

Research Article

Weakly Compatible Maps Using E.A. and (CLR) Properties in Complex Valued G -Metric Spaces

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We introduce the notion of complex valued G -metric spaces and prove common fixed point theorems for weakly compatible maps along with E.A. and (CLR) properties in complex valued G -metric spaces.

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Recently, Mustafa and Sims [1, 2] have shown that most of the results concerning Dhage's D -metric spaces are invalid; therefore, they introduced an improved version of the generalized metric space structure and called it G -metric spaces.

In 2006, Mustafa and Sims [2] introduced the concept of G -metric spaces as follows.

Definition 1. Let X be a nonempty set, and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $0 < G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality);

Then the function G is called a generalized metric or, more specially a G -metric on X , and the pair (X, G) is called a G -metric space.

The idea of complex metric space was initiated by Azam et al. [3] to exploit the idea of complex valued normed spaces and complex valued Hilbert spaces.

Definition 2. Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \quad \text{iff} \quad \begin{aligned} \operatorname{Re}(z_1) &\leq \operatorname{Re}(z_2), \\ \operatorname{Im}(z_1) &\leq \operatorname{Im}(z_2). \end{aligned} \quad (1)$$

That is, $z_1 \preceq z_2$ if one of the following holds:

- (C1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;
- (C2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;
- (C3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$;
- (C4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

In particular, we will write that $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied, and we will write $z_1 < z_2$ if only (C4) is satisfied.

Remark 3. We noted that the following statements hold:

- (i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \preceq bz$ for all $z \in \mathbb{C}$;
- (ii) $0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2|$;
- (iii) $z_1 \preceq z_2$ and $z_2 < z_3 \Rightarrow z_1 < z_3$.

Now we introduce the notion of complex valued G -metric space akin to the notion of complex valued metric spaces [3] as follows.

Definition 4. Let X be a non-empty set. Let $G : X \times X \times X \rightarrow \mathbb{C}$ be a function satisfying the following properties:

- (CG1) $G(x, y, z) = 0$ if $x = y = z$;
- (CG2) $0 < G(x, y, z)$ for all $x, y \in X$ with $x \neq y$;
- (CG3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (CG4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);
- (CG5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a complex valued generalized metric or more specially, a complex valued G -metric on X , and the pair (X, G) is called a complex valued G -metric space.

2. The Complex Valued G -Metric Topology

A point $x \in X$ is called *interior point* of a set $A \subseteq X$, whenever there exists $0 < r \in \mathbb{C}$ such that

$$B_G(x, r) = \{y \in X : G(x, y, y) < r\} \subseteq A. \quad (2)$$

A point $x \in X$ is called *limit point* of a set A , whenever there exists $0 < r \in \mathbb{C}$:

$$B_G(x, r) \cap \left(\frac{A}{X}\right) \neq \emptyset. \quad (3)$$

The set A is called *open*, whenever each element of A is an interior point of A . A subset $B \subseteq X$ is called *closed*, whenever each limit point of B belongs to B .

Proposition 5. Let (X, G) be complex valued G -metric space, then for any $x_0 \in X$ and $r > 0$, one has the following:

- (1) if $G(x_0, x, y) < r$, then $x, y \in B_G(x_0, r)$;
- (2) if $y \in B_G(x_0, r)$, then there exists $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Proposition 6. Let (X, G) be complex valued G -metric space; then for all $x_0 \in X$ and $r > 0$, one has

$$B_G\left(x_0, \frac{1}{3}r\right) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r), \quad (4)$$

where $d_G(x, y) = G(x, y, y) + G(x, x, y)$.

Proposition 7. Let (X, G) be a complex valued G -metric space. Then for any x, y, z , and a in X it follows that:

- (i) if $G(x, y, z) = 0$ and $x = y = z$;
- (ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
- (iii) $G(x, y, y) \leq 2G(y, x, x)$;
- (iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;
- (v) $G(x, y, z) \leq 2/3(G(x, y, a) + G(x, a, z) + G(a, y, z))$;
- (vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Proposition 8. Let (X, G) be a complex valued G -metric space. Then the following are equivalent:

- (i) (X, G) is symmetric;
- (ii) $G(x, y, y) \leq G(x, y, a)$, for all $x, y, a \in X$;
- (iii) $G(x, y, z) \leq G(x, y, a) + G(z, y, b)$ for all $x, y, z, a, b \in X$.

3. Convergence, Continuity, and Completeness in Complex Valued G -Metric Spaces

Now we discuss some definition regarding convergence, continuity, and completeness in complex valued G -metric spaces.

