now suppose that P_1 is Φ-I-embedded in G, that is, there exists a S-quasinormal subgroup T of G such that P_1T is S-quasinormal in G and $P_1 \cap T \leq \Phi(P_1)$. If $T = 1$, then P_1 is S-quasinormal in G. From Lemma 1(3) and the choice of P_1, we deduce that $P_1 \trianglelefteq G$, a contradiction. So $T > 1$. We further assume that $T_G = 1$. By Lemma 1(5), T^G is nilpotent. Combining with (2), T^G is a p-group. Hence $T \leq T^G = N$, which implies that $P_1T \leq P_1N = P$, that is, P_1T is a S-quasinormal p-subgroup of G. From Lemma 1(4)(6), it follows that $P_1T \leq O_p(G) = N$, which shows that $P_1 \leq N$ and consequently $N = P$. In this case, by Lemma 1(3), T is normal in G. Hence, the minimality of N implies that $T = N$. Consequently, $P_1 = P_1 \cap T \leq \Phi(P_1)$, which shows that $P_1 = 1$, a contradiction. Therefore, $T_G > 1$ and the uniqueness of N implies $N \leq T_G \leq T$. Consequently, $P_1 \cap N \leq P_1 \cap T \leq \Phi(P_1)$. Similarly as the above, it is impossible. So we should assume that $O_p(G) = 1$.

(4) Final contradiction.

Assume that $N = G$, that is, G is a simple group. For any maximal subgroup P_1 of P, if T is a subnormal (resp., S-quasinormal) subgroup of G such that P_1 is Φ-I-supplemented (resp., Φ-I-embedded) in G, then $T = G$. As a result, $P_1 = P_1 \cap T \leq \Phi(P_1)$, that is, $P_1 = 1$, a contradiction. Therefore, $N < G$. If $P \leq N$, then N satisfies the hypothesis by Lemma 3(2). So the choice of G and the relationship $N < G$ deduce that N is p-nilpotent, which contradicts (2) and (3). In general, we conclude $1 < P \cap N < P$. Let P_1 be a maximal subgroup of P containing $P \cap N$. Then $(P_1)_G = 1$ and P_1 is either Φ-I-supplemented or Φ-I-embedded in G.

First assume that P_1 is Φ-I-supplemented in G. So there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)$. According to Lemma 6, we have that $N \leq O^p(G) \leq T$. Hence,

$P_1 \cap N \le P_1 \cap T \le \Phi(P_1)$. Note that $P_1 \cap N = P \cap N$ and $\Phi(P_1) \le \Phi(P)$. So $P \cap N \le \Phi(P)$. However, it deduces that N is p-nilpotent by Lemma 7, a contradiction.

If P_1 is Φ-I-embedded in G, then G has a S-quasinormal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \le \Phi(P_1)$. Note that $T = 1$ implies that P_1 is S-quasinormal in G, which contradicts (3) and Lemma 1(4)(6). Moreover, if $T_G = 1$, then by Lemma 1(5) and the uniqueness of N, $N \le T^G$ and N is nilpotent, which contradicts (1) and (3). Therefore, $T_G > 1$ and consequently $N \le T_G \le T$. In this case, we finally conclude that $P \cap N = P_1 \cap N \le P_1 \cap T \le \Phi(P_1) \le \Phi(P)$. By Lemma 7, we also have that N is p-nilpotent, a contradiction. This contradiction completes the proof. □

Proposition 2. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Assume that all cyclic subgroups of P of order p and order 4 (when P is a nonabelian 2-group) are either Φ-I-supplemented or Φ-I-embedded in G. Then, G is p-nilpotent.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order.

Let M be a proper subgroup of G and M_p a Sylow p-subgroup of M. Then, $M_p \le P^g$ for some $g \in G$. Now consider M^g, which has a Sylow p-subgroup $M_p^g \le P$. By Lemma 3(2), M^g satisfies the hypothesis for G. So M^g is p-nilpotent by the minimality of G. Consequently, M is p-nilpotent, and G is a minimal non-p-nilpotent group. By [14], Theorem 3.4.11, the following hold (i) $G = P \rtimes Q$, where $P = G^{\mathfrak{N}}$ and Q is a Sylow q-subgroup of G with $q \ne p$; (ii) $P/\Phi(P)$ is a noncyclic G-chief factor; (iii) the exponent of P is p or 4 (when P is a nonabelian 2-subgroup). Take $x \in P/\Phi(P)$, and denote $H = \langle x \rangle$. Then, H has order p or 4, $H_G \le \Phi(P)$, and $\Phi(H/H_G) = 1$. Moreover, H is either Φ-I-supplemented or Φ-I-embedded in G by the hypothesis.

