Research Article

On Resolvability Parameters of Some Wheel-Related Graphs

Bin Yang,¹ Muhammad Rafiullah ,² Hafiz Muhammad Afzal Siddiqui ,² and Sarfraz Ahmad²

¹Department of Computer Science and Technology, Hefei University, Hefei 230601, Anhui, China

²Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan

Correspondence should be addressed to Hafiz Muhammad Afzal Siddiqui; hmasiddiqui@gmail.com

Received 11 July 2019; Accepted 12 October 2019; Published 28 November 2019

Guest Editor: Shaohui Wang

Copyright © 2019 Bin Yang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $G = (V, E)$ be a simple connected graph, $w \in V$ be a vertex, and $e = uv \in E$ be an edge. The distance between the vertex w and edge e is given by $d(e, w) = \min\{d(w, u), d(w, v)\}$. A vertex w distinguishes two edges $e_1, e_2 \in E$ if $d(w, e_1) \neq d(w, e_2)$. A set S is said to be resolving if every pair of edges of G is distinguished by some vertices of S . A resolving set with minimum cardinality is the basis for G , and this cardinality is the edge metric dimension of G , denoted by $\text{edim}(G)$. It has already been proved that the edge metric dimension is an NP-hard problem. The main objective of this article is to study the edge metric dimension of some families of wheel-related graphs and prove that these families have unbounded edge metric dimension. Moreover, the results are compared with the metric dimension of these graphs.

1. Introduction and Preliminary

The molecular structure of a chemical compound is known as molecular graph (or chemical graph). There are different structures of chemical compounds, and every structure correlated with many chemical properties, and these properties can be calculated through some specific mathematical formulas. Graph theory helps us to study and analyze these dense structures in detail. In graph theory, we consider every atom as a vertex and covalent bond between atoms as an edge. A graph G consists of a set of objects $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ called vertices, set of connections between vertices called edges $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$, and graph G is usually denoted as $G = (V(G), E(G))$. The number of vertices in G denoted by $n = |V(G)|$ is often called the order of G , while the number of edges is known as size of G and denoted by $m = |E(G)|$. The concept of metric dimension was introduced by Slater in 1975 [1], and this concept independently was elaborated for graphs by Harary and Melter in 1976 [2]. After these two papers, lot of research papers have been published related to its theoretical properties and applications. The theoretical studies on metric dimension are highly contributed by several

authors like, for instance, independent resolving sets [3], resolving dominating sets [4], strong resolving sets [5], local resolving sets [6], k -metric generators [7, 8], simultaneous metric generators [9], resolving partitions [10], strong resolving partitions [11], and k -antiresolving sets [12]. The metric dimension has many applications in the digital world and in mathematical chemistry such as network verification [13], robot navigation, image processing and pattern recognition [14], mastermind game [15], chemistry [16, 17, 18], and pharmaceutical chemistry [18].

If G is a simple connected graph and u and v are any two vertices of G , then the distance $d(u, v)$ is the length of a shortest path between u and v . Let $S = \{s_1, s_2, s_3, \dots, s_k\}$ be an ordered set of vertices of G and let v be a vertex of G . The representation $r(v|S)$ of v with respect to S is the ordered k -tuple $(d(v, s_1), d(v, s_2), d(v, s_3), \dots, d(v, s_k))$. If $d(x, v) \neq d(y, v)$, then we shall say that vertex v distinguishes two vertices x and y . If distinct vertices of G have distinct representations with respect to S , then S is called a resolving set for G . A resolving set of minimum cardinality is called a basis for G and the cardinality is known as metric dimension of G , and it is denoted as $\text{dim}(G)$.

Let S be a set of vertices of a simple connected graph G . Then, S is called an edge metric generator for G if every two edges of G are distinguished by some vertices of S . The minimum cardinality of S is called the edge metric dimension, and it is denoted by $\text{edim}(G)$. For an ordered set of vertices $S = \{s_1, s_2, \dots, s_k\}$ of a graph G and any edge e in G , we refer to the k -vector (ordered k -tuple) $r(e | S) = (d(e, s_1), d(e, s_2), \dots, d(e, s_k))$ as the edge metric representation of e with respect to S . The set S is an edge metric generator for G if and only if for every pair of different edges e_1, e_2 of G , $r(e_1 | S) \neq r(e_2 | S)$.

Let $\mathcal{F} = (G_n)_{n \geq 1}$ be a family of connected graphs G_n of order $\varphi(n)$ for which $\lim_{n \rightarrow \infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\text{edim}(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has a bounded edge metric dimension, otherwise $\mathcal{F}$ has unbounded edge metric dimension. Moreover, if $\text{edim}(G_n) = C, \forall n \geq 1$, then $\mathcal{F}$ has a constant edge metric dimension.

There is a controversy in the definition of edge resolving set and edge metric dimension. In the literature, another version of edge metric dimension exists, which is based on the distance between edges, denoted as $d(f, g)$. This version of edge metric dimension follows the following steps.

Step 1. Convert a graph G into a line graph $L(G)$.

Step 2. Find metric dimension, $\dim(L(G))$ of line graph $L(G)$.

According to the above steps, $\dim(L(G)) = \text{edim}(G)$, for reference see [19]. But in our case, the distance is based on vertex to edge, $d(v, e)$, as defined by Kelenc et al. [20].

In 1979, Garey and Johnson [21] pointed out that the finding of the metric dimension of a graph is an NP-hard problem. In 1994, Khuller et al. [22] also showed by another construction that the metric dimension of a graph is an NP-hard problem.

