

## Research Article

# Common Fixed Point Theorems for a Rational Inequality in Complex Valued Metric Spaces

Pankaj Kumar,<sup>1</sup> Manoj Kumar,<sup>1</sup> and Sanjay Kumar<sup>2</sup>

<sup>1</sup> Department of Mathematics, Guru Jambheshwar University of Science and Technology, Hisar 125001, India

<sup>2</sup> Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal 131039, India

Correspondence should be addressed to Sanjay Kumar; sanjuciet@rediffmail.com

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We prove a common fixed point theorem for a pair of mappings. Also, we prove a common fixed point theorem for pairs of self-mappings along with weakly commuting property.

## 1. Introduction

Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point theorems for the mappings satisfying a rational inequality. The definition of a cone metric space banks on the underlying Banach space which is not a division Ring. The idea of rational expressions is not meaningful in cone metric spaces, and therefore many results of analysis cannot be generalized to cone metric spaces. The complex valued metric spaces form a special class of cone metric space, and we can study improvements of host results of analysis involving divisions.

A complex number  $z \in \mathbb{C}$  is an ordered pair of real numbers, whose first coordinate is called  $\text{Re}(z)$  and second coordinate is called  $\text{Im}(z)$ .

Let  $\mathbb{C}$  be the set of complex numbers and  $z_1, z_2 \in \mathbb{C}$ . Define a partial order  $\preceq$  on  $\mathbb{C}$  as follows:  $z_1 \preceq z_2$  if and only if  $\text{Re}(z_1) \leq \text{Re}(z_2)$  and  $\text{Im}(z_1) \leq \text{Im}(z_2)$ ; that is,  $z_1 \preceq z_2$ , if one of the following holds:

- (C1)  $\text{Re}(z_1) = \text{Re}(z_2)$  and  $\text{Im}(z_1) = \text{Im}(z_2)$ ;
- (C2)  $\text{Re}(z_1) < \text{Re}(z_2)$  and  $\text{Im}(z_1) = \text{Im}(z_2)$ ;
- (C3)  $\text{Re}(z_1) = \text{Re}(z_2)$  and  $\text{Im}(z_1) < \text{Im}(z_2)$ ;
- (C4)  $\text{Re}(z_1) < \text{Re}(z_2)$  and  $\text{Im}(z_1) < \text{Im}(z_2)$ .

In particular, we will write  $z_1 \preceq z_2$  if  $z_1 \neq z_2$  and one of (C2), (C3), and (C4) is satisfied, and we will write  $z_1 < z_2$  if only (C4) is satisfied.

*Remark 1.* We note that the following statements hold:

- (i)  $a, b \in \mathbb{R}$  and  $a \leq b \Rightarrow az \preceq bz \forall z \in \mathbb{C}$ ,
- (ii)  $0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2|$ ,
- (iii)  $z_1 \preceq z_2$  and  $z_2 < z_3 \Rightarrow z_1 < z_3$ .

Azam et al. [1] defined the complex valued metric space  $(X, d)$  as follows.

*Definition 2.* Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow \mathbb{C}$  satisfies the following conditions:

- (i)  $0 \preceq d(x, y)$ , for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (iii)  $d(x, y) \preceq d(x, z) + d(z, y)$ , for all  $x, y, z \in X$ .

Then,  $d$  is called a complex valued metric on  $X$ , and  $(X, d)$  is called a complex valued metric space.

*Example 3.* Let  $X = \mathbb{C}$ . Define the mapping  $d : X \times X \rightarrow \mathbb{C}$  by

$$d(x, y) = i|x - y|, \quad \forall x, y \in X. \quad (1)$$

Then,  $(X, d)$  is a complex valued metric space.

*Definition 4.* Let  $(X, d)$  be a complex valued metric space. A sequence  $\{x_n\}$  in  $X$  is said to be

- (i) convergent to  $x$ , if for every  $c \in \mathbb{C}$  with  $0 < c$  there is  $k \in \mathbb{N}$  such that, for all  $n > k$ ,  $d(x_n, x) < c$ . We denote this by  $\{x_n\} \rightarrow x$  as  $n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} x_n = x$ ;
- (ii) Cauchy, if for every  $c \in \mathbb{C}$  with  $0 < c$  there is  $k \in \mathbb{N}$  such that for all  $n > k$ ,  $d(x_n, x_{n+m}) < c$ , where  $m \in \mathbb{N}$ ;
- (iii) complete, if every Cauchy sequence in  $X$  converges in  $X$ .

