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Research Article

Computation of Polynomial Degree-Based Topological Descriptors of Indu-Bala Product of Two Paths

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Cheminformatics is entirely a newly coined term that encompasses a field that includes engineering computer sciences along with basic sciences. As we all know, vertices and edges form a network whereas vertex and its degrees contribute to joining edges. The degree of vertex is very much dependent on a reasonable proportion of network properties. There is no doubt that a network has to have a reliance of different kinds of hub buses, serials, and other connecting points to constitute a system that is the backbone of cheminformatics. The Indu-Bala product of two graphs G_1 and G_2 has a special notation as described in Section 2. The attainment of this product is very much due to related vertices at to different places of $G_1 \vee G_2$. This study states we have found M-polynomial and degree-based topological indices for Indu-Bala product of two paths P_k and P_j for $j, k \geq 2$. We also give some graphical representation of these indices and analyzed them graphically.

1. Introduction

Let $G = (V(G), E(G))$ be a simple and finite graph of order n. We denote the nonempty vertex set by $V(G)$ and edge set by $E(G)$. The fields of chemistry information sciences and mathematics have undoubtedly revolutionized by cheminformatics. It is a new subject that is very much helpful in keeping the data and getting information about chemicals. For this purpose, i.e., keeping the data and storing information, a significant help can be taken from the theory represented by graph in order to make index factors. The study of molecules according to their structures and their different functions based on QSAR models is also called a biological activity. The indictors that represent topology are also known as a subsidiary of the biological activity. Topological indices can be calculated using simply points (atoms) and linkages (chemical bonds) in a graphical representation. A polynomial, numeric number, a sequence of numbers, or an array representing the full graph can be used to identify it, and these representations are meant to be calculated particularly for that graph. The values in mathematics serve as indicators that have a logical connection to the graph and its topology. These are the indicators that give various dimensions and kinds to topological indices from distance based to degree based, counting conjugal polynomials and graphs. In chemistry and especially in graph theory, the degree-based topological indices play an essential role. Precisely, we can say that Σ gives new shape to the index connected with topology from real numbers to its zenith. Various indicator networks are always present in an entangled form of links nodes and hubs in a network. For example, various networks have similarities in atomic structure or molecular structure, such as honeycomb, grid networks, and hexagonal. Topological properties of these networks are very interesting, which are studied in various aspects, such as minimum metric dimension of a honeycomb network in [1] and silicate network in [2], topological properties of this network in [3], and topological indicators of honeycomb, silicate, and hexagonal networks in [4]. As we study the evolution of the things biologically, different kinds of structures having six dimensions and beehive shapes come into our contact. Many authors have researched on this topic; Hayat et al. computed topological indices of some networks in [5] and for some interconnection networks in

[6]. On organized populations, Perc et al. studied the evolutionary dynamics of group interactions in [7] and on coevolutionary games in [8], and Szolnoki et al. further worked on the impact of noise on cooperation in spatial public goods games in topology-independent ways in [9] and on importance of percolation for evolution of cooperation in [10]. Mathematical references have also been found in research of paraffin, Wiener's approach [11]. Wiener invented the index that is also known as the route number. This topological descriptor formed the basis for the topological indices, in terms of theory and application in [12, 13]. Therefore, the topological indices in the chemical and quantitative literature are Weiner in [14], Zagreb in [15], and Randic in [16]. The Indu-Bala product $G_1 \blacktriangledown G_2$ of graphs G_1 and G_2 is obtained from two disjoint copies of the join $G_1 \vee G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2.

In this paper, we calculated some well-known topological indicators based on M-polynomial and degree based indices for Indu-Bala product of two paths.

$$M(G, z_1, z_2) = \sum_{\theta \le i \le j \le \vartheta} m_{ij}(G) z_1^i z_2^j. \tag{1}$$

As in [17], $\theta = \min\{d_s | s \in V(G)\}$, $\vartheta = \mathrm{Max}\{d_s | s \in V(G)\}$; $m_{ij}(G)$ is the edge $z_1 z_2 \in E(G); s \le t$.

Milan Randic in 1975 established the concept of Randic index [18–20], which is represented as $R_{-1/2}(G)$:

$$R_{-1/2}(G) = \sum_{z_1 z_2 \in E(G)} \left(\frac{1}{\sqrt{d_{z_1} d_{z_2}}} \right). \tag{2}$$

The generalized Randic index is defined as [21–28]

$$R_\alpha(G) = \sum_{z_1 z_2 \in E(G)} \left(\frac{1}{\left(d_{z_1} d_{z_2}\right)^\alpha} \right). \tag{3}$$

Two indices $M_1(G), M_2(G)$ were established by Gutman and Trinajstic defined as follows:

$$M_1(G) = \sum_{z_1 z_2 \in E(G)} \left(d_{z_1} + d_{z_2}\right), M_2(G) = \sum_{z_1 z_2 \in E(G)} \left(d_{z_1} d_{z_2}\right). \tag{4}$$

Another form of index is which is known as second Zagreb is define as follows [11, 29–32]:

$${}^m M_2(G) = \sum_{z_1 z_2 \in G} \left(\frac{1}{d(z_1) d(z_2)} \right). \tag{5}$$

The symmetric division index is defined as

$$\mathrm{SDD}(G) = \sum_{z_1 z_2 \in E(G)} \left\{ \frac{\left(\min\left(d_{z_1}, d_{z_2}\right)\right)}{\max\left(d_{z_1}, d_{z_2}\right)} + \frac{\max\left(d_{z_1}, d_{z_2}\right)}{\min\left(d_{z_1}, d_{z_2}\right)} \right\}. \tag{6}$$

The harmonic index is defined as

$$H(G) = \sum_{z_1 z_2 \in E(G)} \left(\frac{2}{d_{z_1} + d_{z_2}} \right). \tag{7}$$

"The inverse sum index is defined as

$$I(G) = \sum_{z_1 z_2 \in E(G)} \left(\frac{d_{z_1} d_{z_2}}{d_{z_1} + d_{z_2}} \right). \tag{8}$$

The augmented Zagreb index is defined as

$$A(G) = \sum_{z_1 z_2 \in E(G)} \left\{ \frac{d_{z_1} d_{z_2}}{d_{z_1} + d_{z_2} - 2} \right\}^3. \tag{9}$$

These indices and deliberations which various researchers laboriously worked on can be seen in [33–37] as authentic references.

$$\begin{aligned} D_\gamma &= \gamma \frac{\partial(f(\gamma, \varepsilon)}{\partial \gamma}, \\ D_\varepsilon &= \varepsilon \frac{\partial(f(\gamma, \varepsilon)}{\partial \varepsilon}, \\ S_\gamma &= \int \frac{f(\gamma, \varepsilon)}{\gamma} \mathrm{d}\gamma, \\ S_\varepsilon &= \int \frac{f(\gamma, \varepsilon)}{\varepsilon} \mathrm{d}\varepsilon, \\ J(h(\gamma, \varepsilon)) &= f(\gamma, \gamma), Q_\alpha(h(\gamma, \varepsilon)) = \gamma^\alpha h(\gamma, \varepsilon). \end{aligned} \tag{10}$$

2. Computational Results on Topological Indices for Indu-Bala Product of Two Paths P_k and P_j When $k > j$

Our fundamental objective of studying M-polynomial and all its related components is to establish a relationship between various affects of M-polynomials and its related things on the Indu-Bala graph, see Figure 1.

2.1. Results. We split vertices and edges degree of the Indu-Bala graph in Table 1. Similarly, we split the edge palpitations of points on the Indu-Bala graph in Table 2.

Theorem 1. *Let* **G** *be a Indu-Bala graph* $P_k \blacktriangledown P_j$, *where* $j \ge 2, k \ge j + 2$. *We have*

$$\begin{aligned} M(G; a, b) = {} & 4a^{j+1} b^{j+2} + 4ja^{j+1} b^{k+2} + 2(k-3) a^{j+1} b^{k+2} \\ & + 2j(k-2) a^{j+2} b^{j+2} + 2ja^{k+2} b^{k+2}. \end{aligned} \tag{11}$$

Proof. As in Figure 1, now we will compute M-polynomial using the values of Tables 1 and 2:

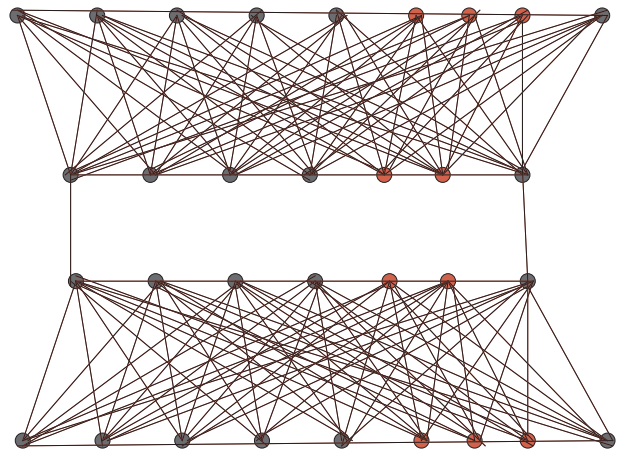


Figure 1: Indu-Bala graph $P_9 \blacktriangledown P_7$.