First assume that H is Φ-I-embedded in G. Let T be a S-quasinormal subgroup of G such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \le \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. Note that $P/\Phi(P)$ is a G-chief factor, so we separate the proof into three cases: (1) $P \cap T_G = P$; (2)$P \cap T^G \le \Phi(P)$; (3) $P \cap T_G \le \Phi(P)$ and $P \cap T^G = P$. If the first case holds, then $P \le T$ and consequently $H/H_G = (H \cap T)H_G/H_G \le \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. From Lemmas 2(1) and 4(1), we further deduce that $1 < H\Phi(P)/\Phi(P) \le \mathrm{Int}_{\mathfrak{u}}(G/\Phi(P)) \le \mathrm{Int}_{\mathfrak{N}^p}(G/\Phi(P))$, where $\mathfrak{N}^p$ denotes the class of all p-nilpotent groups. So $P\Phi(P)/\Phi(P) \le \mathrm{Int}_{\mathfrak{N}^p}(G/\Phi(P))$ by (ii). Together with Lemma 2(3), we finally have that $G/\Phi(P)$ is p-nilpotent, and so is G, a contradiction. In case (2), we have $P \cap T \le \Phi(P)$. Then $H\Phi(P)/\Phi(P) = H(P \cap T)\Phi(P)/\Phi(P) = P/\Phi(P) \cap HT\Phi(P)/\Phi(P)$ is S-quasinormal in G by Lemma 1(2)(4). Note that $P/\Phi(P)$ is abelian. So $H\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ according to Lemma 1(3). Consequently, $P/\Phi(P) = H\Phi(P)/\Phi(P)$ by (ii), a contradiction. Lastly, suppose that case (3) holds, that is, $P \cap T_G \le \Phi(P)$ and $P \le T^G$. If $T^G = G$, then G/T_G is p-nilpotent by Lemma 1(5). By (i), we have that $G/(P \cap T_G)$ is p-nilpotent, and furthermore, $G/\Phi(P)$ is p-nilpotent, which deduces that G is p-nilpotent, a contradiction. So we have $P \le T^G < G$ and T^G is p-nilpotent as G is a minimal non-p-nilpotent group. Note that $O_{p'}(G) = 1$, so $T \le T^G = P$. By Lemma 1(2), we obtain that $T\Phi(P)/\Phi(P)$ is a S-quasinormal subgroup of $G/\Phi(P)$ contained in $P/\Phi(P)$. Consequently, we deduce that $T\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$ from Lemma 1(3). Therefore, $T\Phi(P) = \Phi(P)$ or $T\Phi(P) = P$, that is, $T \le \Phi(P)$ or $T = P$. However, from the proof of cases (1) and (2), we know that it is impossible.

Then assume that H is Φ-I-supplemented in G, and T is a subnormal subgroup of G such that $G = HT$ and $(H \cap T)H_G/H_G \le \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. Clearly, $P \le O^p(G) \le T$ by Lemma 5(1). So we have $H/H_G = (H \cap T)H_G/H_G \le \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. Similarly as the first case above, we know that it is impossible. Thus, the assertion holds.

Now we true to prove Theorems 1 and 2. □

Proof of Theorem 1. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We proceed via the following steps.

(1) E is a p-group.

Assume that $|\pi(E)| > 1$, p is the smallest prime divisor of $|E|$ and E_p is a Sylow p-subgroup of E. If E_p is cyclic, then E is p-nilpotent (see [15], Theorem 10.1.9). Now assume that E_p is noncyclic. From the hypothesis and Lemma 3(2), it follows that E satisfies the hypothesis of Proposition 1. So we have that E is still p-nilpotent. Let $E_{p'}$ be the normal p'-Hall subgroup of E. Then $E_{p'}$ is a normal subgroup of G and $(G, E_{p'})$ satisfies the hypothesis. Hence $E_{p'} \le \mathrm{Int}_{\mathfrak{u}}(G)$ by the choice of (G, E).