**Lemma 5.** Let  $(X, d)$  be a complex valued metric space, and let  $\{x_n\}$  be a sequence in  $X$ . Then,  $\{x_n\}$  converges to  $x$  if and only if  $|d(x_n, x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 6.** Let  $(X, d)$  be a complex valued metric space, and let  $\{x_n\}$  be a sequence in  $X$ . Then,  $\{x_n\}$  is a Cauchy sequence if and only if  $|d(x_n, x_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ , where  $m \in \mathbb{N}$ .

In 1982, Sessa [2] introduced the notion of weak commutativity as follows.

**Definition 7.** Two self-maps  $f$  and  $g$  of a metric space  $(X, d)$  are said to be weakly commuting if  $d(fgx, gfx) \leq d(fx, gx)$ , for all  $x$  in  $X$ .

In a similar mode, we introduce the notion of weak commutativity in complex valued metric spaces as follows.

**Definition 8.** Two self maps  $f$  and  $g$  of a complex valued metric space  $(X, d)$  are said to be weakly commuting if  $d(fgx, gfx) \leq d(fx, gx)$ , for all  $x$  in  $X$ .

**Example 9.** Let  $X = [0, \infty)$ , and define  $d : X \times X \rightarrow \mathbb{C}$  by  $d(x, y) = i|x - y|$ , for all  $x, y \in X$ .

Then,  $(X, d)$  is a complex valued metric space.

Define  $fx = x$  and  $gx = 2x$ .

Then, clearly  $d(fgx, gfx) \leq d(fx, gx)$ , for all  $x$  in  $X$ .

Thus,  $f$  and  $g$  are weakly commuting.

## 2. Main Theorem

**Theorem 10.** Let  $f$  and  $g$  be self mappings of a complex valued metric space  $(X, d)$  satisfying the following:

$$\begin{aligned}
 & d(fx, gy) \\
 & \leq k \left[ \frac{d(x, fx)d(x, gy) + \{d(x, y)\}^2 + d(x, fx)d(x, y)}{d(x, fx) + d(x, y) + d(x, gy)} \right] \\
 & \quad \forall x, y \text{ in } X \text{ with } x \neq y, \quad 0 < k < 1, \\
 & \quad d(x, fx) + d(x, y) + d(x, gy) \neq 0.
 \end{aligned} \tag{2}$$

Then,  $f$  and  $g$  have a common fixed point.

Further, if  $d(x, fx) + d(x, y) + d(x, gy) = 0$  implies that  $d(fx, gy) = 0$ , then  $f$  and  $g$  have a unique common fixed point.

**Proof.** Let  $x_0 \in X$ . Define a sequence  $\{x_n\}$  in  $X$  by  $x_{2n+1} = fx_{2n}$ ,  $x_{2n+2} = gx_{2n+1}$ ,  $n = 0, 1, 2, \dots$

Let  $d(x, fx) + d(x, y) + d(x, gy) \neq 0$ .

From (2), we have

$$\begin{aligned}
 & d(x_{2n+1}, x_{2n+2}) \\
 & = d(fx_{2n}, gx_{2n+1}) \\
 & \leq k \left[ \left( d(x_{2n}, fx_{2n})d(x_{2n}, gx_{2n+1}) + \{d(x_{2n}, x_{2n+1})\}^2 \right. \right. \\
 & \quad \left. \left. + d(x_{2n}, fx_{2n})d(x_{2n}, x_{2n+1}) \right) \right. \\
 & \quad \left. \times (d(x_{2n}, fx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, gx_{2n+1}))^{-1} \right] \\
 & = k \left[ \left( d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + \{d(x_{2n}, x_{2n+1})\}^2 \right. \right. \\
 & \quad \left. \left. + d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}) \right) \right. \\
 & \quad \left. \times (d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}))^{-1} \right] \\
 & = kd(x_{2n}, x_{2n+1}) \\
 & \quad \times \left[ \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})} \right] \\
 & = kd(x_{2n}, x_{2n+1}).
 \end{aligned} \tag{3}$$