Table 1: Number of vertices of different degrees of Indu-Bala graph.

Vertices and edges	Total no. of $V(G)$ and $E(G)$
$V(G)$	$2(j+k)$
$E(G)$	$2jk+2(j+k)-2$

Table 2: Points of edge partitioning of $P_k \blacktriangledown P_j$.

Edges	Deg. of end nodes	Total no. of edges
E_1	$d(h)=j+1, d(i)=j+2$	4
E_2	$d(h)=j+1, d(i)=k+2$	$4j$
E_3	$d(h)=j+2, \mathrm{d}i=j+2$	$2(k-3)$
E_4	$d(h)=j+2, d(i)=k+2$	$2j(k-2)$
E_5	$d(h)=k+2, d(i)=k+2$	$2j$

$$
\begin{aligned}
M(G;a,b) &= \sum_{i\le j} m_{ij}\left(G;a^i b^j\right),\\
&= \sum_{j+1\le j+2} m_{j+1j+2}(G)a^{j+1}b^{j+2} + \sum_{j+1\le k+2} m_{j+1k+2}(G)a^{j+1}b^{k+2}\\
&\quad + \sum_{j+2\le j+2} m_{j+2j+2}(G)a^{j+2}b^{j+2} + \sum_{j+2\le k+2} m_{j+2k+2}(G)a^{j+2}b^{k+2}\\
&\quad + \sum_{k+2\le k+2} m_{k+2k+2}(G)a^{k+2}b^{k+2},\\
&= \sum_{uv\in E_{j+1,j+2}} m_{j+1j+2}(G)a^{j+1}b^{j+2} + \sum_{uv} \in E_{j+1}, k+2m_{j+1k+2}(G)a^{j+1}b^{k+2}\\
&\quad + \sum_{uv} \in E_{j+2,j+2} m_{j+2j+2}(G)a^{j+2}b^{j+2} + \sum_{uv} \in E_{j+2,k+2} m_{j+2k+2}(G)a^{j+2}b^{k+2}\\
&\quad + \sum_{uv\in E_{k+2,k+2}} m_{k+2k+2}(G)a^{k+2}b^{k+2},\\
&= \left|E_{j+1,j+2}\right| a^{j+1}b^{j+2} + \left|E_{j+1,k+2}\right| a^{j+1}b^{k+2} + \left|E_{j+2,j+2}\right| a^{j+2}b^{j+2}\\
&\quad + \left|E_{j+2,k+2}\right| a^{j+2}b^{k+2} + \left|E_{k+2,k+2}\right| a^{k+2}b^{k+2},\\
&= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+1}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2}.
\end{aligned}
\tag{12}
$$

□

Theorem 2. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$. Then,*

$$M_1(G) = 2jk(j+k+8) + 6(k-1). \tag{13}$$

Proof.

$$\begin{aligned}
M(G,a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
(G,a,b) &= f(a,b); D_a f(a,b) = a\frac{\partial f}{\partial a},\\
\frac{\partial f}{\partial a} &= \frac{\partial}{\partial a}\left\{4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2}\right\},\\
a\frac{\partial f}{\partial a} &= 4(j+1)a^{j+1}b^{j+2} + 4j(j+1)a^{j+1}b^{k+2} + 2(j+2)(k-3)a^{j+2}b^{j+2}\\
&\quad + 2j(j+2)(k-2)a^{j+2}b^{k+2} + 2j(k+2)a^{k+2}b^{k+2}; D_b f(a,b) = b\frac{\partial f}{\partial b},\\
\frac{\partial f}{\partial b} &= \frac{\partial}{\partial b}\left\{4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2}\right\},\\
b\frac{\partial f}{\partial b} &= 4(j+2)a^{j+1}b^{j+2} + 4j(k+2)a^{j+1}b^{k+2} + 2(j+2)(k-3)a^{j+2}b^{j+2}\\
&\quad + 2j(k+2)(k-2)a^{j+2}b^{k+2} + 2j(k+2)a^{k+2}b^{k+2},\\
M_1(G) &= (D_a + D_b)(f(a,b)|_{(a=b=1)}
\end{aligned} \tag{14}$$

□

Theorem 3. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$M_2(G) = 8(k-j) + j^2\left(2k^2 + 6k - 6\right) + 2jk(10+3k) - 16. \tag{15}$$

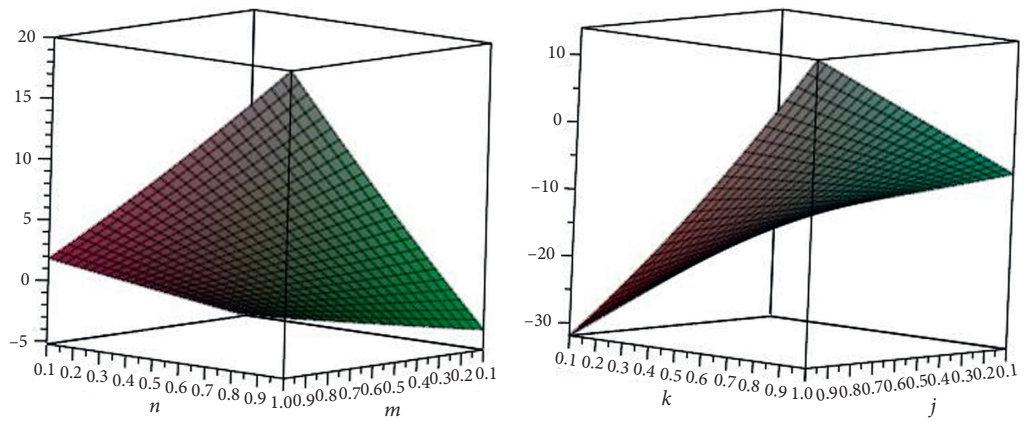
Proof. Suppose

$$\begin{aligned}
M(G,a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
D_a f(a,b) &= 4(j+1)a^{j+1}b^{j+2} + 4j(j+1)a^{j+1}b^{k+2} + 2(j+2)(k-3)a^{j+2}b^{j+2}\\
&\quad + 2j(j+2)(k-2)a^{j+2}b^{k+2} + 2j(k+2)a^{k+2}b^{k+2},\\
D_b D_a f(a,b) &= 4(j+1)(j+2)a^{j+1}b^{j+2} + 4j(j+1)(k+2)a^{j+1}b^{k+2} + 2(j+2)^2(k-3)a^{j+2}b^{j+2}\\
&\quad + 2j(j+2)(k+2)(k-2)a^{j+2}b^{k+2} + 2j(k+2)(k+2)a^{k+2}b^{k+2},\\
M_2(G) &= D_b D_a f(a,b)|_{(a=b=1)}\\
&\quad + 2(j+2)^2(k-3) + 2j(j+2)\left(k^2-4\right) + 2j(k+2)^2,\\
M_2(G) &= 8(k-j) + j^2\left(2k^2+6k-6\right) + 2jk(10+3k) - 16.
\end{aligned} \tag{16}$$

□

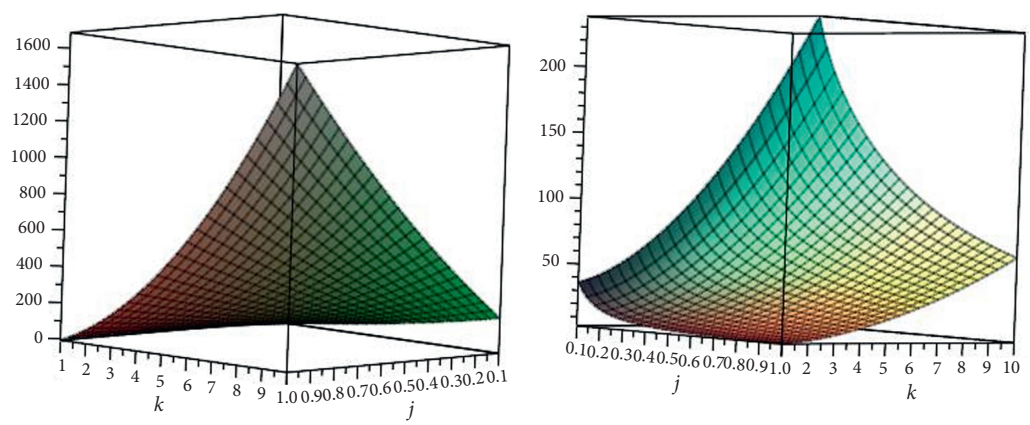
Theorem 4. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then (Figures 2 and 3),*

$$^{m}M_2(G) = \frac{2j^4 + j^3\left(2k^2-2k+4\right) + j^2\left(2k^2-2k+8\right) + j\left(2k^3+8k^2-8k+6\right) + 4(k+1)^2}{j(j+1)^2(k+1)^2}. \tag{17}$$



(a) (b)

Figure 2: (a) First Zagreb index for $j \leq k$. (b) Second Zagreb index for $j \leq k$.



