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Research Article

Synchronization of Nonidentical Coupled Phase Oscillators in the Presence of Time Delay and Noise

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We have studied in this paper the dynamics of globally coupled phase oscillators having the Lorentzian frequency distribution with zero mean in the presence of both time delay and noise. Noise may be Gaussian or non-Gaussian in characteristics. In the limit of zero noise strength, we find that the critical coupling strength (CCS) increases linearly as a function of time delay. Thus the role of time delay in the dynamics for the deterministic system is qualitatively equivalent to the effect of frequency fluctuations of the phase oscillators by additive white noise in absence of time delay. But for the stochastic model, the critical coupling strength grows nonlinearly with the increase of the time delay. The linear dependence of the critical coupling strength on the noise intensity also changes to become nonlinear due to creation of additional phase difference among the oscillators by the time delay. We find that the creation of phase difference plays an important role in the dynamics of the system when the intrinsic correlation induced by the finite correlation time of the noise is small. We also find that the critical coupling is higher for the non-Gaussian noise compared to the Gaussian one due to higher effective noise strength.

1. Introduction

In this paper we have investigated the synchronization behavior of globally coupled phase oscillators. Recent trends in physics [1–6] imply that it is one of the important issues in basic science such as the Brownian motion, nanomaterials, and biophysics. Biological clocks [7, 8], chemical oscillators [9–11], coupled map lattices [12, 13], and coupled random frequency oscillators [14] are examples where the phenomena of synchronization have been observed. To account the phenomenon, coupled phase oscillators model was introduced by Kuramoto [9–11], and it is popularly known as the Kuramoto model. After that, synchronization in nonlinear systems has been systematically studied and attracted much attention. Several reviews on the developments can be found in [1, 2, 5, 6].

Recently there has been considerable interest in some stochastic systems, whose dynamics are determined by both the present state and the state in the past with the time delay ($\tau_d > 0$). It has been considered in visual feedback [15, 16] and brain activity [17, 18] to mention a few. Delay is also studied in the coupled oscillator (CO) model [19–23]. In [19] authors showed that intrinsic frequency of the network of limit cycle oscillators decreases as the time delay grows, and for greater delay there is a metastable synchronized state. However, Nakamura et al. in [20] studied the effect of time delay on the stability of the clusters which are made by nearest neighbor coupling of the oscillators. In these cases identical oscillators were considered. The effect of time delay on the Kuramoto model having frequencies of the Gaussian distribution was investigated by Choi et al. in absence of noise [21]. They have also observed the decrease of the frequency of the coupled oscillator with the time delay. Another observation in this study was that there is a damped oscillation in the plot of the critical coupling strength versus time delay for nonzero mean of intrinsic frequency. This oscillating behavior was reported earlier using the simple Lorentzian [22, 23] or bi-Lorentzian [24] distributions of frequencies. In [22], it is implied that the critical coupling strength would be a diverging function of time delay for zero mean of the intrinsic frequency in the Lorentzian distribution of frequencies (LDF). In Figure 4 of the above-mentioned reference [22] it has been demonstrated that a series of evenly spaced peaks appear at