Similarly, we have

$$\begin{aligned}
 & d(x_{2n}, x_{2n+1}) \\
 & = d(fx_{2n}, gx_{2n-1}) \\
 & \leq k \left[ \left( d(x_{2n}, fx_{2n})d(x_{2n}, gx_{2n-1}) + \{d(x_{2n}, x_{2n-1})\}^2 \right. \right. \\
 & \quad \left. \left. + d(x_{2n}, fx_{2n})d(x_{2n}, x_{2n-1}) \right) \right. \\
 & \quad \left. \times (d(x_{2n}, fx_{2n}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, gx_{2n-1}))^{-1} \right] \\
 & = k \left[ \left( d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n}) + \{d(x_{2n}, x_{2n-1})\}^2 \right. \right. \\
 & \quad \left. \left. + d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n-1}) \right) \right. \\
 & \quad \left. \times (d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n}))^{-1} \right] \\
 & = kd(x_{2n}, x_{2n-1}) \left[ \frac{d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1})} \right] \\
 & = kd(x_{2n}, x_{2n-1}) = kd(x_{2n-1}, x_{2n}).
 \end{aligned} \tag{4}$$

Consequently, it can be concluded that

$$\begin{aligned}
 & d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \\
 & \leq k^2 d(x_{n-2}, x_{n-1}) \\
 & \vdots \\
 & \leq k^n d(x_0, x_1).
 \end{aligned} \tag{5}$$

Now, for all  $m > n$ ,

$$\begin{aligned}
 d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_m, x_{m-1}) \\
 &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\
 &\leq \frac{k^n}{1-k} d(x_0, x_1).
 \end{aligned} \tag{6}$$

Therefore, we have

$$|d(x_m, x_n)| \leq \frac{k^n}{1-k} |d(x_0, x_1)|. \tag{7}$$

Hence,

$$\lim_{n \rightarrow \infty} |d(x_m, x_n)| = 0. \tag{8}$$

Hence,  $\{x_n\}$  is a Cauchy sequence in  $X$ . But  $X$  is complete metric space, so  $\{x_n\}$  is convergent to some point, say  $z$ , in  $X$ , that is,  $x_n \rightarrow z$  as  $n \rightarrow \infty$ .

Now, we will prove that  $z = gz$ .

Let, if possible,  $z \neq gz$ .

Now, using the triangular inequality and (2), we have

$$\begin{aligned}
 d(z, gz) &\leq d(z, x_{2n+1}) + d(x_{2n+1}, gz) \\
 &= d(z, x_{2n+1}) + d(fx_{2n}, gz) \\
 &\leq d(z, x_{2n+1}) \\
 &\quad + k \left[ (d(x_{2n}, fx_{2n}) d(x_{2n}, gz) + \{d(x_{2n}, z)\}^2 \right. \\
 &\quad \left. + d(x_{2n}, fx_{2n}) d(x_{2n}, z)) \right. \\
 &\quad \left. \times (d(x_{2n}, fx_{2n}) + d(x_{2n}, z) + d(x_{2n}, gz))^{-1} \right] \\
 &= d(z, x_{2n+1}) \\
 &\quad + k \left[ (d(x_{2n}, x_{2n+1}) d(x_{2n}, gz) + \{d(x_{2n}, z)\}^2 \right. \\
 &\quad \left. + d(x_{2n}, x_{2n+1}) d(x_{2n}, z)) \right. \\
 &\quad \left. \times (d(x_{2n}, x_{2n+1}) + d(x_{2n}, z) + d(x_{2n}, gz))^{-1} \right].
 \end{aligned} \tag{9}$$

Thus, we have

$$\begin{aligned}
 |d(z, gz)| &\leq |d(z, x_{2n+1})| \\
 &\quad + k \left[ (d(x_{2n}, x_{2n+1}) d(x_{2n}, gz) + \{d(x_{2n}, z)\}^2 \right. \\
 &\quad \left. + d(x_{2n}, x_{2n+1}) d(x_{2n}, z)) \right. \\
 &\quad \left. \times (d(x_{2n}, x_{2n+1}) + d(x_{2n}, z) + d(x_{2n}, gz))^{-1} \right].
 \end{aligned} \tag{10}$$

Letting  $n \rightarrow \infty$ , we have  $|d(z, gz)| \leq 0$ , a contradiction.

Hence, we get  $z = gz$ ; that is,  $z$  is the fixed point of  $g$ .

Again assume that  $z \neq fz$ .