(a) (b)

Figure 3: (a) Randic index for $j \leq k$ and $\alpha = 1$. (b) Modified second Zagreb index for $j \leq k$ and $\alpha = 1$.

Proof. Suppose

$$M(G; a, b) = 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},$$

$$^{m}M_2(G) = S_a S_b f(a, b)|_{(a=b=1)},$$

$$S_a = \int_0^a \frac{f(u, b)}{u} \mathrm{d}u,$$

$$S_b = \int_0^b \frac{f(a, u)}{u} \mathrm{d}u,$$

$$\frac{f(u, b)}{u} = 4u^j b^{j+2} + 4uu^j b^{k+2} + 2(k-3)u^{j+1}b^{j+2} + 2u(k-2)u^{j+1}b^{k+2} + 2uu^{k+1}b^{k+2}, *,$$

$$S_a = \frac{4}{j}a^{j+1}b^{j+2} + 4a^{j+1}b^{k+2} + \frac{2(k-3)}{j+1}a^{j+2}b^{j+2} + \frac{2j(k-2)}{j+1}a^{j+2}b^{k+2} + \frac{2j}{k+1}a^{k+2}b^{k+2},$$

$$\begin{aligned}
\int_0^a \frac{f(a,u)}{u}\,\mathrm{d}t &= \frac{4}{j}a^{j+1}u^{j+1} + 4a^{j+1}u^{k+1} + \frac{2(k-3)}{j+1}a^{j+2}u^{j+1} + \frac{2j(k-2)}{j+1}a^{j+2}u^{k+1} + \frac{2j}{k+1}a^{k+2}u^{k+1},\\
S_bS_af(a,b) &= \int_0^b \frac{S_af(a,u)}{u}\,\mathrm{d}u,\\
S_bS_af(a,b) &= \frac{4}{j(j+1)}a^{j+1}b^{j+2} + \frac{4}{k+1}a^{j+1}b^{k+2} + \frac{2(k-3)}{(j+1)^2}a^{j+2}b^{j+2}\\
&\quad + \frac{2j(k-2)}{(j+1)(k+1)}a^{j+2}b^{k+2} + \frac{2j}{(k+1)^2}a^{k+2}b^{k+2},\\
{}^{m}M_2(G) &= S_bS_af(a,b)|_{(a=b=1)} = \frac{4}{j(j+1)} + \frac{4}{k+1} + \frac{2(k-3)}{(j+1)^2} + \frac{2j(k-2)}{(j+1)(k+1)} + \frac{2j}{(k+1)^2},\\
&= \frac{2j^4 + j^3\left(2k^2-2k+4\right) + j^2\left(2k^2-2k+8\right) + j\left(2k^3+8k^2-8k+6\right) + 4(k+1)^2}{j(j+1)^2(k+1)^2}.
\end{aligned} \tag{18}$$

□

Theorem 5. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$\begin{aligned}
R_\alpha(G) &= 4(j+1)^\alpha\{(j+2)^\alpha + j(k+2)^\alpha\} + 2(j+2)^\alpha\{(k-3)(j+2)^\alpha + 2j(k-2)(k+2)^\alpha\}\\
&\quad + 2j(k+2)^{2\alpha}.
\end{aligned} \tag{19}$$

Proof. Suppose

$$\begin{aligned}
M(G;a;b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
R_\alpha(G) &= D_a^\alpha D_b^\alpha f(a,b)|_{(b=a=1)},\\
D_b^\alpha f(a,b) &= 4(2+j)^\alpha a^{j+1}b^{2+j} + 4j(2+k)^\alpha a^{j+1}b^{2+k} + 2(-3+k)(j+2)^\alpha a^{j+2}b^{j+2}\\
&\quad + 2j(-2+k)(2+k)^\alpha a^{j+2}b^{2+k} + 2j(2+k)^\alpha a^{k+2}b^{2+k},\\
D_a^\alpha D_b^\alpha f(a,b) &= 4(j+1)^\alpha(j+2)^\alpha a^{j+1}b^{j+2} + 4j(j+1)^\alpha(k+2)^\alpha a^{j+1}b^{k+2}\\
&\quad + 2(k-3)(j+2)^{2\alpha}a^{j+2}b^{j+2} + 2j(-2+k)(2+j)^\alpha(2+k)^\alpha a^{j+2}b^{k+2}\\
&\quad + 2j(k+2)^{2\alpha}a^{k+2}b^{k+2},\\
R_\alpha(G) &= D_a^\alpha D_b^\alpha f(a,b)|_{(a=b=1)} = 4(j+1)^\alpha(j+2)^\alpha + 4j(j+1)^\alpha(k+2)^\alpha\\
&\quad + 2(-3+k)(2+j)^{2\alpha} + 2j(-2+k)(2+j)^\alpha(2+k)^\alpha + 2j(2+k)^{2\alpha},\\
R_\alpha(G) &= 4(1+j)^\alpha\{(2+j)^\alpha + j(2+k)^\alpha\} + 2(2+j)^\alpha\{(-3+k)(2+j)^\alpha + 2j(-2+k)(k+2)^\alpha\}\\
&\quad + 2j(2+k)^{2\alpha}.
\end{aligned} \tag{20}$$

□

Theorem 6. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2, k \geq j+2$. Then,*

$$RR_\alpha(G) = \frac{4}{(1+j)^\alpha}\left\{\frac{1}{j^\alpha} + \frac{(k-3)}{2(j+1)^\alpha} + \frac{j(-2+k)}{2(1+k)^\alpha}\right\} + \frac{4}{(1+k)^\alpha}\left\{\frac{j}{2(k+1)^\alpha} + 1\right\}. \tag{21}$$

Proof. Suppose

$$
\begin{aligned}
M(G;a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
RR_\alpha(G) &= S_a^\alpha S_b^\alpha f(a,b)|_{(a=b=1)},\\
S_b &= \frac{4}{j+1}a^{j+1}b^{j+2} + \frac{4j}{k+1}a^{j+1}b^{k+2} + \frac{2(k-3)}{j+1}a^{j+2}b^{j+2} + \frac{2j(k-2)}{k+1}a^{j+2}b^{k+2}\\
&\quad + \frac{2j}{k+1}a^{k+2}b^{k+2},\\
S_b^\alpha &= \frac{4}{(j+1)^\alpha}a^{j+1}b^{j+2} + \frac{4j}{(k+1)^\alpha}a^{j+1}b^{k+2} + \frac{2(k-3)}{(j+1)^\alpha}a^{j+2}b^{j+2}\\
&\quad \frac{2j(k-2)}{(k+1)^\alpha}a^{j+2}b^{k+2} + \frac{2j}{(k+1)^\alpha}a^{k+2}b^{k+2},\\
S_a^\alpha S_b^\alpha f(a,b) &= \frac{4}{(j(j+1))^\alpha}a^{j+1}b^{j+2} + \frac{4}{(k+1)^\alpha}a^{j+1}b^{k+2} + \frac{2(k-3)}{(j+1)^{2\alpha}}a^{j+2}b^{j+2}\\
&\quad + \frac{2j(k-2)}{((j+1)(k+1))^\alpha}a^{j+2}b^{k+2} + \frac{2j}{(k+1)^{2\alpha}}a^{k+2}b^{k+2},\\
S_a^\alpha S_b^\alpha f(a,b)|_{(a=b=1)} &= \frac{4}{(j(j+1))^\alpha} + \frac{4}{(1+k)^\alpha} + \frac{2(k-3)}{(1+j)^{2\alpha}} + \frac{2j(k-2)}{((1+j)(1+k))^\alpha} + \frac{2j}{(1+k)^{2\alpha}},\\
&\quad RR_\alpha(G)\\
&= \frac{4}{(1+j)^\alpha}\left\{\frac{1}{j^\alpha} + \frac{(-3+k)}{2(j+1)^\alpha} + \frac{j(-2+k)}{2(1+k)^\alpha}\right\} + \frac{4}{(1+k)^\alpha}\left\{\frac{j}{2(1+k)^\alpha} + 1\right\}.
\end{aligned}
\tag{22}
$$