Suppose that E_p is cyclic. Then, $E/E_{p'} \le Z_{\mathfrak{u}}(G/E_{p'})$ for the G-isomorphism $E/E_{p'} \cong E_p$. By Lemma 2(5), $E/E_{p'} \le \mathrm{Int}_{\mathfrak{u}}(G/E_{p'})$. Now assume that E_p is noncyclic. By Lemma 3(1), we can easily obtain that $(G/E_{p'}, E/E_{p'})$ satisfies the hypothesis. Analogously, the choice of (G, E) implies that $E/E_{p'} \le \mathrm{Int}_{\mathfrak{u}}(G/E_{p'})$. Therefore, in any case, $E/E_{p'} \le \mathrm{Int}_{\mathfrak{u}}(G/E_{p'})$. Furthermore, $E \le \mathrm{Int}_{\mathfrak{u}}(G)$ by Lemma 2(4), a contradiction. Thus, $|\pi(E)| = 1$, that is, E is a p-group.

(2) $G/E \in \mathfrak{u}$.

Since $E \nleq Int_{\mathfrak{u}}(G)$, there exists a $\mathfrak{u}$-maximal subgroup X of G such that $E \nleq X$. By Lemma 3(2), (EX, E) satisfies the hypothesis for (G, E). If $EX < G$, then the choice of (G, E) implies that $E \le \mathrm{Int}_{\mathfrak{u}}(EX)$. Note that EX/E is supersoluble for the isomorphism $EX/E \cong X/E \cap X$. So $EX \in \mathfrak{u}$ by Lemma 2(3). Furthermore, the choice of X implies $EX = X$, that is, $E \le X$. This contradiction shows that $G = EX$ and, consequently, G/E is supersoluble as $G/E \cong X/E \cap X$.

(3) G has the unique Sylow p-subgroup.

Let q be the largest prime dividing $|G|$ and G_q a Sylow q-subgroup of G. Assume that $q > p$. Note that G/E is supersoluble. So $G_qE/E \trianglelefteq G/E$ and

$G_qE \trianglelefteq G$. Consider (G_qE, E), which satisfies the hypothesis by Lemma 3(2). Note that p is the smallest prime divisor of $|G_qE|$ and E is the Sylow p-subgroup of G_qE. So by Proposition 1, G_qE is p-nilpotent. Therefore, $G_q \trianglelefteq G_qE$ and, consequently, $G_q \trianglelefteq G$. Now consider $(G/G_q, G_qE/G_q)$, which satisfies the hypothesis by Lemma 3(1). So the choice of (G, E) implies that $G_qE/E \leq \mathrm{Int}_{\mathfrak{u}}(G/E)$. Moreover, the isomorphism $(G/G_q)/(G_qE/G_q) \cong (G/E)/(G_qE/E)$ deduces that $(G/G_q)/(G_qE/G_q)$ is supersoluble. Together with Lemma 2(3), we finally obtain G/G_q is supersoluble. Furthermore, G is supersoluble by the isomorphism $G \cong G/(E \cap G_q)$. This contradiction shows $q = p$. So (3) holds.

(4) Final contradiction.

Let N be a minimal normal subgroup of G contained in E. Consider $(G/N, E/N)$, which satisfies the hypothesis by Lemma 3(1). So the choice of (G, E) implies that $E/N \leq \mathrm{Int}_{\mathfrak{u}}(G/N)$. Note that $(G/N)/(E/N)$ is supersoluble by the isomorphism $(G/N)/(E/N) \cong G/E$. Combining with Lemma 2(3), G/N is supersoluble. Therefore, $N \nleq \Phi(G)$ and N is the unique minimal normal subgroup of G contained in E.

Note that $E \cap \Phi(G)$ is a normal subgroup of G contained in E. So the uniqueness of N implies that $E \cap \Phi(G) = 1$. Consequently, E is the direct product of the minimal normal subgroups of G contained in E (see [14], Chap. 1, Lemma 1.8.17). Furthermore, $E = N$ by the uniqueness of N.