From (2) and using the triangular inequality, we have

$$\begin{aligned}
 d(z, fz) &\leq d(z, x_{2n+2}) + d(x_{2n+2}, fz) \\
 &= d(z, x_{2n+2}) + d(fz, x_{2n+2}) \\
 &= d(z, x_{2n+2}) + d(fz, gx_{2n+1}) \\
 &\leq d(z, x_{2n+2}) \\
 &\quad + k \left[ (d(z, fz) d(z, gx_{2n+1}) + \{d(z, x_{2n+1})\}^2 \right. \\
 &\quad \left. + d(z, fz) d(z, x_{2n+1})) \right. \\
 &\quad \left. \times (d(z, fz) + d(z, x_{2n+1}) + d(z, gx_{2n+1}))^{-1} \right] \\
 &= d(z, x_{2n+2}) \\
 &\quad + k \left[ (d(z, fz) d(z, x_{2n+2}) + \{d(z, x_{2n+1})\}^2 \right. \\
 &\quad \left. + d(z, fz) d(z, x_{2n+1})) \right. \\
 &\quad \left. \times (d(z, fz) + d(z, x_{2n+1}) + d(z, x_{2n+2}))^{-1} \right].
 \end{aligned} \tag{11}$$

Thus, we have

$$\begin{aligned}
 |d(z, gz)| &\leq |d(z, x_{2n+2})| \\
 &\quad + k \left[ (d(z, fz) d(z, x_{2n+2}) + \{d(z, x_{2n+1})\}^2 \right. \\
 &\quad \left. + d(z, fz) d(z, x_{2n+1})) \right. \\
 &\quad \left. \times (d(z, fz) + d(z, x_{2n+1}) + d(z, x_{2n+2}))^{-1} \right].
 \end{aligned} \tag{12}$$

Letting  $n \rightarrow \infty$ , we have  $|d(z, gz)| \leq 0$ , a contradiction.

Hence, we get  $z = fz$ ; that is,  $z$  is a fixed point of  $f$ .

Therefore, we find that  $z$  is a common fixed point of  $f$  and  $g$ .  $\square$

*Uniqueness.* Let  $w (\neq z)$  be another fixed point of  $g$ . Suppose that  $d(x, fx) + d(x, y) + d(x, gy) = 0$  implies  $d(fx, gy) = 0$ .

Now,

$$d(z, fz) + d(z, w) + d(z, gw) = 0 \quad \text{implies} \quad d(fz, gw) = 0. \tag{13}$$

Therefore, we get

$$d(z, w) = d(fz, gw) = 0, \quad \text{which implies that } z = w. \tag{14}$$

Hence,  $f$  and  $g$  have a unique common fixed point.

**Corollary 11.** Let  $f$  be a self-map of a complex valued metric space  $(X, d)$  satisfying the following:

$$d(fx, fy) \preceq k \left[ \frac{d(x, fx)d(x, fy) + \{d(x, y)\}^2 + d(x, fx)d(x, y)}{d(x, fx) + d(x, y) + d(x, fy)} \right],$$

$$\forall x, y \text{ in } X \text{ with } x \neq y, \quad 0 < k < 1,$$

$$d(x, fx) + d(x, y) + d(x, fy) \neq 0. \quad (15)$$

Then,  $f$  has a fixed point.

Further, if  $d(x, fx) + d(x, y) + d(x, fy) = 0$  implies that  $d(fx, fy) = 0$  then  $f$  has a unique common fixed point.

*Proof.* By putting  $f = g$  in Theorem 10, we get Corollary 11.  $\square$

### 3. Weakly Commuting Property

**Theorem 12.** Let  $A, B, S$ , and  $T$  be self mappings of a complex valued metric space  $(X, d)$  satisfying the following:

$$(3.1) \quad SX \subseteq BX, \quad TX \subseteq AX,$$

$$(3.2) \quad \text{the pairs } (A, S) \text{ and } (B, T) \text{ are weakly commuting,}$$

$$(3.3) \quad \text{for all } x, y \text{ in } X, \text{ either}$$

$$d(Sx, Ty) \preceq \alpha \left[ (d(Ax, Sx)d(Ax, Ty) + \{d(Ax, By)\}^2 + d(Ax, Sx)d(Ax, By)) \right. \\ \left. \times (d(Ax, Sx) + d(Ax, By) + d(Ax, Ty))^{-1} \right] + \beta d(Ax, By), \quad (16)$$

if  $d(Ax, Sx) + d(Ax, By) + d(Ax, Ty) \neq 0$ , where  $|\alpha + \beta| < 1$  and  $|\beta| < 1$ ;  $d(Sx, Ty) = 0$ , if  $d(Ax, Sx) + d(Ax, By) + d(Ax, Ty) = 0$ .

If any of  $A, B, S$ , or  $T$  is continuous, then  $A, B, S$ , and  $T$  have a unique common fixed point  $z$ .