$\omega_0\tau_d = (2n+1)\pi$, $n = 0, 1, 2\ldots$ in the plot of critical coupling strength versus time delay. Here ω_0 is the mean value of frequencies in the Lorentzian distribution of frequencies. The above relation suggests that the first peak should appear at $\tau_d = \infty$ for $\omega_0 = 0$. In other words, critical coupling strength diverges with increase of time delay for zero centered LDF. Thus effect of time delay on the synchronization phenomenon strongly depends on the mean value of frequencies of nonidentical oscillators. One may now ask what would be the nature of divergence in the variation of critical coupling strength as a function of time delay for zero mean of frequencies of the coupled oscillators. To answer this question we have studied synchronization behavior of coupled nonidentical oscillators with time delay and zero mean of frequencies both in presence and absence of noise. To make the present study general, we have considered both the Gaussian and non-Gaussian characteristics of the noise. It is obvious that both time delay and noise lead to create phase difference among oscillators, and therefore synchronization of the phase oscillators becomes difficult in presence of noise and time delay. Thus present investigation accounts how the critical coupling strength depends on time delay and noise properties. We show that the critical coupling strength (CCS) grows linearly with time delay for the noiseless Kuramoto model. This is similar to the variation of CCS with the strength of the white noise. Thus time delay is qualitatively equivalent to the effect of frequency fluctuations of the phase oscillators by the white noise. In the presence of noise, we have observed that the above divergence becomes slow and nonlinear. As the noise makes it difficult to have the synchronized state, the effect of time delay on synchronization becomes less significant compared to noiseless case. Our other observations show how the transition from the Gaussian to a non-Gaussian character of noise changes the dynamical properties of the system. We will discuss more about this in the next section.

2. The Model

We start considering the following Kuramoto model with time delay in the presence of noise:

$$\frac{d\theta_i(t)}{dt} = \omega_i + \frac{\epsilon}{N}\sum_{i<j}^{N} \sin\left[\theta_j(t-\tau_d) - \theta_i(t)\right] + \eta_i(t), \quad (1)$$

where θ_i and ω_i are, respectively, the phase and the frequency of the ith oscillator ($i = 1, \ldots, N$), ϵ is the coupling constant, and τ_d is the time delay. The independent noise processes $\eta_i(t)$ [25–28] are governed by

$$\frac{d\eta_i(t)}{dt} = -\frac{\eta_i}{\tau\left[1+\alpha(p-1)\eta_i^2/2\right]} + \frac{\sqrt{D}}{\tau}\xi_i(t), \quad (2)$$

with $\alpha = \tau/D$. $\xi_i(t)$ is the Gaussian white noise process defined via $\langle\xi_i(t)\xi_j(t')\rangle = 2\delta_{ij}\delta(t-t')$ and $\langle\xi_i(t)\rangle = 0$. D and τ measure the intensity and the correlation time of the noise process. In [23] we have discussed them in detail. However, in the additive noise term in (1) it is apparent that in the present problem we assume the homogeneous diffusion of phase oscillators as it is independent of the spatial coordinate of the oscillator [29]. In a multiplicative noise driven dynamical system, diffusion depends on the coordinate of the dynamical system [29] and the diffusion becomes nonhomogeneous. Another point to be noted here is that, keeping in mind the Ornstein-Uhlenbeck (OU) noise process [30], the above form of differential equation is very illuminating about non-Gaussian behavior of noise and to attend the Gaussian limit. This form was considered in different contexts in the recent past [31–40].

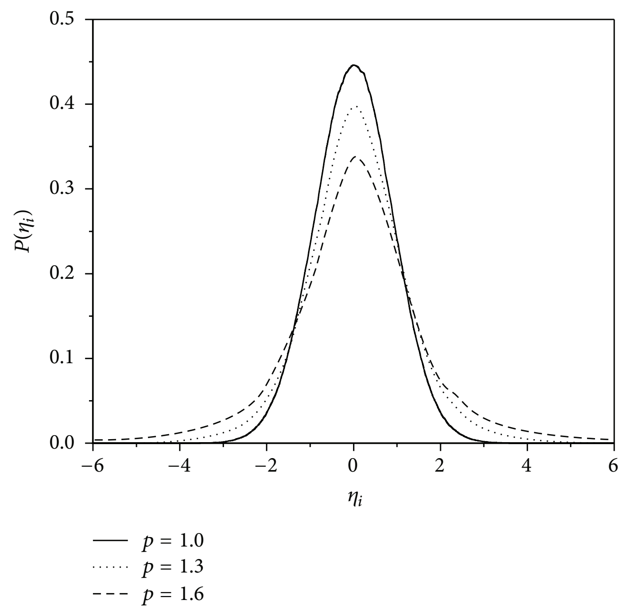

Figure 1: The plot of the probability distribution function $P(\eta_i)$ versus η_i for the parameters set $D = 0.2$ and $\tau = 0.25$.