□

Theorem 7. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$
\mathrm{SSD}(G) = \frac{j^4(k+2) + j^3\left(4k^2+3k+2\right) + j^2\left(6k^2+3k-2\right) + 4(k+1)(3jk+2)}{j(j+1)(k+1)}. \tag{23}
$$

Proof. Suppose

$$
\begin{aligned}
M(G;a;b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2}\\
&\quad + 2ja^{k+2}b^{k+2},\\
\mathrm{SSD}(G) &= \left(S_bD_a + S_aD_b\right)f(a,b)|_{(a=b=1)},\\
S_bD_af(a,b) &= 4a^{j+1}b^{j+2} + \frac{4j(j+1)}{k+1}a^{j+1}b^{k+2} + \frac{2(j+2)(k-3)}{j+1}a^{j+2}b^{j+2}\\
&\quad + \frac{2j(j+2)(k-2)}{k+1}a^{j+2}b^{k+2} + \frac{2j(k+2)}{k+1}a^{k+2}b^{k+2},
\end{aligned}
$$

$$
\begin{aligned}
S_aD_bf(a,b) &= \frac{4(j+2)}{j}a^{j+1}b^{j+2} + 4(k+2)a^{j+1}b^{k+2} + \frac{2(j+2)(k-3)}{j+1}a^{j+2}b^{j+2} \\
&\quad + \frac{2j(k+2)(k-2)}{j+1}a^{j+2}b^{k+2} + \frac{2j(k+2)}{k+1}a^{k+2}b^{k+2}; \left(S_bD_a + S_aD_b\right)f(a,b), \\
&= \left\{4 + \frac{4(j+2)}{j}\right\}a^{j+1}b^{j+2} + \left\{\frac{4j(j+1)}{k+1} + 4(k+2)\right\}a^{j+1}b^{k+2} \\
&\quad + \left\{\frac{2(j+2)(k-3)}{j+1} + \frac{2(j+2)(k-3)}{j+1}\right\}a^{j+2}b^{j+2} + \left\{\frac{2j(j+2)(k-2)}{k+1}\right. \\
&\quad \left. + \frac{2j(k+2)(k-2)}{j+1}\right\}a^{j+2}b^{k+2} + \left\{\frac{2j(k+2)}{k+1} + \frac{2j(k+2)}{k+1}\right\}a^{k+2}b^{k+2}, \\
\mathrm{SSD}(G) &= \left(S_bD_a + S_aD_b\right)f(a,b)|_{(a=b=1)} = \frac{8(j+1)}{j} + \frac{4}{k+1}\left(j^2 + k^2 + j + 3k + 2\right) \\
&\quad + \frac{4(j+2)(k-3)}{j+1} + \frac{j(k-2)}{(j+1)(k+1)}\left(j^2 + k^2 + 3(j+k) + 4\right) + \frac{4j(k+2)}{k+1}, \\
\mathrm{SSD}(G) &= \frac{4j^2k - 4j^2 + 8jk - 8j + 8}{j(j+1)} + \left(4(j+1)\left(j^2 + k^2 + j + 3k + 2\right)\right. \\
&\quad \left. + j(k-2)\left(j^2 + k^2 + 3j + 3k + 4\right) + j(k+2)\right)\frac{1}{(j+1)(k+1)}, \\
&= \frac{4j^2k - 4j^2 + 8jk - 8j + 8}{j(j+1)} \\
&\quad + \frac{j^3(k+2) + j^2\left(4k^2 + 3k + 2\right) + j\left(2k^2 + 3k + 2\right) + 4k(k+3) + 8}{(j+1)(k+1)}, \\
\mathrm{SSD}(G) &= \frac{j^4(k+2) + j^3\left(4k^2 + 3k + 2\right) + j^2\left(6k^2 + 3k - 2\right) + 4(k+1)(3jk+2)}{j(j+1)(k+1)}.
\end{aligned}
\tag{24}
$$

□

Theorem 8. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$
H(G) = \frac{2j^3 + 4j^2\left(k^2 + k\right) + j\left(4k^2 + 10k + 12\right) + 8(k+1)}{j(j+1)(k+1)}.
\tag{25}
$$

Proof. Suppose

$$
\begin{aligned}
M(G;a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2}, \\
H(G) &= 2S_ajf(a,b)|_{a=1}, \\
S_af(a,b) &= \frac{4}{j}a^{j+1}b^{j+2} + 4a^{j+1}b^{k+2} + \frac{2(k-3)}{j+1}a^{j+2}b^{j+2} + \frac{2j(k-2)}{j+1}a^{j+2}b^{k+2} + \frac{2j}{k+1}a^{k+2}b^{k+2}, \\
2S_ajf(a,b) &= \frac{8}{j}a^{2j+3} + 8a^{j+k+3} + \frac{4(k-3)}{j+1}a^{2j+4} + \frac{4j(k-2)}{j+1}a^{j+k+4} + \frac{4j}{k+1}a^{2k+4}, \\
2S_ajf(a,b)|_{a=1} &= \frac{8}{j} + 8 + \frac{4(k-3)}{j+1} + \frac{4j(k-2)}{j+1} + \frac{4j}{k+1}, \\
H(G) &= \frac{2j^3 + 4j^2\left(k^2 + k\right) + j\left(4k^2 + 10k + 12\right) + 8(k+1)}{j(j+1)(k+1)}.
\end{aligned}
\tag{26}
$$

□

Theorem 9. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$I(G) = \frac{4j^5 + 20j^4 + j^3\left(8k^2 + 40\right) + j^2\left(4k^3 + 14k^2 - 8k + 36\right) + j\left(16k^2 - 20k - 20\right) + 8(k+1)}{j(j+1)(k+1)}. \tag{27}$$

Proof. Suppose

$$\begin{aligned}
M(G;a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
I(G) &= S_a jD_aD_b f(a,b)|_{a=1}; D_aD_b f(a,b),\\
&= 4(j+1)(j+2)a^{j+1}b^{j+2} + 4j(j+1)(k+2)a^{j+1}b^{k+2} + 2(j+2)^2(k-3)a^{j+2}b^{j+2}\\
&\quad + 2j(j+2)(k+2)(k-2)a^{j+2}b^{k+2} + 2j(k+2)(k+2)a^{k+2}b^{k+2}; S_a jD_aD_b f(a,b),\\
&= \frac{4(j+1)(j+2)}{j}a^{2j+3} + 4(j+1)(k+2)a^{j+k+3} + \frac{2(j+2)^2(k-3)}{j+1}a^{2j+4}\\
&\quad + \frac{2j(j+2)(k+2)(k-2)}{j+1}a^{j+k+4} + \frac{2j(k+2)(k+2)}{k+1}a^{2k+4}; S_a jD_aD_b f(a,b)|_{a=1},\\
&= \frac{4(j+1)(j+2)}{j} + 4(j+1)(k+2) + \frac{2(j+2)^2(k-3)}{j+1}\\
&\quad + \frac{2j(j+2)\left(k^2-4\right)}{j+1} + \frac{2j(k+2)(k+2)}{k+1},\\
I(G) &= \frac{4j^5 + 20j^4 + j^3\left(8k^2 + 40\right) + j^2\left(4k^3 + 14k^2 - 8k + 36\right) + j\left(16k^2 - 20k - 20\right) + 8(k+1)}{j(j+1)(k+1)}.
\end{aligned} \tag{28}$$

□

Theorem 10. *In Indu-Bala graph $P_k \blacktriangledown P_j$, $j \geq 2$, $k \geq j+2$; then,*

$$I(G) = 2(j+2)^3\left\{\frac{2(j+1)^3}{(2j+1)^3} + \frac{(k-3)(j+2)^3}{(2j+2)^3}\right\} + \frac{2j(k+2)^3}{(j+k+1)^3}\left\{2(j+1)^3 + (k-2)(j+2)^3\right\}. \tag{29}$$