*Proof.* Let  $x_0 \in X$ . Since  $SX \subseteq BX$ , so there exists a point  $x_1$  in  $X$  such that  $Sx_0 = Bx_1$ . Also, since  $TX \subseteq AX$ , we can choose a point  $x_2$  in  $X$  such that  $Tx_1 = Ax_2$ .

Continuing this process, we have  $Sx_{2n} = Bx_{2n+1}$  and  $Tx_{2n+1} = Ax_{2n+2}$ , for  $n = 0, 1, 2, \dots$

Define  $d_{2n} = d(Sx_{2n}, Tx_{2n+1})$  and  $d_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1})$ .

Suppose  $d_{2n} \neq 0$  and  $d_{2n+1} \neq 0$  for  $n = 1, 2, 3, \dots$

From (3.3), we have

$$d_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1}) \\ \preceq \alpha \left[ (d(Ax_{2n+2}, Sx_{2n+2})d(Ax_{2n+2}, Tx_{2n+1}) + \{d(Ax_{2n+2}, Bx_{2n+1})\}^2 + d(Ax_{2n+2}, Sx_{2n+2})d(Ax_{2n+2}, Bx_{2n+1})) \right. \\ \left. \times (d(Ax_{2n+2}, Sx_{2n+2}) + d(Ax_{2n+2}, Bx_{2n+1}) + d(Ax_{2n+2}, Tx_{2n+1}))^{-1} \right] \\ + \beta d(Ax_{2n+2}, Bx_{2n+1}) \\ = \alpha \left[ (d(Tx_{2n+1}, Sx_{2n+2})d(Tx_{2n+1}, Tx_{2n+1}) + \{d(Tx_{2n+1}, Sx_{2n})\}^2 + d(Tx_{2n+1}, Sx_{2n+2})d(Tx_{2n+1}, Sx_{2n})) \right. \\ \left. \times (d(Tx_{2n+1}, Sx_{2n+2}) + d(Tx_{2n+1}, Sx_{2n}))^{-1} \right] \\ + \beta d(Tx_{2n+1}, Sx_{2n}) \\ = \alpha \left[ (d(Tx_{2n+1}, Sx_{2n}) \times \{d(Tx_{2n+1}, Sx_{2n}) + d(Tx_{2n+1}, Sx_{2n+2})\}) \right. \\ \left. \times (d(Tx_{2n+1}, Sx_{2n+2}) + d(Tx_{2n+1}, Sx_{2n}))^{-1} \right] \\ + \beta d(Tx_{2n+1}, Sx_{2n}) \\ = (\alpha + \beta) d(Tx_{2n+1}, Sx_{2n}) = (\alpha + \beta) d_{2n}. \quad (17)$$

Thus, we have

$$|d_{2n+1}| \leq |\alpha + \beta| |d_{2n}| = k |d_{2n}|, \quad \text{where } k = |\alpha + \beta| < 1. \quad (18)$$

In general, we have

$$|d_{2n+1}| \leq k |d_{2n}| \\ \leq k^2 |d_{2n-1}| \\ \vdots \quad (19)$$

$$\leq k^{2n+1} |d_0|, \quad \text{that is,}$$

$$|d_{2n+1}| \leq k^{2n+1} |d_0|.$$

Letting  $n \rightarrow \infty$ , we have

$$|d_{2n+1}| \leq 0, \quad \text{which implies that } d_{2n+1} = 0. \quad (20)$$

Therefore,  $d(Sx_{2n+2}, Tx_{2n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ .

We get the following sequence:

(3.4)  $\{Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots\}$ , which is a Cauchy sequence in the complete complex valued metric space  $(X, d)$ , therefore converges to a limit point  $z$  in  $X$ .

Therefore, the sequences  $\{Sx_{2n}\} = \{Bx_{2n+1}\}$  and  $\{Tx_{2n+1}\} = \{Ax_{2n}\}$ , which are the subsequences of (3.4) hence also converge to the same point  $z$  in  $X$ .

Now, suppose that  $A$  is continuous so that the sequences  $\{A^2x_{2n}\}$  and  $\{ASx_{2n}\}$  converge to the same point  $Az$ . Since  $A$  and  $S$  are weakly commuting, we have

$$d(SAx_{2n}, ASx_{2n}) \lesssim d(Ax_{2n}, Sx_{2n}), \quad \text{which implies that}$$

$$|d(SAx_{2n}, ASx_{2n})| \leq |d(Ax_{2n}, Sx_{2n})|. \quad (21)$$

Letting  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} |d(SAx_{2n}, Az)| \leq |d(z, z)| = 0, \quad \text{which implies that}$$

$$\lim_{n \rightarrow \infty} |d(SAx_{2n}, Az)| = 0; \quad \text{that is,}$$

$$SAx_{2n} \longrightarrow z \quad \text{as } n \longrightarrow \infty. \quad (22)$$

Now, we will show that  $Az = z$ . Let, if possible,  $Az \neq z$ .