The stationary probability distribution function of the noise process is given by [41]

$$P(\eta_i) = \frac{1}{Z_{ip}}\left[1+\alpha(p-1)\frac{\eta_i^2}{2}\right]^{-1/(p-1)}, \quad (3)$$

where Z_{ip} is the normalization constant which equals to

$$\begin{aligned} Z_{ip} &= \int_{-\infty}^{\infty} d\eta_i \left[1+\alpha(p-1)\frac{\eta_i^2}{2}\right]^{-1/(p-1)} \\ &= \sqrt{\frac{\pi}{\alpha(p-1)}}\frac{\Gamma_1(1/(p-1)-1/2)}{\Gamma_1(1/(p-1))}, \end{aligned} \quad (4)$$

with Γ_1 being the gamma function. This distribution can be normalized only for $p < 3$. The normalized distribution function is plotted in Figure 1 for different values of p. It implies that the noise is deviated more from the Gaussian behavior with increase of tail of the symmetric distribution function. Because of the symmetric distribution function, odd moments of η_i are zero. But nonvanishing even moments increase as p grows since the tail of the distribution function becomes larger. We have demonstrated these aspects in

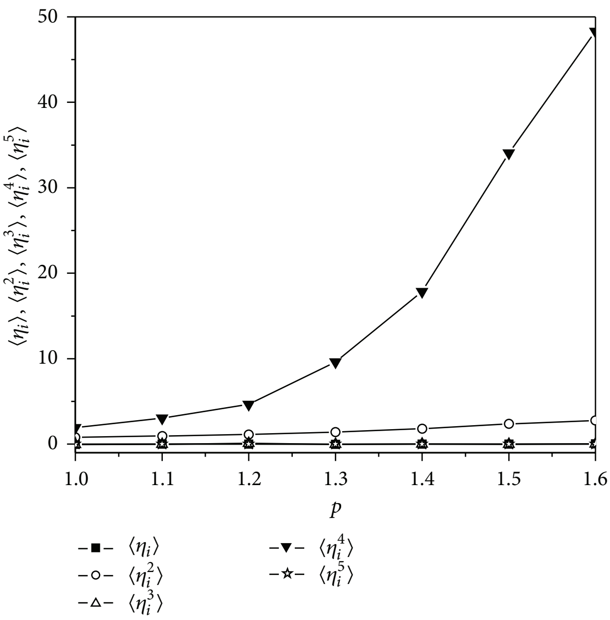

FIGURE 2: The plot of first five moments ($\langle \eta_i \rangle, \langle \eta_i^2 \rangle, \langle \eta_i^3 \rangle, \langle \eta_i^4 \rangle$, and $\langle \eta_i^5 \rangle$) versus noise parameter p for the parameters set $D = 0.2$ and $\tau = 0.25$.

Figure 2 for $\langle \eta_i \rangle, \langle \eta_i^2 \rangle, \langle \eta_i^3 \rangle, \langle \eta_i^4 \rangle$, and $\langle \eta_i^5 \rangle$, respectively. In this figure it is implied that odd moments are almost zero. They are merged to single line which is parallel to the p-axis, and therefore it seems that three data sets are plotted in the figure. Furthermore, the correlation time of non-Gaussian noise τ_p at the stationary regime of the process $\eta_i(t)$ diverges near $p = 5/3$, and it can be approximated [41] over the range $1 \le p < 5/3$ as

$$\tau_p \simeq \frac{2\tau}{(5-3p)}. \tag{5}$$

However, for this approximate correlation time, the variance of the noise process is given by

$$\left\langle \eta_{ip}^2 \right\rangle \simeq \frac{4D}{\tau_p (5-3p)^2}. \tag{6}$$

Before leaving this issue we would like to emphasize that both the intrinsic frequency (ω_i) and nonlinear coupling are affected due to colored noise (for details we refer to [39, 42, 43]). In the present problem, the effect depends on the gradient of coupling as well as noise correlation time. Thus the transition from Gaussian to non-Gaussian character of noise changes the dynamical properties of the system as their noise correlation time and noise strength are different.