Note that $E \cap Z(G_p)$ is a nontrivial normal subgroup of G. So $E \cap Z(G_p) = E$, that is, $E \leq Z(G_p)$. Take P_1 be an arbitrary maximal subgroup of E. Clearly, $(P_1)_G = 1$, $\Phi(P_1) = 1$ and by the hypothesis, P_1 is either Φ-I-supplemented or Φ-I-embedded in G.

Assume that P_1 is Φ-I-supplemented in G. Let T be a subnormal subgroup of G such that $G = P_1T$ and $P_1 \cap T \leq \mathrm{Int}_{\mathfrak{u}}(G)$. Then, $1 < E \cap T \trianglelefteq G$ and, consequently, $E \cap T = E$ by the minimality of E. In this case, $P_1 = P_1 \cap T \leq \mathrm{Int}_{\mathfrak{u}}(G)$, which implies $E \leq \mathrm{Int}_{\mathfrak{u}}(G)$ by the minimality of E again, a contradiction.

Now suppose that P_1 is Φ-I-embedded in G and T is a S-quasinormal subgroup of G such that P_1T is S-quasinormal in G and $P_1 \cap T \leq \mathrm{Int}_{\mathfrak{u}}(G)$. It is easy to show that the above holds if T is replaced by $T \cap E$. So, without loss of generality, assume that $T \leq E$. Since T is S-quasinormal in G, we have $T \trianglelefteq G$ by Lemma 1(3) and the relationship $E \leq Z(G_p)$. Therefore, $T = 1$ or $T = E$. If $T = 1$, then P_1 is S-quasinormal in G and, similarly as above, $P_1 \trianglelefteq G$, which contradicts the minimality of E. But if $T = E$, then $P_1 = P_1 \cap T \leq \mathrm{Int}_{\mathfrak{u}}(G)$, which also deduces a contradiction as above. So the proof is completed. □

Proof of Theorem 2. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Then, G is not supersolvable. Similarly as steps (1) and (2) in the proof of Theorem 1, assume that $G^{\mathfrak{u}} \leq E$ and E is a p-group.

Let M be any proper subgroup of G. Consider $(M, E \cap M)$, which satisfies the hypothesis for (G, E) by Lemma 3(2). So the minimality of (G, E) deduces that $E \cap M \leq \mathrm{Int}_{\mathfrak{u}}(M)$. Note that $M/E \cap M$ is supersolvable by the isomorphism $M/E \cap M \cong ME/E \leq G/E$. So M is supersolvable by Lemma 2(3). Consequently, G is a minimal nonsupersolvable group and from ([14], Theorem 3.4.2), we deduce that (i) $E = G^{\mathfrak{u}}$ is a p-subgroup of G; (ii) $E/\Phi(E)$ is a noncyclic G-chief factor; (iii) the exponent of E is p or 4 (when E is a non-abelian 2-group). Similarly, as step (3) of the proof of Theorem 1, the Sylow p-subgroup G_p of G is normal in G. Note that $E/\Phi(E) \cap Z(G_p/\Phi(E))$ is a nontrivial normal subgroup of $G/\Phi(E)$, so $E/\Phi(E) \leq Z(G_p/\Phi(E))$.

Take $x \in E/\Phi(E)$, and denote $H = \langle x \rangle$. Then, H has order p or 4, $H_G \leq \Phi(E)$, and $\Phi(H/H_G) = 1$. By the hypothesis, H is either Φ-I-supplemented or Φ-I-embedded in G.