Proof. Suppose

$$\begin{aligned}
M(G;a,b) &= 4a^{j+1}b^{j+2} + 4ja^{j+1}b^{k+2} + 2(k-3)a^{j+2}b^{j+2} + 2j(k-2)a^{j+2}b^{k+2} + 2ja^{k+2}b^{k+2},\\
A(G) &= S_a^3 Q_{-2} jD_a^3 D_b^3 f(a,b)|_{a=1},\\
D_a^\alpha D_b^\alpha f(a,b) &= 4(j+1)^\alpha (j+2)^\alpha a^{j+1}b^{j+2} + 4j(j+1)^\alpha (k+2)^\alpha a^{j+1}b^{k+2}\\
&\quad + 2(k-3)(j+2)^{2\alpha}a^{j+2}b^{j+2} + 2j(k-2)(j+2)^\alpha (k+2)^\alpha a^{j+2}b^{k+2},
\end{aligned}$$

$$\begin{aligned}
D_a^3 D_b^3 f(a,b) &= 4(j+1)^3(j+2)^3 a^{j+1}b^{j+2} + 4j(j+1)^3(k+2)^3 a^{j+1}b^{k+2} \\
&\quad + 2(k-3)(j+2)^6 a^{j+2}b^{j+2} + 2j(k-2)(j+2)^3(k+2)^3 a^{j+2}b^{k+2}, \\
JD_a^3 D_b^3 f(a,b) &= 4(j+t1)^3(j+t2)^3 a^{2j+3} + 4j(j+t1)^3(k+t2)^3 a^{j+k+3} + 2(k-3)(j+t2)^6 a^{2j+4} \\
&\quad + 2j(k-2)(j+2)^3(k+2)^3 a^{j+k+4}; Q_{-2} jD_a^3 D_b^3 f(a,b), \\
&= 4(j+1)^3(j+2)^3 a^{2j+1} + 4j(j+1)^3(k+2)^3 a^{j+k+1} + 2(k-3)(j+2)^6 a^{2j+2} \\
&\quad + 2j(k-2)(j+2)^3(k+2)^3 a^{j+k+2}; S_a^3 Q_{-2} jD_a^3 D_b^3 f(a,b), \\
&= \frac{4(j+1)^3(j+2)^3}{(2j+1)^3} a^{2j+1} + \frac{4j(j+1)^3(k+2)^3}{(j+k+1)^3} a^{j+k+1} + \frac{2(k-3)(j+2)^6}{(2j+2)^3} a^{2j+2} \\
&\quad + \frac{2j(k-2)(j+2)^3(k+2)^3}{(j+k+1)^3} \Bigg\} a^{j+k+2}; S_a^3 Q_{-2} jD_a^3 D_b^3 f(a,b)|_{a=1} \\
&= \frac{4(j+1)^3(j+2)^3}{(2j+1)^3} + \frac{4j(j+1)^3(k+2)^3}{(j+k+1)^3} + \frac{2(k-3)(j+2)^6}{(2j+2)^3} \\
&\quad + \frac{2j(k-2)(j+2)^3(k+2)^3}{(j+k+1)^3}, \\
A(G) &= 2(j+2)^3 \left\{ \frac{2(j+1)^3}{(2j+1)^3} + \frac{(k-3)(j+2)^3}{(2j+2)^3} \right\} + \frac{2j(k+2)^3}{(j+k+1)^3} \left\{ 2(j+1)^3 + (k-2)(j+2)^3 \right\}.
\end{aligned} \tag{30}$$

□

3. Computational Results on Topological Indices for Indu-Bala Product of Two Paths P_j and P_k When $j > k$

Our fundamental objective of studying M-polynomial and its related all components is to establish a relationship between various affects of M-polynomials and its related things on the Indu-Bala graph, see Figure 4.

3.1. Results. We split vertices and edge degree of the Indu-Bala graph in Table 3. Similarly, we split the edge palpitations of points on the Indu-Bala graph in Table 4.

Theorem 11. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$\begin{aligned}
M(G;a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} \\
&\quad + k(2j-4)a^{j+2}b^{k+2} \\
&\quad + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}.
\end{aligned} \tag{31}$$

Proof. As in Figure 4, now we will compute M-polynomial using the values of Tables 3 and 4:

$$\begin{aligned}
M(G;a,b) &= \sum_{i \leq j} m_{ij}\left(G; a^i b^j\right), \\
&= \sum_{j+2 \leq j+2} j_{(j+2)(j+2)}(G) a^{j+2}b^{j+2} + \sum_{j+2 \leq k+1} j_{(j+2)(k+1)}(G) a^{j+2}b^{k+1} \\
&\quad + \sum_{j+2 \leq k+2} j_{(j+2)(k+2)}(G) a^{j+2}b^{k+2} + \sum_{k+1 \leq k+2} j_{(k+1)(k+2)}(G) a^{k+1}b^{k+2} \\
&\quad + \sum_{k+2 \leq k+2} j_{(k+2)(k+2)}(G) a^{k+2}b^{k+2}, \\
&= \sum_{uv \in E_{tj+2,j+2}} j_{(j+2)(j+2)}(G) a^{j+2}b^{j+2} + \sum_{uv \in E_{j+2,k+1}} j_{(j+2)(k+1)}(G) a^{j+2}b^{k+1} \\
&\quad + \sum_{uv \in E_{j+2,k+2}} j_{(j+2)(k+2)}(G) a^{j+2}b^{k+2} + \sum_{uv \in E_{k+1,k+2}} j_{(k+1)(k+2)}(G) a^{k+1}b^{k+2} \\
&\quad + \sum_{uv \in E_{k+2,k+2}} j_{(k+2)(k+2)}(G) a^{k+2}b^{k+2}, \\
&= \left|E_{j+2,j+2}\right| a^{j+2}b^{j+2} + \left|E_{j+2,k+1}\right| a^{j+2}b^{k+1} + \left|E_{j+2,k+2}\right| a^{j+2}b^{k+2} \\
&\quad + \left|E_{k+1,k+2}\right| a^{k+1}b^{k+2} + \left|E_{k+2,k+2}\right| a^{k+2}b^{k+2}, \\
&= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}.
\end{aligned} \tag{32}$$

□

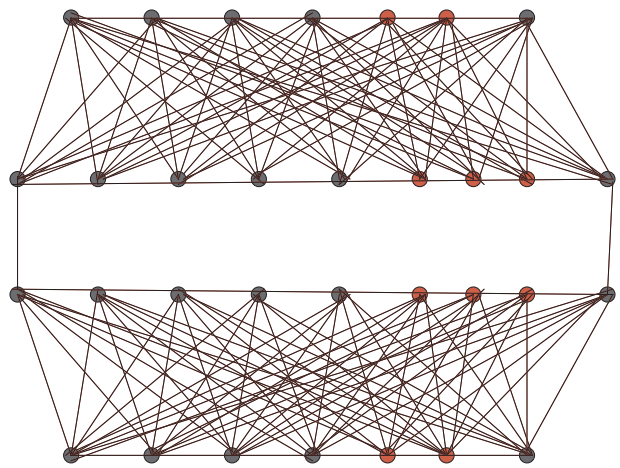


Figure 4: Indu-Bala graph $P_7 \blacktriangledown P_9$.

Table 3: Number of vertices of different degrees of the Indu-Bala graph.

Vertices	Total no. of $V(G)$ and $E(G)$
$V(G)$	$2(j+k)$
$E(G)$	$2jk+2(j+k)-2$

Table 4: Points of edge partitioning of $P_j \blacktriangledown P_k$.

Edges	Deg. of end nodes	Total no. of edges
E_1	$d(h)=j+2, d(i)=j+2$	$2k$
E_2	$d(h)=j+2, d(i)=k+1$	$4k$
E_3	$d(h)=j+2, d(i)=k+2$	$k(2j-4)$
E_4	$d(h)=k+1, d(i)=k+2$	4
E_5	$d(h)=k+2, d(i)=k+2$	$2j-6$

Theorem 12. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$M_1(G) = j\left(2jk + 2k^2 + 16k + 8\right) - 12. \tag{33}$$

Proof.

$$M(G,a,b) = 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},$$

$$(G,a,b) = f(a,b); D_a f(a,b) = a\frac{\partial f}{\partial a},$$

$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}\left\{2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}\right\},$$

$$a\frac{\partial f}{\partial a} = 2k(j+2)a^{j+2}b^{j+2} + 4k(j+2)a^{j+2}b^{k+1} + k(j+2)(2j-4)a^{j+2}b^{k+2}$$

$$+ 4(k+1)a^{k+1}b^{k+2} + (k+2)(2j-6)a^{k+2}b^{k+2}; D_b f(a,b) = b\frac{\partial f}{\partial b},$$

$$\frac{\partial f}{\partial b} = \frac{\partial}{\partial b}\left\{2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}\right\},$$

$$\begin{aligned} b\frac{\partial f}{\partial b} &= 2k(j+2)a^{j+2}b^{j+2} + 4k(k+1)a^{j+2}b^{k+1} + k(k+2)(2j-4)a^{j+2}b^{k+2} + 4(k+2)a^{k+1}b^{k+2} \\ &\quad + (k+2)(2j-6)a^{k+2}b^{k+2}; M_1(G) = (D_a + D_b)(f(a,b)|_{(a=b=1)}, \\ &= 4k(j+2) + 4k(j+k+3) + k(2j-4)(j+k+4) + 4(2k+3) + 2(k+2)(2j-6), \\ M_1(G) &= j\left(2kj + 2k^2 + 16k + 8\right) - 12. \end{aligned} \tag{34}$$