The quantity of interest in the present study is

$$Z = \Gamma e^{i\Theta} = \frac{1}{N}\sum_{i=1}^{N} e^{i\theta_i}, \tag{7}$$

which is the order parameter that measures the extent of synchronization in the system of N phase oscillators. We have discussed more about this in [23]. However, for the initial distribution of frequencies we choose the following Lorentzian distribution function with zero mean:

$$g(w) = \frac{1}{\pi}\frac{\lambda}{\omega^2 + \lambda^2}. \tag{8}$$

λ in the above equation measures the width of the distribution function. Consideration of the above distribution function is a generalization of the Kuramoto model with the identical oscillators. To mention the possibility of different frequencies of the oscillators, we refer to the nice demonstration about the synchronization in the introduction of the recent review [6]. However, the above symmetric distribution function has been considered in many contexts of theoretical physics to describe the effect of frequency distribution whenever is necessary. Even in the case of the Brownian motion in the presence of non-Markovian bath, people consider the Lorentzian distribution of frequency of the bath modes which leads to having exponentially decaying memory kernel [44]. Authors in the field of synchronization also consider this distribution [22]. The reason may be that it is simple to be implemented and can account the experimental results satisfactorily. However, in a recent paper [45] the Gaussian distribution function of frequencies has been used. The Lorentzian distribution leads to longer tail compared to the Gaussian one. Thus both distribution functions would give qualitatively same result, and we have chosen the above distribution function just to study the effect of distribution of frequencies in presence of time delay and noise.

Before leaving this section we would like to mention that the noise $\eta_i(t)$ in (1) makes the frequency of the oscillator random. Study of stochastic version of the Kuramoto model becomes an important part of complex system [6]. Very recently a linear stability analysis of the incoherent state has been done in a system of globally coupled identical phase oscillators subject to colored noise [46]. The relevance of noise and time delay has been discussed in our recent papers [23, 36]. Keeping in mind all this we are motivated to study the above model.

3. Results and Discussion

3.1. Calculation Procedure. We have studied the present problem solving (1) and (2) simultaneously using Heun's method. It is a stochastic version of the Euler method [47, 48] which reduces to the second-order Runge-Kutta method in the absence of noise. However, based on the above method, we have studied time evolution of $N = 5000$ coupled phase oscillators. In [36], we have mentioned that this large value corresponds to the thermodynamic limit ($N \to \infty$). This limit means that the result is independent of number of oscillators. We have checked that in the present calculation $N = 5000$ is a very good number to satisfy the thermodynamic limit. System with around $N = 4000$ starts to obey the limit.

It is well known that if the coupling strength exceeds a threshold value then the stationary state is the coherent one. The threshold value is called critical coupling strength (ϵ_c). The main objective of the present paper is the investigation

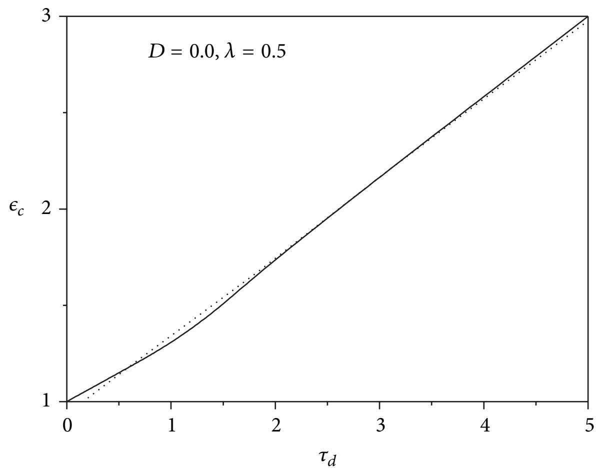

Figure 3: The plot of critical value of coupling strength ϵ_c versus the time delay τ_d in the absence of noise $D = 0$. The Lorentzian distribution width is $\lambda = 0.5$.