Recently, Kelenc et al. [20] have introduced the notion of edge metric dimension of graph, denoted by $\text{edim}(G)$, and they presented some results relating the notions of metric dimension and edge metric dimension for some families of graph such as for path P_n : $\text{edim}(P_n) = \dim(P_n) = 1$ where $n \geq 2$, for cycle C_n : $\text{edim}(C_n) = \dim(C_n) = 2$, for complete graph K_n : $\text{edim}(K_n) = \dim(K_n) = n - 1$, where $n \geq 2$, for complete bipartite graph $K_{r,t}$: $\text{edim}(K_{r,t}) = \dim(K_{r,t}) = r + t - 2$, where $r, t \geq 2$, for tree T_n which is not a path: $\text{edim}(T) = \dim(T) = \sum_{v \in V, L_v > 1} (L_v - 1)$, for grid graph $G = P_r \square P_t$ (where $r \geq t \geq 2$): $\text{edim}(G) = \dim(G) = 2$, for wheel graph $(G = S_{1,n})$: $\text{edim}(G) > \dim(G)$, and for the grid graph $G = C_r \square C_t$: $\text{edim}(C_r \square C_t) < \dim(C_r \square C_t)$. They also proved that the edge metric dimension is an NP-hard problem. Moreover, they have raised many open problems related to edge metric dimension. Furthermore, in 2019, Zubrilina has classified some graphs on the same topic which has the $\text{edim}(G) = |V| - 1$ in her paper [23]. Recently, in 2019, Rafiullah et al. studied the edge metric dimensions of wheel-related convex polytopes in [24] and characterized these graphs.

This paper is based on the question raised in [20] to characterize the families of graphs which observe one of the

relations $\dim(G) = \text{edim}(G)$, $\dim(G) < \text{edim}(G)$, or $\dim(G) > \text{edim}(G)$.

We consider Jahangir graph J_{2n} , helm graph H_n , sunflower graph Sf_n , and friendship graph F_n for edge metric dimension (NP-hard problem). Our aim is to characterize these families with respect to the nature of edge metric dimension. Moreover, we present closed formulas for edge metric dimension of these graphs. For further reading on metric dimension and edge metric dimension, we refer [19, 24, 25, 26, 27].

2. Main Results

2.1. Jahangir Graph (J_{2n}). The Jahangir graph J_{2n} obtained from wheel graph W_{2n} by alternately deleting n spokes. The order of J_{2n} is $2n + 1$, and the size is $3n$. The Jahangir graph is also referred to as gear graph G_n [28]. The metric dimension of Jahangir graph $\dim(J_{2n})2n/3$ is given in [29]. Now, we present some observations for J_{2n} in the form of lemmas which help us to compute its edge metric dimension.

Lemma 1. *The central vertex v does not belong to any edge metric basis S of J_{2n} , for $n \geq 6$.*

Proof. Let $J_{2n} = C_{2n} + K_1$ be a gear graph of order $2n + 1$. The vertices of C_{2n} are labeled as $v_1, v_2, v_3, \dots, v_{2n}$, and edges of J_{2n} are labeled as $e_j = vv_{2j-1}$ and $e'_i = v_i v_{i+1}$ for all $j \in \{1, 2, \dots, n\}$ and $i \in \{1, 2, \dots, 2n - 1\}$ as shown in Figure 1. The vertices v_{2j-1} and v_{2j} are named as major and minor vertices, respectively. Since $d(e_j, v) = 0$ and $d(e'_i, v) = 1$, v does not belong to any basis of J_{2n} .

Let $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\}$ be a basis of J_{2n} and $r = |S| \geq 2$ number of vertices on C_{2n} for $n \geq 6$. We say that the pairs of vertices $\{v_{i_a}, v_{i_{a+1}}\}$ for $1 \leq i \leq r - 1$ and $\{v_{i_r}, v_{i_1}\}$ are pair of neighboring vertices. We define the gap g_a for $1 \leq a \leq r - 1$ as the set of vertices $\{v_j | i_a < j < i_{a+1}\}$ and $g_r = \{v_j | 1 \leq j < i_r \text{ or } i_r < j \leq 2n\}$. The number of gaps is r and some of which may be empty. If $|a - b| = 1$ or $r - 1$, then the gaps g_a and g_b are known as the neighboring gaps. Let $d(u) = \alpha$ and $d(v) = \beta$ or $d(u) = \beta$ and $d(v) = \alpha$ with $\alpha \leq \beta$. Then, the gap between u and v is denoted by $\alpha - \beta$. The number of vertices between u and v is known as the size of gap. There are three types of gaps observed on C_{2n} of J_{2n} , which are $2 - 2$, $2 - 3$, and $3 - 3$. We represent the major vertices by $\hat{x}$ in the following proofs. $\square$

Lemma 2. *If S is a basis of J_{2n} ($n \geq 6$), then every $2 - 2$, $2 - 3$, and $3 - 3$ gap of S contains at most 5, 4, and 3 vertices, respectively.*

Proof. Let $e'_i = v_i v_{i+1}$ be edges on C_{2n} where $i \in \{1, 2, \dots, 2n - 1\}$ and $e_j = vv_{2j-1}$ be spokes edges of J_{2n} where $j \in \{1, 2, \dots, n\}$. Suppose that there is a $2 - 2$ gap of S containing seven consecutive vertices $\hat{x}_{i+1}, \hat{x}_{i+2}, \dots, \hat{x}_{i+7}$ of C_{2n} , determined by x_i and x_{i+8} with $d(\hat{x}_{i+1}) = d(\hat{x}_{i+7}) = 3$. Then, we have $r(e_{i+2} | S) = r(e_{i+3} | S)$, a contradiction. The existence of a $2 - 3$ gap containing six consecutive vertices of C_{2n} : $\hat{x}_{i+1}, \hat{x}_{i+2}, \dots, \hat{x}_{i+6}$, determined by x_i and $\hat{x}_{i+7}$ with

FIGURE 1: Jahangir graph (J_{2n}). (a) J_8 . (b) J_{10} .

$d(x_{i+1}^*) = 3$ and $d(x_{i+6}) = 2$, it follows that $r(e'_{i+4} | S) = r(e'_{i+5} | S)$ and $r(e_i | S) = r(e_{i+1} | S)$.