□

Theorem 13. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$M_2(G) = j^2\left(2k^2 + 6k\right) + j\left(6k^2 + 20k + 8\right) - 2k^2 - 12k - 6. \tag{35}$$

Proof.

$$\begin{aligned} M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}, \\ D_a f(a,b) &= 2k(j+2)a^{j+2}b^{j+2} + 4k(j+2)a^{j+2}b^{k+1} + k(j+2)(2j-4)a^{j+2}b^{k+2} \\ &\quad + 4(k+1)a^{k+1}b^{k+2} + (k+2)(2j-6)a^{k+2}b^{k+2}, \\ D_b D_a f(a,b) &= 2k(j+2)^2 a^{j+2}b^{j+2} + 4k(j+2)(k+1)a^{j+2}b^{k+1} + k(j+2)(k+2)(2j-4)a^{j+2}b^{k+2} \\ &\quad + 4(k+1)(k+2)a^{k+1}b^{k+2} + (k+2)^2(2j-6)a^{k+2}b^{k+2}, \\ M_2(G) &= D_b D_a f(a,b)|_{(a=b=1)} = 2k(j+2)^2 + 4k(j+2)(k+1) + k(j+2)(k+2)(2j-4) \\ &\quad 4(k+1)(k+2) + (k+2)^2(2j-6), \\ M_2(G) &= j^2\left(2k^2 + 6k\right) + j\left(6k^2 + 20k + 8\right) - 2k^2 - 12k - 6. \end{aligned} \tag{36}$$

□

Theorem 14. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$^{m}M_2(G) = \frac{2k^4 + k^3\left(2j^2 - 6j\right) + k^2\left(6j^2 - 2j\right) + k\left(2j^3 + 2j^2 - 10j - 6\right) + 4j^2 + 8j + 4}{k(j+1)^2(k+1)^2}. \tag{37}$$

Proof.

$$\begin{aligned} M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}, \\ {}^{m}M_2(G) &= S_a S_b f(a,b)|_{(a=b=1)}, \\ S_a &= \int_0^a \frac{f(x,b)}{x}\,\mathrm{d}x; S_b = \int_0^b \frac{f(a,x)}{x}\,\mathrm{d}x, \\ \frac{f(x,b)}{x} &= 2kx^{j+1}b^{j+2} + 4kx^{j+1}b^{k+1} + k(2j-4)x^{j+1}b^{k+2} + 4x^k b^{k+2} + (2j-6)x^{k+1}b^{k+2}, \\ S_a &= \frac{2k}{j+1}a^{j+2}b^{j+2} + \frac{4k}{j+1}a^{j+2}b^{k+1} + \frac{k(2j-4)}{j+1}a^{j+2}b^{k+2} + \frac{4}{k}a^{k+1}b^{k+2} + \frac{2j-6}{k+1}a^{k+2}b^{k+2}, \end{aligned}$$

$$\begin{aligned}
\frac{S_a(a,x)}{x} &= \frac{2k}{j+1}a^{j+2}x^{j+1} + \frac{4k}{j+1}a^{j+2}x^{k+1} + \frac{k(2j-4)}{j+1}a^{j+2}x^{k+1} + \frac{4}{k}a^{k+1}x^{k+1} + \frac{2j-6}{k+1}a^{k+2}x^{k+1},\\
S_bS_af(a,b) &= \int_0^b \frac{S_af(a,x)}{x}\mathrm{d}x,\\
S_bS_af(a,b) &= \frac{2k}{(j+1)^2}a^{j+2}b^{j+2} + \frac{4k}{(j+1)(k+1)}a^{j+2}b^{k+2} + \frac{k(2j-4)}{(j+1)(k+1)}a^{j+2}b^{k+2}\\
&\quad + \frac{4}{k(k+1)}a^{k+1}b^{k+2} + \frac{2j-6}{(k+1)^2}a^{k+2}b^{k+2},\\
{}^{m}M_2(G) &= S_bS_af(a,b)|_{(a=b=1)} = \frac{2k}{(j+1)^2} + \frac{4k}{(j+1)(k+1)} + \frac{k(2j-4)}{(j+1)(k+1)} + \frac{4}{k(k+1)} + \frac{2j-6}{(k+1)^2},\\
&= \frac{2k^4 + k^3\left(2j^2-6j\right) + k^2\left(6j^2-2j\right) + k\left(2j^3+2j^2-10j-6\right) + 4j^2 + 8j + 4}{k(j+1)^2(k+1)^2}.
\end{aligned} \tag{38}$$

□

Theorem 15. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$\begin{aligned}
R_\alpha(G) &= 2k(j+1)^\alpha\{(j+2)^\alpha + 2(k+1)^\alpha + k(j-2)(k+2)^\alpha\}\\
&\quad + 2(k+2)^\alpha\{2(k+1)^\alpha + j(k+2)^\alpha\}.
\end{aligned} \tag{39}$$

Proof.

$$\begin{aligned}
M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},\\
R_\alpha(G) &= D_a^\alpha D_b^\alpha f(a,b)|_{(a=b=1)},\\
D_b^\alpha f(a,b) &= 2k(j+2)^\alpha a^{j+2}b^{j+2} + 4k(k+1)^\alpha a^{j+2}b^{k+1} + k(2j-4)(k+2)^\alpha a^{j+2}b^{k+2}\\
&\quad + 4(k+2)^\alpha a^{k+1}b^{k+2} + (2j-6)(k+2)^\alpha a^{k+2}b^{k+2},\\
D_a^\alpha D_b^\alpha f(a,b) &= 2k(j+2)^{2\alpha}a^{j+2}b^{j+2} + 4k(j+2)^\alpha(k+1)^\alpha a^{j+2}b^{k+1} + 4(k+1)^\alpha(k+2)^\alpha a^{k+1}b^{k+2}\\
&\quad + k(2j-4)(j+2)^\alpha(k+2)^\alpha a^{j+2}b^{k+2} + (2j-6)(k+2)^{2\alpha}a^{k+2}b^{k+2},\\
R_\alpha(G) &= D_a^\alpha D_b^\alpha f(a,b)|_{(a=b=1)} = 2k(j+2)^{2\alpha} + 4k(j+2)^\alpha(k+1)^\alpha + k(2j-4)(j+2)^\alpha(k+2)^\alpha\\
&\quad + 4(k+1)^\alpha(k+2)^\alpha + (2j-6)(k+2)^{2\alpha},\\
R_\alpha(G) &= 2k(j+1)^\alpha\{(j+2)^\alpha + 2(k+1)^\alpha + k(j-2)(k+2)^\alpha\}\\
&\quad + 2(k+2)^\alpha\{2(k+1)^\alpha + j(k+2)^\alpha\}.
\end{aligned} \tag{40}$$

□

Theorem 16. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$\begin{aligned}
RR_\alpha(G) &= \frac{2k}{\left((j+1)\left(k^2+k\right)\right)^\alpha}\left\{(kj+k)^\alpha + (2j+2)(k+1)^\alpha + (j-2)\left(\left(k^2+k\right)(j+1)\right)^\alpha\right\}\\
&\quad + \frac{2}{k^\alpha(k+1)^{2\alpha}}\{2(k+1)^\alpha + (j-3)k^\alpha\}.
\end{aligned} \tag{41}$$

Proof.