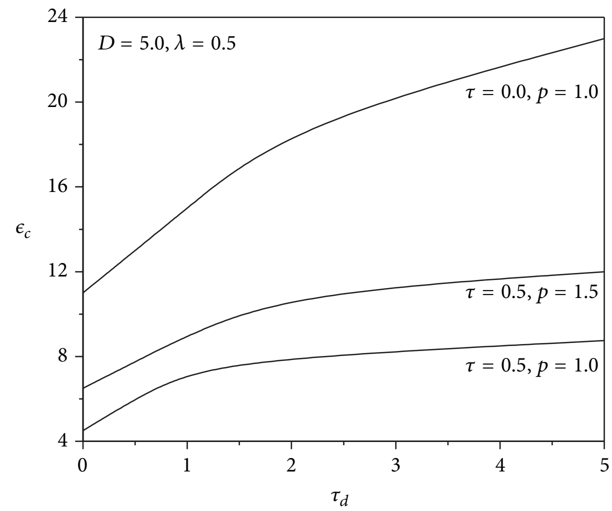

Figure 4: The plot of critical value of coupling strength ϵ_c versus the time delay τ_d in the presence of noise. The Lorentzian distribution width is $\lambda = 0.5$.

of dependence of ϵ_c on the noise properties and time delay. Therefore, to make the present paper self-sufficient we would like to present how one can determine the critical coupling strength numerically. First, we determine the ensemble average of Γ ($\langle\Gamma\rangle$) at a given time as it is a statistical quantity due to noise. The ensemble corresponds to consider all possible realizations of noise. Each member of the ensemble is formed by coupling of system with its environment. However, for different values of coupling strength (ϵ) we determine $\langle\Gamma\rangle$ at long time. From this we calculate numerically the derivative of $\langle\Gamma\rangle$ with respect to ϵ at stationary state. It depends on the coupling strength (CS). At low ϵ, the order parameter increases slowly as a function of ϵ since the formation of coherent state is difficult. But after a threshold value of CS, it grows rapidly when coherent state formation is possible. Beyond the threshold value again it increases slowly to approach the perfectly coherent state. We have demonstrated this aspect in [36]. Thus there would be a maximum in the variation of the derivative with the coupling strength. The maximum corresponds to the sharp change of state from incoherent to coherent one. ϵ, corresponding to the maximum value of the derivative, is identified as the critical coupling strength. Validity of this numerical scheme has been already tested in our recent papers [23, 36]. Another point to be mentioned here is that relaxation time would depend on the initial condition. If we start from a state which is closed to the synchronized state then it will synchronize immediately. However, we have chosen initial value of phase of the N-oscillators in such a way that the order parameter is zero. To satisfy this we distribute phase of N-oscillators between 0 and 2π in such a way that the phase difference between ith and $(i + 1)$th oscillators is $2\pi/(N - 1)$. Thus we start simulation from perfectly incoherent state as it is necessary in the present problem. We chose stationary time to be a linear function of time delay (τ_d) with the proportionality constant 100 and integration step length $h = 0.01$. We have checked that this is sufficiently long time to attain the stationary state.