Definition 9. Let (X, G) be a complex valued G -metric space. Let $\{x_n\}$ be a sequence of points of X ; we say that $\{x_n\}$ is complex valued G -convergent to x if, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$. We refer to x as the limit of the sequence $\{x_n\}$, and we write $x_n \xrightarrow{(G)} x$.

Proposition 10. Let (X, G) be complex valued G -metric spaces. For a sequence $\{x_n\} \subseteq X$ and point $x \in X$, the following are equivalent:

- (1) $\{x_n\}$ is complex valued G -convergent to x ;
- (2) $|G(x_n, x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$;
- (3) $|G(x_n, x, x)| \rightarrow 0$ as $n \rightarrow \infty$;
- (4) $|G(x_m, x_n, x)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Definition 11. Let (X, G) and (X', G') be two complex valued G -metric spaces. Then a function $f : X \rightarrow X'$ is complex valued G -continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0), r)) \in \tau(G)$ for all $r > 0$. We say f is complex valued G -continuous if it is complex valued G -continuous at all points of X ; that is, continuous as a function from X with the $\tau(G)$ -topology to X' with $\tau(G')$ -topology.

Since complex valued G -metric topologies are metric topologies, therefore we have some proposition in this regard.

Proposition 12. Let (X, G) and (X', G') be two complex valued G -metric spaces. Then a function $f : X \rightarrow X'$ is complex valued G -continuous at a point $x \in X$ if and only if it is complex valued G -sequentially continuous at x ; that is, whenever $\{x_n\}$ is complex valued G -convergent to x , one has $(f\{x_n\})$ is complex valued G -convergent to $f(x)$.

Proposition 13. Let (X, G) be a complex valued G -metric spaces. The function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proof. Suppose that $\{x_k\}$, $\{y_m\}$, and $\{z_n\}$ are complex valued G -convergent to x , y , and z , respectively. Then by (CG5), we have

$$\begin{aligned} G(x, y, z) &\leq G(y, y_m, y_m) + G(y_m, x, z), \\ G(z, x, y_m) &\leq G(x, x_k, x_k) + G(x_k, y_m, z), \\ G(z, x_k, y_m) &\leq G(z, z_n, z_n) + G(z_n, y_m, x_k), \end{aligned} \quad (5)$$

so

$$\begin{aligned} G(x, y, z) - G(x_k, y_m, z_n) \\ \leq G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n). \end{aligned} \quad (6)$$

Similarly,

$$\begin{aligned} G(x_k, y_m, z_n) - G(x, y, z) \\ \leq G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z). \end{aligned} \quad (7)$$

Combining the above inequality and using (iii) of Proposition 7, we have

$$\begin{aligned} |G(x_k, y_m, z_n) - G(x, y, z)| \\ \leq 2(G(x, x_k, x_k) + G(y, y_m, y_m) + G(z, z_n, z_n)), \end{aligned} \quad (8)$$

$|G(x_k, y_m, z_n) - G(x, y, z)| \rightarrow 0$, as $k, m, n \rightarrow \infty$, and the result follows by Proposition 12. \square

Definition 14. Let (X, G) be a complex valued G -metric space. A sequence $\{x_n\}$ is complex valued G -Cauchy if, given $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$.

Definition 15. A complex valued G -metric space (X, G) is said to be complex valued G -complete if every complex valued G -Cauchy sequence is complex valued G -convergent in (X, G) .

Proposition 16. Let (X, G) be a complex valued G -metric space. Then the following are equivalent:

- (1) the sequence $\{x_n\}$ is a complex valued G -Cauchy in X ;
- (2) for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq k$;
- (3) $\{x_n\}$ is a Cauchy sequence in the complex valued metric space (X, d_G) .

Proposition 17. Let (X, G) be a complex valued G -metric space, and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is complex valued G -convergent to x if and only if $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. Suppose that $\{x_n\}$ is complex valued G -convergent to x . For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}. \quad (9)$$

Then $0 < r \in \mathbb{C}$, and there is a natural number k , such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$.