$$M(G,a,b) = 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},$$

$$RR_\alpha(G) = S_a^\alpha S_b^\alpha f(a,b)|_{(a=b=1)},$$

$$S_b = \frac{2k}{j+1}a^{j+2}b^{j+2} + \frac{4k}{k}a^{j+2}b^{k+1} + \frac{k(2j-4)}{k+1}a^{j+2}b^{k+2} + \frac{4}{k+1}a^{k+1}b^{k+2} + \frac{2j-6}{k+1}a^{k+2}b^{k+2},$$

$$S_b^\alpha = \frac{2k}{(j+1)^\alpha}a^{j+2}b^{j+2} + \frac{4k}{k^\alpha}a^{j+2}b^{k+1} + \frac{k(2j-4)}{(k+1)^\alpha}a^{j+2}b^{k+2} + \frac{4}{(k+1)^\alpha}a^{k+1}b^{k+2} + \frac{2j-6}{(k+1)^\alpha}a^{k+2}b^{k+2},$$

$$S_a^\alpha S_b^\alpha f(a,b) = \frac{2k}{(j+1)^{2\alpha}}a^{j+2}b^{j+2} + \frac{4k}{(k(j+1))^\alpha}a^{j+2}b^{k+1} + \frac{k(2j-4)}{((j+1)(k+1))^\alpha}a^{j+2}b^{k+2} + \frac{4}{(k(k+1))^\alpha}a^{k+1}b^{k+2} + \frac{2j-6}{(k+1)^{2\alpha}}a^{k+2}b^{k+2}, \tag{42}$$

$$S_a^\alpha S_b^\alpha f(a,b)|_{(a=b=1)} = \frac{2k}{(j+1)^{2\alpha}} + \frac{4k}{(k(j+1))^\alpha} + \frac{k(2j-4)}{((j+1)(k+1))^\alpha} + \frac{4}{(k(k+1))^\alpha} + \frac{2j-6}{(k+1)^{2\alpha}},$$

$$RR_\alpha(G) = \frac{2k}{(j+1)^\alpha}\left\{\frac{1}{(j+1)^\alpha} + \frac{2}{k^\alpha} + \frac{(j-2)}{(j+1)^\alpha}\right\} + \frac{2}{(k+1)^\alpha}\left\{\frac{2}{k^\alpha} + \frac{j-3}{(k+1)^\alpha}\right\},$$

$$RR_\alpha(G) = \frac{2k}{\left((j+1)\left(k^2+k\right)\right)^\alpha}\left\{(kj+k)^\alpha + (2j+2)(k+1)^\alpha + (j-2)\left(\left(k^2+k\right)(j+1)\right)^\alpha\right\} + \frac{2}{k^\alpha(k+1)^{2\alpha}}\left\{2(k+1)^\alpha + (j-3)k^\alpha\right\}.$$

□

Theorem 17. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$\mathrm{SSD}(G) = \frac{4}{j+1}\left(4j^2 + k^2 + jk + 3j + 3k + 2\right) + \frac{1}{k(j+1)(k+1)}\left(2k^2j^3 + j^2\left(6k^2 + 8k\right) + j\left(2k^4 + 6k^3 - 28k^2 + 8k + 16\right) - 12k^3 - 28k^2 - 10k + 16\right). \tag{43}$$

Proof.

$$
\begin{aligned}
M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},\\
\mathrm{SSD}(G) &= \left(S_bD_a + S_aD_b\right)f(a,b)|_{(a=b=1)},\\
S_bD_af(a,b) &= \frac{2k(j+2)}{j+1}a^{j+2}b^{j+2} + \frac{4k(j+2)}{k}a^{j+2}b^{k+1} + \frac{k(2j-4)(j+2)}{k+1}a^{j+2}b^{k+2}\\
&\quad + \frac{4(k+1)}{k+1}a^{k+1}b^{k+2} + \frac{(2j-6)(k+2)}{k+1}a^{k+2}b^{k+2},\\
S_aD_bf(a,b) &= \frac{2k(j+2)}{j+1}a^{j+2}b^{j+2} + \frac{4k(k+1)}{j+1}a^{j+2}b^{k+1} + \frac{k(k+2)(2j-4)}{j+1}a^{j+2}b^{k+2}\\
&\quad + \frac{4(k+2)}{k}a^{k+1}b^{k+2} + \frac{(k+2)(2j-6)}{k+1}a^{k+2}b^{k+2};\ \left(S_bD_a + S_aD_b\right)f(a,b),\\
&= \frac{4k(j+2)}{j+1}a^{j+2}b^{j+2} + \frac{k(2j-4)\left(j^2+k^2+3j+3k+4\right)}{(j+1)(k+1)}a^{j+2}b^{k+2}\\
&\quad + \frac{4\left(j^2+k^2+3j+k+2\right)}{j+1}a^{j+2}b^{j+1} + \frac{8(k+1)}{k}a^{k+1}b^{k+2} + \frac{4(j-3)(k+2)}{k+1}a^{k+2}b^{k+2},\\
\mathrm{SSD}(G) &= \left(S_bD_a + S_aD_b\right)f(a,b)|_{(a=b=1)} = \frac{4k(j+2)}{j+1} + \frac{k(2j-4)\left(j^2+k^2+3j+3k+4\right)}{(j+1)(k+1)}\\
&\quad + \frac{4\left(j^2+k^2+3j+k+2\right)}{j+1} + \frac{8(k+1)}{k} + \frac{4(j-3)(k+2)}{k+1},\\
\mathrm{SSD}(G) &= \frac{4}{j+1}\left(4j^2+k^2+jk+3j+3k+2\right) + \frac{1}{k(j+1)(k+1)}\left(2k^2j^3 + j^2\left(6k^2+8k\right)\right.\\
&\quad \left. + j\left(2k^4+6k^3-28k^2+8k+16\right) - 12k^3 - 28k^2 - 10k + 16\right).
\end{aligned}
\tag{44}
$$

□

Theorem 18. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$
H(G) = \frac{4\left\{(j+7)\left(k^3+k^2\right) + k\left(j^2+2\right) - k + 2\right\}}{k(j+1)(k+1)}. \tag{45}
$$

Proof.

$$
\begin{aligned}
M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},\\
H(G) &= 2S_ajf(a,b)|_{a=1},\\
S_af(a,b) &= \frac{2k}{j+1}a^{j+2}b^{j+2} + \frac{4k}{j+1}a^{j+2}b^{k+1} + \frac{k(2j-4)}{j+1}a^{j+2}b^{k+2} + \frac{4}{k}a^{k+1}b^{k+2}\\
&\quad + \frac{2j-6}{k+1}a^{k+2}b^{k+2},
\end{aligned}
$$

$$\begin{aligned}
2S_a jf(a,b) &= \frac{4k}{j+1}a^{2j+4} + \frac{8k}{j+1}a^{j+k+3} + \frac{k(4j-8)}{j+1}a^{j+k+4} + \frac{8}{k}a^{2k+3} + \frac{4j-12}{k+1}a^{2k+4} \\
2S_a jf(a,b)|_{a=1} &= \frac{4k}{j+1} + \frac{8k}{j+1} + \frac{k(4j-8)}{j+1} + \frac{8}{k} + \frac{4j-12}{k+1}, \\
H(G) &= \frac{4\left\{(j+7)\left(k^3+k^2\right)+k\left(j^2+2\right)-k+2\right\}}{k(j+1)(k+1)}.
\end{aligned} \tag{46}$$

□

Theorem 19. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$I(G) = \frac{j^2\left(2k^3+10k^2+18k+16\right)+j\left(4k^3+16k^2+12k\right)-2k^2-12k-16}{(j+1)(k+1)}. \tag{47}$$

Proof.

$$\begin{aligned}
M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2}, \\
I(G) &= S_a jD_aD_b f(a,b)|_{a=1}; D_aD_b f(a,b), \\
&= 2k(j+2)^2a^{j+2}b^{j+2} + 4k(j+2)(k+1)a^{j+2}b^{k+1} + k(j+2)(k+2)(2j-4)a^{j+2}b^{k+2} \\
&\quad + 4(k+1)(k+2)a^{k+1}b^{k+2} + (k+2)^2(2j-6)a^{k+2}b^{k+2}; S_a jD_aD_b f(a,b), \\
&= \frac{2k(j+2)^2}{j+1}a^{2j+4} + \frac{4k(j+2)(k+1)}{j+1}a^{j+k+3} + \frac{2k(k+2)(j+2)(j-2)}{j+1}a^{j+k+4} \\
&\quad + \frac{4(k+1)(k+2)}{k+1}a^{2k+3} + \frac{(2j-6)(k+2)^2}{k+1}a^{2k+4}; S_a jD_aD_b f(a,b)|_{a=1}, \\
&= \frac{2k(j+2)^2}{j+1} + \frac{4k(j+2)(k+1)}{j+1} + \frac{2k(k+2)(j+2)(j-2)}{j+1} \\
&\quad + \frac{4(k+1)(k+2)}{k+1} + \frac{(2j-6)(k+2)^2}{k+1}, \\
I(G) &= \frac{j^2\left(2k^3+10k^2+18k+16\right)+j\left(4k^3+16k^2+12k\right)-2k^2-12k-16}{(j+1)(k+1)}.
\end{aligned} \tag{48}$$