3.2. Calculated Results. The critical coupling strength as mentioned above implies the range of coupling strength in which the incoherent state can survive at long time. This range corresponds to the stability zone of the incoherent state of the coupled phase oscillators. Thus critical coupling strength is a measure to imply how the stability of incoherent or coherent state of the coupled oscillators depends on time delay and other parameters of the system. The dependence of the critical coupling strength ϵ_c on the time delay τ_d in the absence of noise ($D = 0$) is demonstrated (solid line) in Figure 3. The dotted line is corresponding to fitting of the solid line. It suggests a linear relationship of the coupling strength with the time delay, $\epsilon_c \simeq 2 \times \lambda + 0.4 \times \tau_d$. This is very different from Figures 2 and 4 of our earlier study [23] in which the peak appears at regular interval. Appearance of peak can be understood in the following way. We now invoke the coupling term in (1). For some time delay depending on the mean frequency of the phase oscillators (POS), it may be negative. Then the coupling becomes repulsive (it is equivalent to changing ϵ for $-\epsilon$). In [22] it has been shown that at $\tau_D = (2n + 1)\pi/\omega_0$, $n = 0, 1, 2 \ldots$, synchronization is difficult. In other words, peaks appear in the plot of critical coupling strength versus time delay when this relation holds. Similar phenomenon also occurs for identical phase oscillators [22]. Coupling term implies that if the phase angles evolve with time with a narrow distribution then there is a possibility to vary it periodically with increase of time delay. Narrow distribution would be favored for identical oscillators or nonidentical (POS) with nonzero ω_0. Thus there is a possibility of disappearance of the above-mentioned periodic nature for $\omega_0 = 0$, and the time delay may introduce additional phase diffusion by weakening the coupling strength. This is very similar to the effect of white noise strength on the coupling strength ($\epsilon_c = 2 \times \lambda + 2 \times D$) in the absence of time delay (Figure 5). Thus one may conclude qualitatively that the time delay is equivalent to

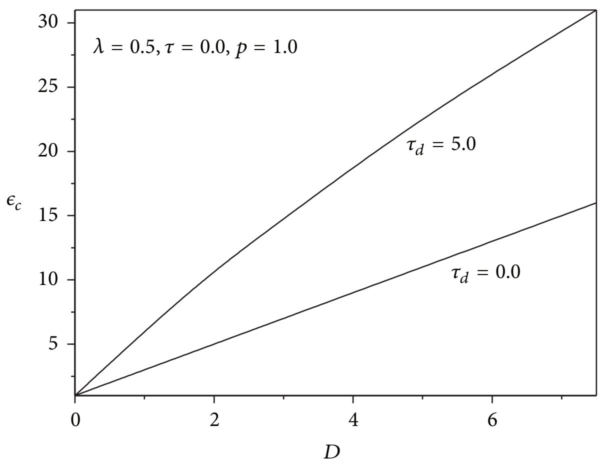

Figure 5: The critical value of coupling strength ϵ_c versus the strength of noise D for Gaussian ($p = 1$) white ($\tau = 0$) noise. The Lorentzian distribution width is $\lambda = 0.5$.

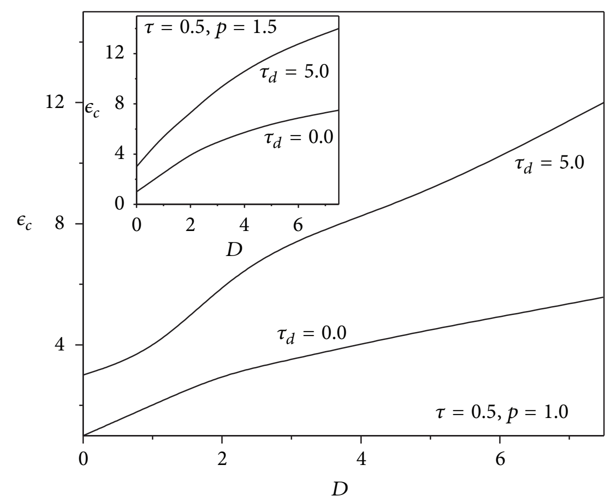

Figure 6: The critical value of coupling strength ϵ_c versus the strength of noise D for colored Gaussian noise. The Lorentzian distribution width is $\lambda = 0.5$. In the inset same is plotted for colored non-Gaussian noise.

the fluctuation of frequency of the phase oscillators by white noise. Here it is to be noted that the rate of increase for the former case is slower compared to the latter one.