If there is a 3-3 gap of S containing five consecutive vertices $x_{i+1}^*, x_{i+2}^*, \dots, x_{i+5}^*$ of C_{2n} and determined by x_i^* and x_{i+6}^* with $d(x_{i+1}) = d(x_{i+5}) = 2$ would imply $r(e_{i+1} | S) = r(e_{i+2} | S)$ and $r(e'_{i+2} | S) = r(e'_{i+3} | S)$, a contradiction.

Hence, it is proved that every 2-2, 2-3, and 3-3 gap of S contains at most 5, 4, and 3 vertices, respectively.

Note that these 2-2, 2-3, and 3-3 gaps are the known major gaps. $\square$

Lemma 3. If S is a basis of J_{2n} ($n \geq 6$), then it contains at most one major gap.

Proof. Suppose that S contains two distinct major gaps g_a and g_b .

- (i) 3-3 and 3-3: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*$, it follows $r(e_{i+1} | S) = r(e_{i+3} | S)$ and $r(e'_{i+3} | S) = r(e'_{i+4} | S)$
- (ii) 3-3 and 2-2: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*$, y_{j+4}^*, y_{j+5}^* ; in this case $r(e_{i+1} | S) = r(e_{i+4} | S)$
- (iii) 3-3 and 2-3: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*$, y_{j+5}^* ; we have $r(e_{i+1} | S) = r(e_{i+3} | S)$ and $r(e'_{i+3} | S) = r(e'_{i+4} | S)$
- (iv) 2-2 and 2-2: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*, x_{i+5}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*, y_{j+4}^*, y_{j+5}^*$; in this case, $r(e_{i+2} | S) = r(e_{i+5} | S)$, $r(e_{i+3} | S) = r(e_{i+4} | S)$, and $r(e'_{i+4} | S) = r(e'_{i+9} | S)$
- (v) 2-2 and 2-3: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*, x_{i+5}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*, y_{j+4}^*$; in this case, $r(e_{i+2} | S) = r(e_{i+5} | S)$, $r(e_{i+3} | S) = r(e_{i+4} | S)$, and $r(e'_{i+4} | S) = r(e'_{i+9} | S)$
- (vi) 2-3 and 2-3: $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*$ and $y_{j+1}^*, y_{j+2}^*, y_{j+3}^*, y_{j+4}^*$; in this case, $r(e_i | S) = r(e_{i+1} | S)$, $r(e_{i+2} | S) = r(e_{i+5} | S)$, and $r(e'_i | S) = r(e'_{i+1} | S)$, which contradicts the hypothesis. $\square$

Lemma 4. If $n \geq 6$ and S be a basis of J_{2n} . Then, any two neighboring gaps, one of which being a major gap, contain together at most six vertices.

Proof. If we take a 3-3 gap which contains three vertices, then its neighboring gap should contain three vertices that is only possible when we take 3-3 as a neighboring gap, which is not possible because Lemma 2 says that the basis only contains one major gap.

If we take major 2-2 gap with five vertices, then the neighboring cannot be a minor 2-2 gap with three vertices or minor 2-3 gap with two vertices, we show them below case wise:

Case 1. The vertices of gaps g_a and g_b are $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*, x_{i+5}^*$ and $x_{i+7}^*, x_{i+8}^*, x_{i+9}^*$, respectively, and determined by x_i^* , x_{i+6}^* , and x_{i+10}^* , then we have $r(e_{i+3} | S) = r(e_{i+4} | S)$ and $r(e'_{i+6} | S) = r(e'_{i+8} | S)$, a contradiction.

Case 2. The vertices of gaps g_a and g_b are $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*, x_{i+5}^*$ and x_{i+7}^*, x_{i+8}^* , respectively, determined by x_i^* , x_{i+6}^* , and x_{i+9}^* , then we have $r(e_i | S) = r(e_{i+1} | S)$ and $r(e'_i | S) = r(e'_{i+1} | S)$.

If the gap is 2-3 gap with four vertices, then Lemma 2 is sufficient to show the neighboring gap cannot be minor 2-2 gap with three vertices. In this case, the gaps g_a and g_b vertices are $x_{i+1}^*, x_{i+2}^*, x_{i+3}^*, x_{i+4}^*$ and $x_{i-1}^*, x_{i-2}^*, x_{i-3}^*$, respectively, so $r(e_i | S) = r(e_{i+1} | S)$ and $r(e'_{i+2} | S) = r(e'_{i+3} | S)$. $\square$

Lemma 5. If S is a basis of J_{2n} ($n \geq 6$), then any two minor neighboring gaps contain together at most four vertices.

Proof. Since by Lemma 2 any minor 2-2, 2-3, and 3-3 gap contains 3, 2, and 1 vertexes, respectively, it is sufficient to consider the following cases:

Case 1. A 2 – 2 gap with 3 vertices has a neighboring 2 – 2 gap with 3 vertices.

Case 2. A 2 – 2 gap with 3 vertices having a neighboring 2 – 3 gap with 2 vertices cannot occur.

If Case 1 holds, then $r(e_i | S) = r(e_{i+1} | S)$ and $r(e'_i | S) = r(e'_{i+1} | S)$, and if Case 2 holds, then $r(e_i | S) = r(e_{i+1} | S)$ and $r(e'_i | S) = r(e'_{i+1} | S)$, so there are contradictions. $\square$

Theorem 1. If $n \geq 6$, then $\text{edim}(J_{2n}) = 2n/3$.

Proof. We can easily see that $\text{edim}(J_8) = 2$, in this case, the possible resolving sets are $\{v_2, v_4\}$, $\{v_4, v_6\}$, and $\{v_6, v_8\}$. Similarly, we can also see that $\text{edim}(J_{10}) = 3$. First, we show that $\text{edim}(J_{2n}) \leq 2n/3$ by constructing a resolving set in J_{2n} with $2n/3$ vertices. We consider three cases according to the residue classes of modulo 3, where n belongs to the following cases.

Case 1. $n \equiv 0 \pmod{3}$. Let $2n = 3k$, where k is even, $k \geq 4$, and $2n/3 = k$. In this case, $S = \{v_{6i-4}, v_{6i} : 1 \leq i \leq k/2 - 1\} \cup \{v_1, v_{2n-4}\}$.