Therefore,

$$|G(x, x_n, x_m)| < |\epsilon| = \epsilon, \quad \forall n, m \geq k. \quad (10)$$

It follows that $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Conversely, suppose that $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$. Then given $r \in \mathbb{C}$ with $0 < c$, there exists a real number $\delta > 0$, such that, for $z \in \mathbb{C}$,

$$|z| < \delta \implies z < c. \quad (11)$$

For this δ , there is a natural number k such that

$$|G(x, x_n, x_m)| < \delta, \quad \forall n, m \geq k. \quad (12)$$

This means that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$. Hence $\{x_n\}$ is complex valued G -convergent to x . \square

Proposition 18. Let (X, G) be a complex valued G -metric space, and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is complex valued G -Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. Suppose that $\{x_n\}$ is complex valued G -Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}. \quad (13)$$

Then $0 < r \in \mathbb{C}$, and there is a natural number k , such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m \geq k$.

Therefore,

$$|G(x_n, x_m, x_l)| < |\epsilon| = \epsilon, \quad \forall n, m \geq k. \quad (14)$$

It follows that $|G(x_n, x_m, x_l)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Conversely, suppose that $|G(x_n, x_m, x_l)| \rightarrow 0$ as $n, m \rightarrow \infty$. Then given $c \in \mathbb{C}$ with $0 < c$, there exists a real number $\delta > 0$, such that, for $z \in \mathbb{C}$,

$$|z| < \delta \implies z < c. \quad (15)$$

For this δ , there is a natural number k such that

$$|G(x_n, x_m, x_l)| < \delta, \quad \forall n, m \geq k. \quad (16)$$

This means that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m \geq k$. Hence $\{x_n\}$ is complex valued G -Cauchy sequence. \square

4. Weakly Compatible Maps

In 1996, Jungck [4] introduced the concept of weakly compatible maps as follows.

Definition 19. Two self-maps f and g are said to be weakly compatible if they commute at coincidence points.

Now we prove our main result for a pair of self-mappings.

Theorem 20. Let (X, G) be a complete complex valued G -metric space. Let $S, T : X \rightarrow X$ be self-mappings satisfying the following conditions:

- (2.1) $S(X) \subseteq T(X)$;
 (2.2) any one of the subspace $S(X)$ or $T(X)$ is complete;
 (2.3) $G(Sx, Sy, Sz) \leq k G(Tx, Ty, Tz)$ for all $x, y, z \in X$, where $0 \leq k < 1$;
 (2.4) S and T are weakly compatible self-maps.

Then S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point in X . By (2.1), one can choose a point x_1 in X such that $Sx_0 = Tx_1$. In general, choose x_{n+1} such that

$$y_n = Sx_n = Tx_{n+1}. \quad (17)$$

Now, we prove that $\{y_n\}$ is a complex valued G -Cauchy sequence in X .

Putting $x = x_n$, $y = x_{n+1}$, and $z = x_{n+1}$ in (2.3), we have

$$\begin{aligned} G(Sx_n, Sx_{n+1}, Sx_{n+1}) \\ \leq kG(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ = kG(Sx_{n-1}, Sx_n, Sx_n). \end{aligned} \quad (18)$$

Continuing in the same way, we have

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq k^n G(Sx_0, Sx_1, Sx_1). \quad (19)$$

This implies that $G(y_n, y_{n+1}, y_{n+1}) \leq k^n G(y_0, y_1, y_1)$.

Then, for all $n, m \in \mathbb{N}$, $n < m$, we have by (CG5)

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) \\ &\quad + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + G(y_{n+2}, y_{n+3}, y_{n+3}) \\ &\quad + \cdots + G(y_{m-1}, y_m, y_m) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}) \\ &\quad \times G(y_0, y_1, y_1) \\ &\leq \frac{k^n}{1-k} G(y_0, y_1, y_1). \end{aligned} \quad (20)$$

Therefore,

$$|G(y_n, y_m, y_m)| \leq \left(\frac{k^n}{1-k} \right) |G(y_0, y_1, y_1)|. \quad (21)$$

Proceeding limit as $n, m \rightarrow \infty$ and since $0 \leq k < 1$, then $(k^n/(1-k))|G(y_0, y_1, y_1)| \rightarrow 0$; that is, $|G(y_n, y_m, y_m)| \rightarrow 0$.

For $n, m, l \in \mathbb{N}$, (CG5) implies that

$$G(y_n, y_m, y_l) \leq G(y_n, y_m, y_m) + G(y_l, y_m, y_m). \quad (22)$$

Therefore,

$$|G(y_n, y_m, y_l)| \leq |G(y_n, y_m, y_m)| + |G(y_l, y_m, y_m)|. \quad (23)$$

Taking limit as $n, m, l \rightarrow \infty$, we get $|G(y_n, y_m, y_l)| \rightarrow 0$; that is, $G(y_n, y_m, y_l) \rightarrow 0$. So $\{y_n\}$ is complex valued G -Cauchy sequence. Since either $S(X)$ or $T(X)$ is complete.