In the presence of noise, the effect of τ_d on ϵ_c is demonstrated in Figure 4. It shows that for the nonzero noise strength the critical coupling strength varies nonlinearly with time delay in the interplay of D and τ_d. The variation is slower for the colored noise compared to the white noise case. It may be due to modification of intrinsic frequency as well as coupling by virtue of colored noise as we have mentioned earlier. The strong coupling in presence of colored noise would lead to slow variation of critical coupling strength with time delay. Weak diffusion in the presence of colored noise compared to white noise may be another reason for the above-mentioned slow variation. Because of higher variance for the non-Gaussian noise compared to the Gaussian one (which is implied in (6) and Figures 1 and 2), the critical coupling strength is always higher for the former than the latter for a given set of parameters. Before leaving this part we would mention that the divergence of critical coupling strength with time delay is implied in [22] for zero mean of the frequency distribution function (8). But the nature of divergence was not implied both for absence and presence of noise in [22].

As the next step, we have studied how the onset of synchronization is affected by the noise strength in the presence of time delay. The results are presented in Figure 5. It shows that the well-known linear relationship between ϵ_c and D for white noise [22] is broken due to the creation of additional phase difference by the time delay. The effect of time delay is becoming strong with increase of noise strength. Similarly critical coupling strength increases more rapidly with D for colored noises for nonzero τ_d compared to $\tau_d = 0$ case. It is depicted in Figure 6.

In Figure 7, we have presented results of investigation of the interplay of memory effect of the non-Gaussian noise process and time delay on the critical coupling strength. It shows that the numerical result is fitted well by the biexponentially decaying function. We have represented it by dots. Although ϵ_c decays biexponentially with the increase in noise correlation time both in the absence and the presence of τ_d, the decay rate constant corresponding to low τ is surprisingly affected by the time delay. In the presence of time delay, the rate constant becomes double compared to the rate constant for $\tau_d = 0$ for the Gaussian noise for the given parameter set. In the case of non-Gaussian noise the increment of rate constant is four times. However, on increasing noise correlation time, the difference in magnitude of the critical coupling strength for $\tau_d = 0$ and $\tau_d = 3.0$ for cases decays rapidly, and the decay rate constants at large τ become almost equal. Thus it is apparent in Figure 7 that as the variance decreases and the intrinsic correlations among the phases grow due to the increase of two times or cross-correlation times for the phases with increase of noise correlation time, the effect of time delay on the creation of phase difference among the phase oscillators becomes less important in the dynamics.

Finally, we have investigated how the critical coupling strength depends on p which accounts the deviation of noise properties from the Gaussian characteristics. In Figure 8 we have demonstrated this aspect. The critical coupling strength grows with the increase of p both in presence and absence of time delay as the noise strength becomes higher for more deviation from Gaussian characteristic. The rate of increase is higher for nonzero τ_d compared to $\tau_d = 0$ case as we expect from the earlier discussion.

We now compare our earlier results in [23] with the present study related to effect of noise on critical coupling strength in presence of time delay. It is apparent in Figures 5–8 of the present study that the ratio of critical coupling strengths $\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_d \neq 0)$ decreases with the increase of noise strength (D) or noise parameter p and it increase as the noise correlation time grows. These observations are reverse situations of the previous results discussed in [23]. It