Case 2. $n \equiv 1 \pmod{3}$. Let $2n = 3k + 2$, where k is even, $k \geq 4$, and $2n/3 = k$. We define $S = \{v_{6i-2}, v_{6i+2} : 1 \leq i \leq k/2 - 1\} \cup \{v_2, v_{2n-4}\}$.

Case 3. $n \equiv 2 \pmod{3}$. Let $2n = 3k + 1$, where k is odd, $k \geq 5$, and $2n/3 = k$ and $S = \{v_{6i-2}, v_{6i} : 1 \leq i \leq (k-1)/2\} \cup \{v_1\}$.

The set S contains only one major gap, and remaining gaps are minor gaps which are 2 – 2, 2 – 3, and 3 – 3 gaps containing 3, 2, and 1 number of vertices, respectively. The total number of vertices in any two neighboring gaps, one of which is major, are at most 6 (Lemma 4). Any two neighboring gaps contain at most 4 vertices (Lemma 5). Now, we show that $\text{edim}(J_{2n}) \geq 2n/3$. Let S be a basis of J_{2n} and $|S| = r$. Then, S induces r gaps on C_{2n} , denoted as $g_1, \dots, g_r$ such that g_i and g_{i+1} for every $1 \leq i \leq r-1$ and also g_1 and g_r are neighboring gaps. By Lemma 3, at most one of them, say g_1 is a major gap. By Lemma 4, we can write $|g_1| + |g_2| \leq 6$, $|g_r| + |g_1| \leq 6$, and by Lemma 5, the relation for the remaining neighboring gaps is $|g_i| + |g_{i+1}| \leq 4$ for every $i = 2, \dots, r-1$.

So, we have

$$\begin{aligned} 2(2n - r) &= 2 \sum_{i=1}^r |g_i| \leq 4(r-2) + 12, \\ 2(2n - r) &\leq 4r + 4, \\ r &\geq \frac{2n-2}{3} \geq \lfloor \frac{2n}{3} \rfloor. \end{aligned} \quad (1)$$

Since $r \in \mathbb{N}$, for each $n \equiv 0, 1$ or $2 \pmod{3}$, we obtain $r \geq 2n/3$. $\square$

2.2. Helm Graph (H_n). The helm graph H_n is obtained from wheel graph W_n by adding a single pendant edge at each vertex of cycle C_n . The number of vertices in the helm graph

are $|V| = 2n + 1$, and the number of edges are $|E| = 3n$. We labeled the central vertex, cycle vertices, and pendant vertices as v , v_i , and v'_i , respectively, where $1 \leq i \leq n$. There are three types of edges, and we labeled them in such a way $e_i = vv_i$, $e'_i = v_i v_{i+1}$, and $e''_i = v_i v'_i$, see Figure 2.

In [30], it was proved that the metric dimension of the helm graph is unbounded, i.e., $\text{dim}(H_n) = 2n + 2/5$.

Theorem 2. Let H_n be a helm graph. Then,

$$\text{edim}(H_n) = \begin{cases} n, & n = 3, 4, \\ n - 1, & n \geq 5. \end{cases} \quad (2)$$

Proof. If $n = 3$ or 4 , then the proof is simple. Let $n \geq 5$ and $V(H_n) = \{v, v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$, where $d(v) = n$, vertices v_i are on C_n and they have degree 4, i.e., $d(v_i) = 4$, and the v'_i are the pendant vertices having $d(v'_i) = 1$. To prove that S is a minimum edge resolving set, we consider the following three cases:

- (1) If $e_i = vv_i$, then e_i has distance 0 to v_i and distance 1 to v_j , where $j \in \{1, 2, \dots, n\}$ and $j \neq i$, distance 1 to v'_i and distance 2 to v'_j , where $j \in \{1, 2, \dots, n\}$ and $j \neq i$.
- (2) If $e'_i = v_i v_{i+1}$, then e'_i has distance 0 to v_i and v_{i+1} , distance 1 to $\{v, v_{i-1}, v_{i+2}\}$, distance 2 to all other vertices on C_n , distance 1 to $\{v'_i, v'_{i+1}\}$, distance 2 to $\{v'_{i-1}, v'_{i+2}\}$, and distance 3 to all other pendant vertices.
- (3) If $e''_i = v_i v'_i$, then e''_i has distance 0 to v_i and v'_i , distance 1 to $\{v, v_{i-1}, v_{i+1}\}$, distance 2 to $\{v'_{i-1}, v'_{i+1}\}$, and distance 3 to all other pendant vertices.

The above cases show that the edge metric representation with any $n-2$ vertices or less number of vertices of H_n is same for some edges. So, S is the edge metric generator with $n-1$ vertices, and therefore,

$$\text{edim}(H_n) \leq n - 1. \quad (3)$$

We can also show that the $\text{edim}(H_n) \not\leq n-2$ by considering S as a basis of H_n with $|S| = n-2$. For this we have the following cases:

- (i) If we take $S = \{v_i, v_{i+1}, \dots, v_{n-2}\}$, then $r(e_{i-1} | S) = r(e_{n-1} | S)$.
- (ii) If we suppose $S = \{v'_i, v'_{i+1}, \dots, v'_{n-2}\}$, then $r(e_{i-1} | S) = r(e_{n-1} | S)$.
- (iii) If we consider $S = \{v'_i, \dots, v'_{i+n-4}, v'_{i+n-3}\}$ and $S = \{v'_i, v'_{i+1}, \dots, v'_{i+n-3}\}$, then $r(e_i | S) = r(e'_i | S)$ and $r(e_{i+1} | S) = r(e_{i+2} | S)$, respectively.

So, set S is not the basis of H_n . This implies that

$$\text{edim}(H_n) \geq n - 1. \quad (4)$$

Hence, from inequalities (3) and (4), we have $\text{edim}(H_n) = n - 1$.

Note. The total number of edge resolving sets for the helm graph (H_n) is $n + 1$. $\square$

FIGURE 2: Helm graph (H_n). (a) H_5 . (b). H_n .