□

Theorem 20. *In Indu-Bala graph $P_j \blacktriangledown P_k$, $j \geq 2$, $k \geq j+2$; then,*

$$\begin{aligned}
A(G) &= 2k(j+2)^3\left\{\frac{(j+2)^3}{(2j+1)^3} + \frac{2(k+1)^3}{(j+k)^3} + \frac{(j-2)(k+2)^3}{(j+k+1)^3}\right\} \\
&\quad + 2(k+2)^3\left\{\frac{2(k+1)^3}{(2k)^3} + \frac{(j-3)(k+2)^3}{(2j+1)^3}\right\}.
\end{aligned} \tag{49}$$

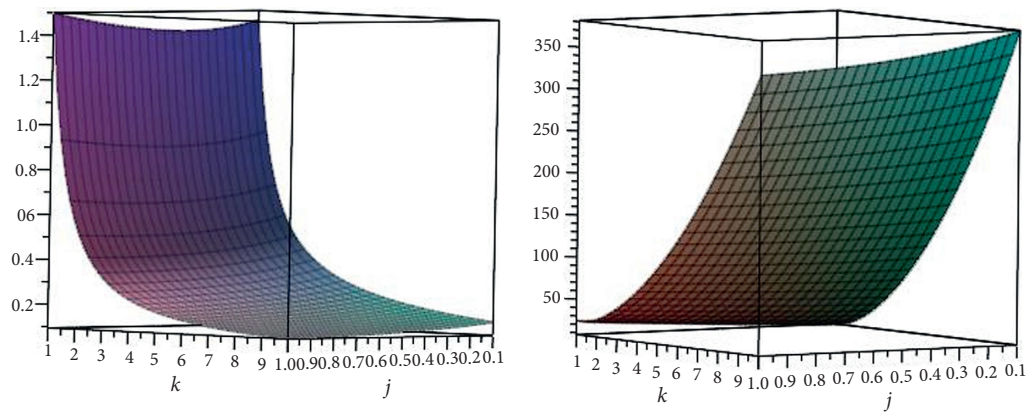
Proof.

$$
\begin{aligned}
M(G,a,b) &= 2ka^{j+2}b^{j+2} + 4ka^{j+2}b^{k+1} + k(2j-4)a^{j+2}b^{k+2} + 4a^{k+1}b^{k+2} + (2j-6)a^{k+2}b^{k+2},\\
A(G) &= S_a^3 Q_{-2} j D_a^3 D_b^3 f(a,b)|_{a=1},\\
D_a^\alpha D_b^\alpha f(a,b) &= 2k(j+2)^{2\alpha}a^{j+2}b^{j+2} + 4k(j+2)^\alpha(k+1)^\alpha a^{j+2}b^{k+1} + 4(k+1)^\alpha(k+2)^\alpha a^{k+1}b^{k+2}\\
&\quad + k(2j-4)(j+2)^\alpha(k+2)^\alpha a^{j+2}b^{k+2} + (2j-6)(k+2)^{2\alpha}a^{k+2}b^{k+2},\\
D_a^3 D_b^3 f(a,b) &= 2k(j+2)^6 a^{j+2}b^{j+2} + 4k(j+2)^3(k+1)^3 a^{j+2}b^{k+1} + 4(k+1)^3(k+2)^3 a^{k+1}b^{k+2}\\
&\quad + k(2j-4)(j+2)^3(k+2)^3 a^{j+2}b^{k+2} + (2j-6)(k+2)^6 a^{k+2}b^{k+2},\\
JD_a^3 D_b^3 f(a,b) &= 2k(j+2)^6 a^{2j+4} + 4k(j+2)^3(k+1)^3 a^{j+k+3} + 4(k+1)^3(k+2)^3 a^{2k+3}\\
&\quad + k(2j-4)(j+2)^3(k+2)^3 a^{j+k+4} + (2j-6)(k+2)^6 a^{2k+4}; Q_{-2} j D_a^3 D_b^3 f(a,b),\\
&= 2k(j+2)^6 a^{2j+2} + 4k(j+2)^3(k+1)^3 a^{j+k+1} + 4(k+1)^3(k+2)^3 a^{2k+1}\\
&\quad + k(2j-4)(j+2)^3(k+2)^3 a^{j+k+2} + (2j-6)(k+2)^6 a^{2k+2}; S_a^3 Q_{-2} j D_a^3 D_b^3 f(a,b),\\
&= \frac{2k(j+2)^6}{(2j+1)^3}a^{2j+2} + \frac{4k(j+2)^3(k+1)^3}{(j+k)^3}a^{j+k+1} + \frac{4(k+1)^3(k+2)^3}{(2k)^3}a^{2k+1}\\
&\quad + \frac{k(2j-4)(j+2)^3(k+2)^3}{(j+k+1)^3}a^{j+k+2} + \frac{(2j-6)(k+2)^6}{(2k+1)^3}a^{2k+2}; S_a^3 Q_{-2} j D_a^3 D_b^3 f(a,b)|_{a=1},\\
&= \frac{2k(j+2)^6}{(2j+1)^3} + \frac{4k(j+2)^3(k+1)^3}{(j+k)^3} + \frac{4(k+1)^3(k+2)^3}{(2k)^3} + \frac{(2j-6)(k+2)^6}{(2k+1)^3}\\
&\quad + \frac{k(2j-4)(j+2)^3(k+2)^3}{(j+k+1)^3},\\
A(G) &= 2k(j+2)^3\left\{\frac{(j+2)^3}{(2j+1)^3} + \frac{2(k+1)^3}{(j+k)^3} + \frac{(j-2)(k+2)^3}{(j+k+1)^3}\right\}\\
&\quad + 2(k+2)^3\left\{\frac{2(k+1)^3}{(2k)^3} + \frac{(j-3)(k+2)^3}{(2j+1)^3}\right\}.
\end{aligned}
\tag{50}
$$

□

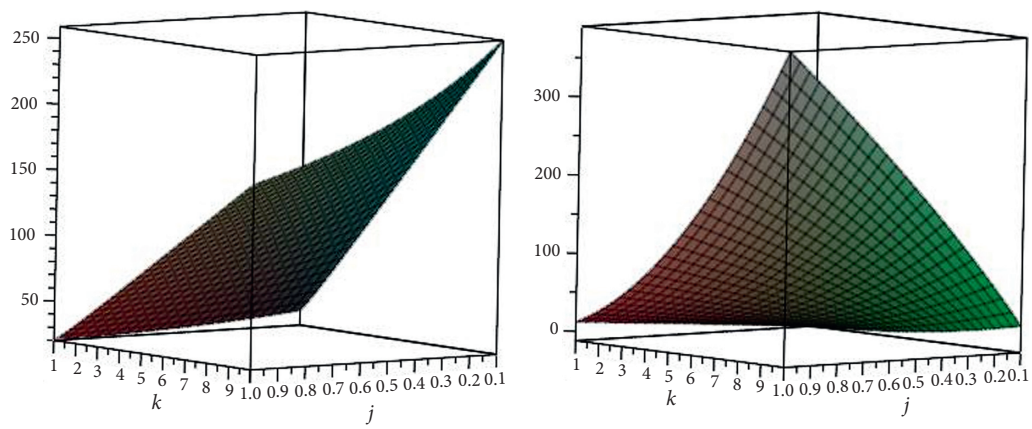
4. Graphical Results and Their Discussion

In this section, we present some graphical results which are related to M-polynomial and their degree-based indices for the Indu-Bala product of two paths when one path is greater than another path and vice versa. We have used different values of k and j and drawn their respective graphs as shown in Figure 5(a) (inverse Randic index for $j \geq k$ and $\alpha = 1$) and Figure 5(b) (symmetric division index for $j \geq k$ and $\alpha = 1$). We have observed from Figure 6(a) (harmonic index for $j \geq k$ and $\alpha = 1$) and Figure 6(b) (inverse sum index for $j \geq k$ and $\alpha = 1$) that the overall structure of indices increase with the increase of the value of j and k.



(a) (b)

Figure 5: (a) Inverse Randic index for $j \geq k$ and $\alpha = 1$. (b) Symmetric division index for $j \geq k$ and $\alpha = 1$.



(a) (b)

Figure 6: (a) Harmonic index for $j \geq k$ and $\alpha = 1$. (b) Inverse sum index for $j \geq k$ and $\alpha = 1$.

5. Conclusions

It is important to research the network through charts, and topological indicators are important for understanding its basic topology. This type of research finds global application in computer science, networks, and communication systems, which uses various indexes based on graph invariance to consider some stimulation summary. The Indu-Bala networks we studied in this paper are used to optimize (minimized) the operational cost of the network and find the shortest linkage between the connectors.In this article, we present some product for M-polynomial and nine different degree-based topological descriptors as discussed above for the Indu-Bala product of two paths.

Data Availability

There are no data used in this research.

Conflicts of Interest

The authors declare no conflicts of interest.

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