Without loss of generality, we assume that $T(X)$ is complete subspace of X , and then the subsequence of $\{y_n\}$ must get a limit in $T(X)$, say z . Then $Tu = z$ for some $u \in X$, as $\{y_n\}$ is a complex valued G -Cauchy sequence containing a convergent subsequence.

Next we show that $Su = z$. On setting $x = u$, $y = x_n$, and $z = x_n$ in (2.3), we have

$$G(Su, Sx_n, Sx_n) \leq kG(Tu, Tx_n, Tx_n). \quad (24)$$

Taking limit as $n \rightarrow \infty$, we have $G(Su, z, z) \leq kG(Tu, z, z)$.

Therefore, $|G(Su, z, z)| \leq k|G(Tu, z, z)|$ implies that $Su = z$.

Therefore, $Su = Tu = z$. That is, u is coincidence point of S and T . Since S and T are weakly compatible, it follows that $STu = TSu$; that is, $Sz = Tz$.

We now show that $Sz = z$. Suppose that $Sz \neq z$; therefore $0 < G(Sz, z, z)$ implies that $|G(Sz, z, z)| > 0$.

Putting $x = z$, $y = u$, and $z = u$ in (2.3), we have

$$G(Sz, Su, Su) \leq kG(Tz, Tu, Tu) = kG(Sz, z, z); \quad (25)$$

That is, $|G(Sz, z, z)| \leq k|G(Sz, z, z)| < |G(Sz, z, z)|$, which is a contradiction; therefore $Sz = z$. Thus $Sz = Tz = z$; that is, z is a common fixed point of S and T .

Uniqueness. To prove uniqueness, suppose that $w \neq z$ be another common fixed point of S and T . Then $0 < G(z, w, w)$ implies that $|G(z, w, w)| > 0$.

Putting $x = z$, $y = u$, and $z = u$ in (2.3), we have

$$\begin{aligned} G(z, w, w) &= G(Sz, Sw, Sw) \leq kG(Tz, Tw, Tw) \\ &= kG(z, w, w); \end{aligned} \quad (26)$$

that is, $|G(z, w, w)| \leq k|G(z, w, w)| < |G(z, w, w)|$, which is a contradiction; therefore $z = w$. Thus $Sz = Tz = z$; that is, z is a unique common fixed point of S and T . \square

Example 21. Let $X = [-1, 1]$, and let $G : X \times X \times X \rightarrow \mathbb{C}$ be complex valued G -metric space defined as follows:

$$\begin{aligned} G(x, y, z) &= |x - y| + |y - z| + |z - x|, \\ &\quad \forall x, y, z \in X. \end{aligned} \quad (27)$$

Then (X, G) is complex valued G -metric space. Define $S, T : X \rightarrow X$ as $Sx = x/6$ and $Tx = x/2$. Here we note that (2.1) $S(X) \subseteq T(X)$, (2.3) $G(Sx, Sy, Sz) \leq kG(Tx, Ty, Tz)$ holds for all $x, y, z \in X$, $1/3 \leq k < 1$, and (2.4) S and T are weakly compatible because S and T commute at their coincidence point, that is, at $x = 0$, and $x = 0$ is the unique common fixed point of S and T and S and T also satisfy the condition (2.2).

5. E.A. Property and Weakly Compatible Maps

In 2002, Aamri and Moutawakil [5] introduced the notion of E.A. property as follows.

Definition 22. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a

sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = t$ for some t in X .

In a similar mode, we use these notions in complex valued G -metric spaces.

Example 23. Let $X = \mathbb{C}$. Let $G : X \times X \times X \rightarrow \mathbb{C}$ be complex valued G -metric space defined as follows:

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad \forall x, y, z \in X. \quad (28)$$

Then (X, G) is complex valued G -metric space. Define $S, T : X \rightarrow X$ as $Sx = x + 1$ and $Tx = 2x$; for all $x \in X$.

Consider a sequence $\{x_n\} = \{1 - 1/n\}$, $n \in \mathbb{N}$, in X ; then

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} (x_n + 1) = \lim_{n \rightarrow \infty} 1 - \frac{1}{n} + 1 = 2, \quad (29)$$

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} 2x_n = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right) = 2.$$

Thus S and T satisfy E.A. property.

Now we prove a common fixed point theorem for weakly compatible maps along with E.A. property.