2.3. Sunflower Graph (Sf_n). The sunflower graph Sf_n obtained from wheel graph (W_n) having a cycle C_n ($v_1, v_2, \dots, v_n$) and by adding n new vertices $v'_1, v'_2, \dots, v'_n$ with new edges (v'_i, v_i) and (v'_i, v_{i+1}) where $1 \leq i \leq n$ are shown in Figure 3. The number of vertices in the sunflower graph are $|V| = 2n + 1$, and the number of edges are $|E| = 4n$. We labeled the central vertex, cycle vertices, and the newly added vertices by v , v_i , and v'_i , respectively, where $1 \leq i \leq n$. We denote the cycle $v_1, v'_1, v_2, v'_2, \dots, v_n, v'_n, v_1$ by C'_{2n} . There are three types of edges and we label them in the following manner, $e_i = vv_i$, $e'_i = v_i v_{i+1}$, and $e''_i = v_i v'_i$ or $e''_i = v_{i+1} v'_i$, consider Figure 3 for reference.

The metric dimension of the sunflower graph for $n \geq 6$,

$$\dim(Sf_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \pmod{3}, \\ \left. \begin{aligned} &\frac{n-1}{3}, \text{ if } n \text{ is odd,} \\ &\frac{n+1}{3}, \text{ if } n \text{ is even,} \end{aligned} \right\}, & n \equiv 1 \pmod{3}, \\ \frac{n+1}{3}, & n \equiv 2 \pmod{3}. \end{cases} \quad (5)$$

Now, we present some observations for Sf_n to prove its edge metric dimension.

- (i) There are two edge disjoint cycles: $C_n : v_1 e'_1 v_2 e'_2 v_3 \dots e'_{n-1} v_n e'_n v_1$ and $C'_{2n} : v_1 e''_1 v'_1 e''_2 v_2 \dots e''_{2n-1} v'_n e''_{2n} v_1$.
- (ii) Every 2 – 2 gap of S contains one vertex v_i or three vertices v_i, v'_i, v_{i+1} of C_n and C'_{2n} , respectively.
- (iii) There are three types of gaps, namely, 2 – 2, 2 – 5, and 5 – 5 with the maximum gap of sizes 3, 2, and 0, respectively.

- (iv) The resolving set S contains two 2 – 5 gaps of sizes 2 and 0, when $n \equiv 2 \pmod{3}$.
- (v) The resolving set S mostly contains 2 – 2 gaps.
- (vi) Any two 2 – 2 neighboring gaps contain together at most 4 number of vertices.
- (vii) The neighboring 2 – 2 and 2 – 5 gaps contain together at most 3 number of vertices.
- (viii) Only one vertex of degree 5 from C_n belongs to edge resolving set S of Sf_n , when $n \equiv 2 \pmod{3}$ and all remaining vertices belong to C'_{2n} . If we consider any two consecutive vertices of the degree 5 in basis set S such as v_i and v_{i+1} , then $r(e'_{i+1} | S) = r(e'_{i+2} | S) = (1, 0, 2, 3, \dots, 3)$ and $r(e'_i | S) = r(e'_{i-1} | S) = (0, 1, 3, \dots, 3, 2)$. So, the 5 – 5 gap is not possible in the sunflower graph for edge metric dimension.
- (ix) Any two 2 – 5 neighboring gaps contain together at most 2 vertices. If there are 4 vertices in 2 – 5 neighboring gaps for $v_i \in S$ where $1 \leq i \leq 2n$, then $r(e'_{i+1} | S) = r(e'_{i+3} | S) = (1, 1, 2, \dots, 2)$, $r(e'_{i-1} | S) = r(e'_i | S) = (1, 2, \dots, 1, 2)$, and $r(e''_{2n} | S) = r(e'_1 | S) = (0, 2, 3, \dots, 3)$.
- (x) The central vertex v does not belong to any edge metric basis S of Sf_n because $d(e_i, v) = 0$, $d(e'_i, v) = 1$, and $d(e''_j, v) = 1$ where $1 \leq i \leq n$ and $1 \leq j \leq 2n$.

We can also define the selection of basis S of Sf_n by the following Claims.

Claim 1. The resolving set S mostly contains 2 – 2 gaps having one vertex.

If a gap g of the size 1 with vertex v_{i+1} on C_n and $v'_i, v'_{i+1}, v'_{i+3}, v'_{i+4} \in S$, then g contains two edges $e''_{i+1} = v'_i v_{i+1}$ and $e''_{i+2} = v'_{i+1} v'_{i+1}$ on C'_{2n} . If $z \in \{e'_{i+1}, e'_{i+2}\}$ then

$$\begin{aligned} r(z | S) &= (1, 0, 1, 2, \dots, 2, \overline{0, 1, 2, 3, \dots, 3, 2}), \\ r(z | S) &= (1, 0, 1, 2, \dots, 2, \overline{1, 0, 2, 3, \dots, 3, 2}), \end{aligned} \quad (6)$$

FIGURE 3: Sunflower graph (Sf_6).

where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i types of vertices, respectively.

By observation v and Claim 1, there is no other edge in Sf_n having these types of codes.

Claim 2. At most one 2 – 2 gap of S has at most 3 number of vertices.

If a gap g of the size 3 with vertices $v_{i+3}, v'_{i+3}, v_{i+4}$ on C'_{2n} and $v'_i, v'_{i+1}, v'_{i+2}, v'_{i+4}, v'_{i+5} \in S$, then the g contains 5 number of edges, one of them $e'_{i+3} = v_{i+3}v_{i+4}$ on C_n and 4 of them $e''_{i+5} = v'_{i+2}v_{i+3}$, $e''_{i+6} = v'_{i+3}v'_{i+4}$, $e''_{i+7} = v'_{i+3}v'_{i+4}$, $e''_{i+8} = v'_{i+4}v'_{i+4}$ on C'_{2n} . If $z \in \{e'_{i+3}, e'_{i+5}, e'_{i+6}, e'_{i+7}, e'_{i+8}\}$, then

$$\begin{aligned} r(z|S) &= (2, 2, 1, 0, 0, 1, 2, \dots, 2, \overline{3, 2, 1, 1, 1, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 2, 0, 1, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 2, 1, 0, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 3, 2, 0, 1, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 3, 2, 1, 0, 2, 3, \dots, 3}), \end{aligned} \quad (7)$$

where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i types of vertices, respectively.