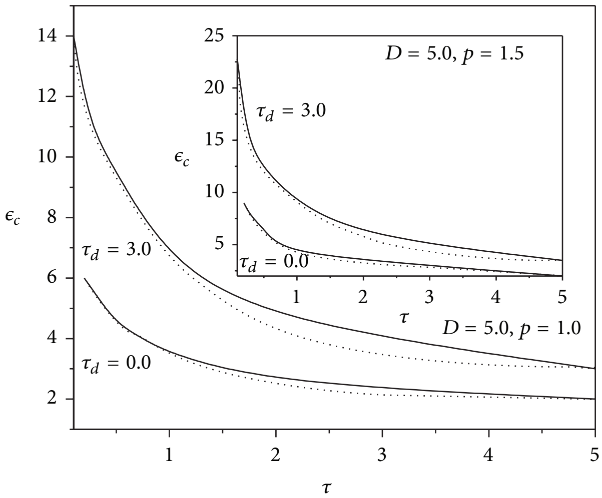

Figure 7: The critical value of coupling strength ϵ_c versus the correlation time τ for Gaussian noise for the parameters set $\lambda = 0.5$ and $D = 5$. In the inset same is plotted for colored non-Gaussian noise.

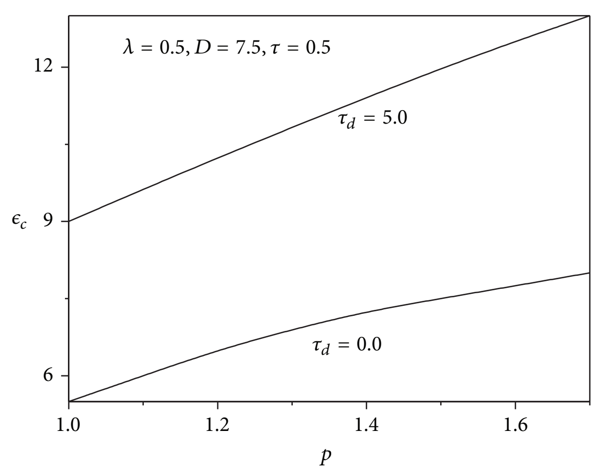

Figure 8: The critical value of coupling strength ϵ_c versus p which measures the deviation of noise behavior from Gaussian characteristics for the parameters set $\lambda = 0.5$, $D = 7.5$, and $\tau = 0.5$.

suggests that the critical coupling strength varies rapidly with the noise parameter in presence of time delay in case of $\omega_0 = 0$ compared to $\omega_0 \neq 0$. This is because noise is less effective in presence of time delay when it induces strong repulsive coupling (as mentioned earlier) in the case of nonzero mean of frequencies of nonidentical oscillators.

4. Conclusion

We have studied synchronization behavior of the noise driven coupled phase oscillators with time delay. The Lorentzian distribution of frequency with zero mean has been considered in the present model. To make it general with respect to noise properties both Gaussian and non-Gaussian noises are used. However, we have demonstrated that the effect of time delay for the deterministic Kuramoto model is qualitatively equivalent to the effect of frequency fluctuations of the phase oscillators by white additive noise in presence of time delay. This is highly contrast to our earlier result in [23] where peak appears at regular interval in the variation of critical coupling as a function of time delay. Meanwhile for the stochastic Kuramoto model, the critical coupling grows nonlinearly with the increase of the time delay. The dependence of the critical coupling on the noise intensity becomes nonlinear for white noise due to creation of the additional phase difference among the oscillators by the time delay. We have found that it plays an important role in the dynamics of the system when the intrinsic correlations induced by the finite correlation time of the noise are small. We have also observed that the critical coupling is higher for the non-Gaussian noise compared to the Gaussian one due to higher effective noise strength. Our other observation is that in the presence of time delay the critical coupling strength grows at faster rate with the noise parameter (which accounts for the deviation of noise properties from the Gaussian characteristics) in presence of time delay compared to its absence. Finally, the ratio of critical coupling strengths $\epsilon_c(\tau_d = 0)/\epsilon_c(\tau_d \neq 0)$ decreases with increase in noise strength (D) or noise parameter p, and it increases as the noise correlation time grows. These observations represent the reverse situations of the previous results presented in [23].

Acknowledgment

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