Theorem 24. Let S and T be self-mappings of a complex valued G -metric space (X, G) satisfying (2.3), (2.4), and the following:

(3.1) S and T satisfy E.A. property;

(3.2) $T(X)$ is a closed subspace of X .

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy the E.A. property, therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = u \in X$. Since $T(X)$ is a closed subspace of X , therefore every convergent sequence of points of $T(X)$ has a limit point in $T(X)$.

Then $\lim_{n \rightarrow \infty} Sx_n = u = Tv = \lim_{n \rightarrow \infty} Tx_n$ for some v in X . This implies that $u = Tv \in T(X)$.

On setting $x = v$, $y = x_n$, and $z = x_n$, in (2.3), we have

$$G(Sv, Sx_n, Sx_n) \preceq kG(Tv, Tx_n, Tx_n). \quad (30)$$

Taking limit as $n \rightarrow \infty$, we have $G(Sv, u, u) \preceq kG(Tv, u, u)$.

Therefore, $|G(Sv, u, u)| \leq k|G(Tv, u, u)|$ implies that $Sv = u$.

Therefore, $Sv = Tv = u$. That is, v is coincidence point of S and T . Since S and T are weakly compatible, it follows that $STv = TSv$; that is, $Su = Tu$.

Next we show that $Su = u$. On setting $x = u$, $y = v$, and $z = v$ in (2.3), we have

$$G(Su, Sv, Sv) \preceq kG(Tu, Tv, Tv). \quad (31)$$

Therefore, $|G(Su, u, u)| \leq k|G(Tu, u, u)|$ implies that $Su = u$. Hence u is a common fixed point of S and T . \square

Uniqueness easily follows from Theorem 20. Hence u is a unique common fixed point of S and T .

6. (CLR) Property and Weakly Compatible Maps

In 2011, Kumam and Sintunavarat [6] introduced the notion of (CLR_g) property as follows.

Definition 25. Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = gx$ for some x in X .

In a similar mode, we use these notions in complex valued G -metric spaces.

Example 26. Let $X = \mathbb{C}$. Let $G : X \times X \times X \rightarrow \mathbb{C}$ be complex valued G -metric space defined as follows:

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad \forall x, y, z \in X. \quad (32)$$

Then (X, G) is complex valued G -metric space. Define $S, T : X \rightarrow X$ as $Sx = x + 1$ and $Tx = 2x$ for all $x \in X$.

Consider a sequence $\{x_n\} = \{1 - 1/n\}$, $n \in \mathbb{N}$, in X ; then

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} (x_n + 1) = \lim_{n \rightarrow \infty} 1 - \frac{1}{n} + 1 = 2, \quad (33)$$

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} 2x_n = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right) = 2.$$

Also, we have

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = 2 = S(1). \quad (34)$$

Thus S and T satisfy (CLR_S) property

Now we prove a common fixed point theorem for weakly compatible maps along with (CLR_T) property.

Theorem 27. Let S and T be self-mappings of a complex valued G -metric space (X, G) satisfying (2.1), (2.3), (2.4), and the following:

(4.1) S and T satisfy (CLR_T) property.

Then S and T have a unique common fixed point.

Proof. Since S and T satisfy the (CLR_T) property, therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = u \in X$. Then $\lim_{n \rightarrow \infty} Sx_n = u = Tv = \lim_{n \rightarrow \infty} Tx_n$ for some v in X . This implies that $u = Tv \in T(X)$.

On setting $x = v$, $y = x_n$, and $z = x_n$ in (2.3), we have

$$G(Sv, Sx_n, Sx_n) \preceq kG(Tv, Tx_n, Tx_n). \quad (35)$$

Taking limit as $n \rightarrow \infty$, we have $G(Sv, u, u) \preceq kG(Tv, u, u)$.

Therefore, $|G(Sv, u, u)| \leq k|G(Tv, u, u)|$ implies that $Sv = u$.

Therefore, $Sv = Tv = u$. That is, v is coincidence point of S and T . Since S and T are weakly compatible, it follows that $STv = TSv$; that is, $Su = Tu$.

Next we show that $Su = u$. On setting $x = u$, $y = v$, and $z = v$ in (2.3), we have

$$G(Su, Sv, Sv) \preceq kG(Tu, Tv, Tv). \quad (36)$$

Therefore, $|G(Su, u, u)| \leq k|G(u, u, u)|$ implies that $Su = u$. Hence u is a common fixed point of S and T . \square

Uniqueness easily follows from Theorem 20. Hence u is a unique common fixed point of S and T .

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