By observation iii and Claim 2, there is no other edge in Sf_n with these types of codes.

Claim 3. At most one 2 – 5 gap of S has 2 vertices.

If a gap g of the size 2 with vertices v_n, v'_n and $v'_{n-7}, v'_{n-6}, v'_{n-4}, v'_{n-2}, v'_{n-1}, v_{n+1} = v_i \in S$, then the g contains 4 number of edges, one of them $e'_n = v_n v_{n+1}$ on C_n and 3 of them $e''_{2n-2} = v'_{n-1}v_n$, $e''_{2n-1} = v_n v'_n$, $e''_{2n-1} = v'_n v_{n+1}$ on C'_{2n} . If $z \in \{e'_n, e''_{2n-2}, e''_{2n-1}, e''_{2n}\}$, then

$$\begin{aligned} r(z|S) &= (0, 1, 2, \dots, 2, 1, 0, \overline{1, 2, 3, \dots, 3, 2, 1, 1}), \\ r(z|S) &= (1, 2, \dots, 2, 1, 0, \overline{2, 3, \dots, 3, 2, 0, 1}), \\ r(z|S) &= (1, 2, \dots, 2, 1, 0, \overline{2, 3, \dots, 3, 2, 1, 0}), \\ r(z|S) &= (0, 1, 2, \dots, 2, 0, 1, \overline{1, 2, 3, \dots, 3, 2, 1}), \end{aligned} \quad (8)$$

where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i types of vertices, respectively.

By observations iii and iv and Claim 3, there is no other edge in Sf_n with these type of codes.

Claim 4. Any two, 2 – 2 neighboring gaps contain together at most 4 number of vertices.

Let two 2 – 2 neighboring gaps g_a and g_b have 1 and 3 number of vertices, respectively. The gaps g_a and g_b are determined by v'_i, v'_{i+2} , and v'_{i+3} . The gap g_a has one vertex v_{i+2} and two edges $e''_{i+3} = v'_{i+1}v_{i+2}$, $e''_{i+4} = v_{i+2}v'_{i+2}$ on C'_{2n} . The g_b has 3 vertices v_{i+3}, v'_{i+3} , and v_{i+4} and 5 number of edges, one of them $e'_{i+3} = v_{i+3}v_{i+4}$ on C_n and 4 of them $e''_{i+5} = v'_{i+2}v_{i+3}$, $e''_{i+6} = v_{i+3}v'_{i+3}$, $e''_{i+7} = v'_{i+3}v_{i+4}$, $e''_{i+8} = v_{i+4}v'_{i+4}$ on C'_{2n} . If $v'_i, v'_{i+1}, v'_{i+2}, v'_{i+3}, v'_{i+4}, v'_{i+5} \in S$ and $z \in \{e'_{i+3}, e'_{i+5}, e'_{i+6}, e'_{i+7}, e'_{i+8}\}$, then

$$\begin{aligned} r(z|S) &= (2, 1, 0, 1, 2, \dots, 2, \overline{2, 0, 1, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 1, 0, 0, 1, 2, \dots, 2, \overline{3, 2, 1, 1, 1, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 1, 0, 1, 2, \dots, 2, \overline{2, 1, 0, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 3, 2, 0, 1, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 2, 1, 0, 2, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 3, 2, 0, 1, 3, \dots, 3}), \\ r(z|S) &= (2, 2, 2, 1, 0, 1, 2, \dots, 2, \overline{3, 3, 2, 1, 0, 2, 3, \dots, 3}), \end{aligned} \quad (9)$$

where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i types of vertices, respectively.

By observations iii, v, and vi and Claim 4, there is no other edge in Sf_n having these types of codes.

Claim 5. The neighboring 2 – 2 and 2 – 5 gaps contain together at most 3 number of vertices.

Let g_a and g_b be 2 – 2 and 2 – 5 neighboring gaps having 1 and 2 vertices, respectively. These gaps can be determined by v'_{i-7}, v'_{n-1} , and v_{n+1} . The gap g_a has one vertex v_{n-1} on C_n or two edges $e''_{2n-3} = v'_{n-1}v_{n-1}$, $e''_{2n-4} = v_{n-1}v'_n$ on C'_{2n} . The g_b has 2 vertices v_n, v'_n and 4 number of edges, one of them $e'_n = v_n v_{n+1}$ on C_n and 3 of them $e''_{2n-2} = v_{n-1}v_n$, $e''_{2n-1} = v_n v'_n$, and $e''_{2n} = v'_n v_{n+1}$ on C'_{2n} . If $v'_{n-7}, v'_{n-6}, v'_{n-4}, v'_{n-2}, v'_{n-1}, v'_{n+1} = v_i \in S$ and $z \in \{e'_n, e''_{2n-4}, e''_{2n-3}, e''_{2n-2}, e''_{2n-1}, e''_{2n}\}$, then

$$\begin{aligned}
r(z|S) &= (0, 1, 2, \dots, 2, 1, 0, \overline{1, 2, 3, \dots, 3, 2, 1, 1}), \\
r(z|S) &= (2, \dots, 2, 1, 0, 1, \overline{3, \dots, 3, 2, 0, 1, 2}), \\
r(z|S) &= (2, \dots, 2, 1, 0, 1, 1, 0, \overline{3, \dots, 3, 2, 1, 0, 2}), \\
r(z|S) &= (1, 2, \dots, 2, 1, 0, \overline{2, 3, \dots, 3, 2, 0, 1}), \\
r(z|S) &= (1, 2, \dots, 2, 1, 0, \overline{2, 3, \dots, 3, 2, 1, 0}), \\
r(z|S) &= (0, 1, 2, \dots, 2, 0, 1, \overline{1, 2, 3, \dots, 3, 2, 1}),
\end{aligned} \tag{10}$$

where the normal part of the code and overlined part of the code represent distance of z from v_i and v'_i types of vertices, respectively.