Assume that H is Φ-I-embedded in G. Let T be a S-quasinormal subgroup of G such that HT is S-quasinormal in G and $(H \cap T)H_G/H_G \leq \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. Clearly, $T \cap E$ is another S-quasinormal subgroup of G such that H is Φ-I-embedded in G. So without loss of generality, assume that $T \leq E$. Then, $T\Phi(E)/\Phi(E)$ is a S-quasinormal subgroup of $G/\Phi(E)$ contained in $E/\Phi(E)$. Together with Lemma 1(3) and the relationship $E/\Phi(E) \leq Z(G_p/\Phi(E))$, we have $T\Phi(E)/\Phi(E) \trianglelefteq G/\Phi(E)$. Thus, $T \leq \Phi(E)$ or $T = E$. If $T \leq \Phi(E)$, then $H\Phi(E)/\Phi(E) = HT\Phi(E)/\Phi(E)$ is S-quasinormal in G and then $H\Phi(E)/\Phi(E) \trianglelefteq G/\Phi(E)$. The choice of H shows that $E/\Phi(E) = H\Phi(E)/\Phi(E)$, which contradicts (ii). Assume that $T = E$. Then, $H/H_G = (H \cap T)H_G/H_G \leq \mathrm{Int}_{\mathfrak{u}}(G/H_G)$ and by Lemma 2(2), $H\Phi(E)/\Phi(E) \leq \mathrm{Int}_{\mathfrak{u}}(G/\Phi(E))$. Together with (ii), we have $E/\Phi(E) \leq \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. Recall that G/E is supersoluble. Therefore, $G/\Phi(E)$ is supersoluble by the isomorphism $(G/\Phi(E))/(E/\Phi(E)) \cong G/E$ and Lemma 2(3). Furthermore, we have G is supersoluble, a contradiction.

Now assume that H is Φ-I-supplemented in G and T is a subnormal subgroup of G such that $G = HT$ and $(H \cap T)H_G/H_G \leq \mathrm{Int}_{\mathfrak{u}}(G/H_G)$. It is easy to show that $(E \cap T)\Phi(E) \trianglelefteq G$. So $E \cap T \leq \Phi(E)$ or $E \leq T$. Similarly as the above, $E \leq T$ is impossible. However, $E \cap T \leq \Phi(E)$ implies that $E = E \cap HT = H(E \cap T) = H$, which contradicts (ii). Then, we complete the proof. □

4. Some Applications

The following result follows directly from Lemma 2(3) and Theorems 1 and 2.

Corollary 1. *Let E be a normal subgroup of G such that G/E is supersoluble. Then, G is supersoluble if and only if for every prime $p \in \pi(E)$ and every noncyclic Sylow p-subgroup P of E, one of the following holds*

(1) *All maximal subgroups of P are either Φ-I-supplemented or Φ-I-embedded in G*

(2) *All cyclic subgroups of P with order p and 4 (when P is a non-abelian 2-group) are either Φ-I-supplemented or Φ-I-embedded in G*

Recall also that a subgroup H of G is called as follows: c-normal in G [16] if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$; $\mathfrak{u}_c$-normal in G [17] if G has a subnormal subgroup T such that $G = HT$ and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{u}}(G/H_G)$; $S\Phi$-supplemented in G [18] if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)$. Obviously, c-normal subgroups, $\mathfrak{u}_c$-normal subgroups, and $S\Phi$-supplemented subgroups of G are all Φ-I-supplemented in G. Hence, we have the following.

Corollary 2. *G is supersoluble, if one of the following holds*

(a) *Every maximal subgroup of every Sylow subgroup of G is $\mathfrak{u}_c$-normal in G ([17], Corollary 1.3)*

(b) *All cyclic subgroups of G with prime order or order 4 are $\mathfrak{u}_c$-normal in G ([17], Corollary 1.5)*

(c) *Every maximal subgroup of every Sylow subgroup of G is c-normal in G ([16], Theorem 4.1)*

(d) *All cyclic subgroups of G with prime order or order 4 are c-normal in G ([16], Theorem 4.2)*

From Proposition 2, we obtain the following.

Corollary 3 ([19], Lemma 3.1). *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If all subgroups of P with order p or order 4 are c-normal in G, then G is p-nilpotent.*

Corollary 4 ([18], Theorem 3.1). *Let P be a Sylow p-subgroup of G, where p is a prime dividing $|G|$ such that $(|G|, p - 1) = 1$. If every maximal subgroup of P is $S\Phi$-supplemented in G, then G is p-nilpotent.*

Moreover, Theorem 3 in [20] and Theorems 3.3 and 3.4 in [21] follow directly from Theorem 1.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest regarding the content and implications of this manuscript.

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