By observations iii, v, and vii and Claim 5, there is no other edge in Sf_n with these types of codes.

Theorem 3. *If $n \geq 6$, then $\text{edim}(Sf_n) = 2n/3$.*

Proof. The resolving set for Sf_5 is $S = \{v_2, v_3, v_7, v_9, v_{10}\}$.

First, we show that $\text{edim}(Sf_n) \leq 2n/3$ for $n \geq 6$ by constructing a resolving set for Sf_n with $2n/3$ vertices. We consider three cases according to the residue class modulo 3 to which n belongs.

Case 1. $n \equiv 0 \pmod{3}$. Let $2n = 3k$, where k is even, $k \geq 4$, and $2n/3 = k$. In this case, $S = \{v'_{3i-1}, v'_{3i} : 1 \leq i \leq (k/2) - 1\} \cup \{v'_{n-1}, v'_n\}$.

Case 2. $n \equiv 1 \pmod{3}$. Let $2n = 3k + 2$, where k is even, $k \geq 4$, and $2n/3 = k$. We define $S = \{v'_{3i}, v'_{3i+1} : 1 \leq i \leq (k/2) - 1\} \cup \{v'_{n+2}\}$.

Case 3. $n \equiv 2 \pmod{3}$. Let $2n = 3k + 1$, where k is odd, $k \geq 5$, and $2n/3 = k$ and $S = \{v'_{3i+2}, v'_{3i+2} : 1 \leq i \leq (k/2) - 1\} \cup \{v'_{n+2}\}$.

To show that $\text{edim}(Sf_n) \geq 2n/3$, we consider the aforementioned observations.

Let S be a basis of Sf_n and $|S| = r$. Then, S induces r gaps which are denoted as $g_1, \dots, g_r$ such that g_i and g_{i+1} for every $1 \leq i \leq r - 1$ and also g_1 and g_r are neighboring gaps. We can write $|g_1| + |g_2| \leq 3$, $|g_r| + |g_1| \leq 3$ and the relation for the remaining neighboring gaps is $|g_i| + |g_{i+1}| \leq 3$ for every $i = 2, \dots, r - 1$.

So, we have

$$\begin{aligned}
2(2n - r) &= 2 \sum_{i=1}^r |g_i| \leq 4(r - 2) + 6, \\
2(2n - r) &\leq 4r - 2,
\end{aligned} \tag{11}$$

$$r \geq \frac{4n + 2}{6} = \frac{2n}{3} + \frac{1}{3} \geq \left\lceil \frac{2n}{3} \right\rceil.$$

Since $r \in \mathbb{N}$, for each $n \equiv 0, 1$ or $2 \pmod{3}$, we obtain $r \geq 2n/3$. Hence, the theorem is proved. $\square$

2.4. Friendship Graph (F_n). The friendship graph F_n can be obtained from the wheel graph by removing the alternative edges of the C_n cycle of W_n . The number of vertices in the friendship graph are $|V| = 2n + 1$, and the number of edges

FIGURE 4: Friendship graph (F_4).

are $|E| = 3n$. We label the central vertex and remaining vertices as v and v_i , respectively, where $1 \leq i \leq 2n$. There are two types of edges, and we label them in such a way $e_i = vv_i$ and $e'_j = v_{2j-1}v_{2j}$ where $j \in \{1, 2, \dots, n\}$, for reference, see Figure 4

The metric dimension of friendship graph $\text{dim}(F_n) = n$ for $n \geq 2$, computed by Mulyono and Wulandari in [31].

Theorem 4. *Let F_n be a friendship graph, then $\text{edim}(F_n) = 2n - 1$ for $n \geq 3$.*

Proof. Let $n \geq 3$ and $V(F_n) = \{v, v_1, v_2, \dots, v_{2n}\}$, where $d(v) = 2n$ and $d(v_i) = 2$, where $i \in \{1, 2, \dots, 2n\}$. To prove that S is a minimum edge resolving set, we consider the following three cases:

Case 1. If $e_i = vv_i$, then e_i has distance 0 to $\{v, v_i\}$ and distance 1 to all other vertices, where $i \in \{1, 2, \dots, 2n\}$.

Case 2. If $e'_i = v_i v_{i+1}$, then e'_i has distance 0 to v_i and v_{i+1} distance 1 to v , distance 2 to all other vertices.

The above cases show that the edge metric representation of edges of $G \cong F_n$ with respect to a set with $2n - 2$ vertices is not distinct. So S can be edge metric generator if we take $2n - 1$ number of vertices. Therefore, the edge metric dimension of the friendship graph is $2n - 1$.

We also show that the by considering the resolving set, S of the size $2n - 2$. Let $S = \{v_i, v_{i+1}, \dots, v_{2n-2}\}$, then $r(e_{n-1} | S) = r(e_n | S)$ or $r(e_i | S) = r(e_n | S)$. So set S is not the basis of F_n . Hence, it is proved that $\text{edim}(F_n) = 2n - 1$.

Note. The total number of edge resolving sets for a fan graph (F_n) is $2n$. $\square$

3. Conclusion

In this article, the edge metric dimension for certain families of wheel-related graphs, like Jahangir graph J_{2n} , helm graph H_n , sunflower graph Sf_n , and friendship graph F_n , have been studied and proved that all these families have

unbounded edge metric dimensions. We also compared the results with their metric dimensions and concluded that $\dim(G) = \text{edim}(G)$ if $G \cong J_{2n}, H_n$, and $\dim(G) < \text{edim}(G)$ if $G \cong Sf_n, F_n$.

4. Open Problems

It is natural to ask about the characterization of the graph families on the bases of the nature of their edge metric dimension. To answer the aforementioned question, the readers are invited to study the following problems.

- (1) Characterize the nanostructures, nanocones, nanotubes, nanotorus, and chemical trees on the basis of edge metric dimension.
- (2) Is it the case that the other families of graphs generated by wheel also have unbounded edge metric dimension?

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors contributed equally to preparing this article.

Acknowledgments

The authors are grateful to Prof. Jia-Bao Liu for his useful suggestions and help to improve and complete this manuscript. This study was supported by Major University Science Research Project of Anhui Province (KJ2016A605) and Major Nature Science Project of Hefei University Research and Development Foundation (16ZR13ZDA).

References

- [1] P. J. Slater, "Leaves of trees," *Congressus Numerantium*, vol. 14, pp. 549–559, 1975.
- [2] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.
- [3] G. Chartrand, V. Saenpholphat, and P. Zhang, "The independent resolving number of a graph," *Mathematica Bohemica*, vol. 128, no. 4, pp. 379–393, 2003.
- [4] R. C. Brigham, G. Chartrand, R. D. Dutton, and P. Zhang, "Resolving domination in graphs," *Mathematica Bohemica*, vol. 128, no. 1, pp. 25–36, 2003.
- [5] O. R. Oellermann and J. Peters-Fransen, "The strong metric dimension of graphs and digraphs," *Discrete Applied Mathematics*, vol. 155, no. 3, pp. 356–364, 2007.
- [6] F. Okamoto, S. Phinezyn, and P. Zhang, "The local metric dimension of a graph," *Mathematica Bohemica*, vol. 135, no. 3, pp. 239–255, 2010.
- [7] A. Estrada-Moreno, J. A. Rodríguez-Velázquez, and I. G. Yero, "The k-metric dimension of a graph," *Applied Mathematics & Information Sciences*, vol. 9, no. 6, pp. 2829–2840, 2015.
- [8] I. G. Yero, A. Estrada-Moreno, and J. A. Rodríguez-Velázquez, "Computing the k-metric dimension of graphs," *Applied Mathematics and Computation*, vol. 300, pp. 60–69, 2017.
- [9] Y. Ramírez-Cruz, O. R. Oellermann, and J. A. Rodríguez-Velázquez, "The simultaneous metric dimension of graph families," *Discrete Applied Mathematics*, vol. 198, pp. 241–250, 2016.
- [10] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of a graph," *Aequationes Mathematicae*, vol. 59, no. 1, pp. 45–54, 2000.
- [11] I. G. Yero, "On the strong partition dimension of graphs," *The Electronic Journal of Combinatorics*, vol. 21, no. 3, 3.14 pages, 2014.
- [12] R. Trujillo-Rasua and I. G. Yero, "k-metric antidimension: a privacy measure for social graphs," *Information Sciences*, vol. 328, pp. 403–417, 2016.
- [13] Z. Beerliova, F. Eberhard, T. Erlebach et al., "Network discovery and verification," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 12, pp. 2168–2181, 2006.
- [14] R. A. Melter and I. Tomescu, "Metric bases in digital geometry," *Computer Vision, Graphics, and Image Processing*, vol. 25, no. 1, pp. 113–121, 1984.
- [15] J. Cáceres, C. Hernando, M. Mora et al., "On the metric dimension of cartesian products of graphs," *SIAM Journal on Discrete Mathematics*, vol. 21, no. 2, pp. 423–441, 2007.
- [16] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, no. 1–3, pp. 99–113, 2000.
- [17] G. Chartrand, C. Poisson, and P. Zhang, "Resolvability and the upper dimension of graphs," *Computers & Mathematics with Applications*, vol. 39, no. 12, pp. 19–28, 2000.
- [18] M. Johnson, "Structure-activity maps for visualizing the graph variables arising in drug design," *Journal of Biopharmaceutical Statistics*, vol. 3, no. 2, pp. 203–236, 1993.
- [19] J.-B. Liu, Z. Zahid, R. Nasir, and W. Nazeer, "Edge version of metric dimension and doubly resolving sets of the necklace graph," *Mathematics*, vol. 6, no. 11, p. 243, 2018.
- [20] A. Kelenc, N. Tratnik, and I. G. Yero, "Uniquely identifying the edges of a graph: the edge metric dimension," *Discrete Applied Mathematics*, vol. 251, pp. 204–220, 2018.
- [21] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, NY, USA, 1979.
- [22] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217–229, 1996.
- [23] N. Zulbrilina, "On edge metric dimension of a graph," *Discrete Mathematics*, vol. 341, pp. 2083–2088, 2018.
- [24] M. Rafiullah, H. M. A. Siddiqui, and A. Sarfraz, "Resolvability of some convex polytopes," *Utilitas Mathematica*, vol. 111, pp. 309–328, 2019.
- [25] M. Imran and H. M. A. Siddiqui, "Computing the metric dimension of convex polytopes generated by wheel related graphs," *Acta Mathematica Hungarica*, vol. 149, no. 1, pp. 10–30, 2016.
- [26] J.-B. Liu, A. Kashif, T. Rashid, and M. Javaid, "Fractional metric dimension of generalized jahangir graph," *Mathematics*, vol. 7, no. 1, p. 100, 2019.

- [27] H. M. A. Siddiqui and M. Imran, "Computing the metric dimension of wheel related graphs," *Applied Mathematics and Computation*, vol. 242, pp. 624–632, 2014.
- [28] J. A. Gallian, "A dynamic survey of graph labeling," *The Electronic Journal of Combinatorics*, vol. 16, no. 6, pp. 1–219, 2009.
- [29] I. Tomescu and I. Javaid, "On the metric dimension of the Jahangir graph," *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 50, no. 98, pp. 371–376, 2007.
- [30] I. Javaid, *On metric and partition dimension of some wheel related graphs*, Ph.D. thesis, GC University Lahore, Lahore, Pakistan, 2007.
- [31] W. Mulyono, "The metric dimension of friendship graph F_n , lollipop graph $L_{m,n}$ and peterson graph, $P_{n,m}$," *Bulletin of Mathematics*, vol. 8, no. 2, pp. 117–124, 2016.