Now, using the triangle inequality and (3.3), we get

$$\begin{aligned} d(Az, z) &\lesssim d(Az, SAx_{2n}) + d(SAx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, z) \\ &\lesssim d(Az, SAx_{2n}) \\ &\quad + \alpha \left[ \left( d(A^2x_{2n}, SAx_{2n}) d(A^2x_{2n}, Tx_{2n+1}) \right. \right. \\ &\quad \left. \left. + \{d(A^2x_{2n}, Bx_{2n+1})\}^2 \right. \right. \\ &\quad \left. \left. + d(A^2x_{2n}, SAx_{2n}) d(A^2x_{2n}, Bx_{2n+1}) \right) \right. \\ &\quad \left. \times \left( d(A^2x_{2n}, SAx_{2n}) + d(A^2x_{2n}, Bx_{2n+1}) \right. \right. \\ &\quad \left. \left. + d(A^2x_{2n}, Tx_{2n+1}) \right)^{-1} \right] \\ &\quad + \beta d(A^2x_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, z). \end{aligned} \quad (23)$$

Thus, we have

$$\begin{aligned} |d(Az, z)| &\leq |d(Az, SAx_{2n})| \\ &\quad + \left| \alpha \left[ \left( d(A^2x_{2n}, SAx_{2n}) d(A^2x_{2n}, Tx_{2n+1}) \right. \right. \right. \\ &\quad \left. \left. + \{d(A^2x_{2n}, Bx_{2n+1})\}^2 \right. \right. \right. \\ &\quad \left. \left. + d(A^2x_{2n}, SAx_{2n}) d(A^2x_{2n}, Bx_{2n+1}) \right) \right. \right. \\ &\quad \left. \left. \times \left( d(A^2x_{2n}, SAx_{2n}) + d(A^2x_{2n}, Bx_{2n+1}) \right. \right. \right. \\ &\quad \left. \left. \left. + d(A^2x_{2n}, Tx_{2n+1}) \right)^{-1} \right] \right. \\ &\quad \left. + \beta d(A^2x_{2n}, Bx_{2n+1}) \right| + |d(Tx_{2n+1}, z)|. \end{aligned} \quad (24)$$

Letting  $n \rightarrow \infty$ , we have

$$\begin{aligned} |d(Az, z)| &\leq |d(Az, Az)| \\ &\quad + \left| \alpha \left[ \left( d(Az, Az) d(Az, z) + \{d(Az, z)\}^2 \right. \right. \right. \\ &\quad \left. \left. + d(Az, Az) d(Az, z) \right) \right. \right. \\ &\quad \left. \left. \times (d(Az, Az) + d(Az, z) + d(Az, z))^{-1} \right] \right. \\ &\quad \left. + \beta d(Az, z) \right| + |d(z, z)| \\ &= \left| \frac{\alpha}{2} + \beta \right| |d(Az, z)|, \quad \text{which implies that} \\ &\quad \left| \frac{\alpha}{2} + \beta \right| \geq 1, \quad \text{a contradiction to } |\alpha + \beta| < 1. \end{aligned} \quad (25)$$

Hence,  $Az = z$ .

Now, we will prove that  $Sz = z$ .

Again, using the triangle inequality and (3.3), we have

$$\begin{aligned} d(Sz, z) &\lesssim d(Sz, Tx_{2n+1}) + d(Tx_{2n+1}, z) \\ &\lesssim \alpha \left[ \left( d(Az, Sz) d(Az, Tx_{2n+1}) + \{d(Az, Bx_{2n+1})\}^2 \right. \right. \\ &\quad \left. \left. + d(Az, Sz) d(Az, Bx_{2n+1}) \right) \right. \\ &\quad \left. \times \left( d(Az, Sz) + d(Az, Bx_{2n+1}) \right. \right. \\ &\quad \left. \left. + d(Az, Tx_{2n+1}) \right)^{-1} \right] \\ &\quad + \beta d(Az, Bx_{2n+1}) + d(Tx_{2n+1}, z). \end{aligned} \quad (26)$$

Thus, we have

$$\begin{aligned}
 & |d(Sz, z)| \\
 & \leq \left| \alpha \left[ \left( d(Az, Sz) d(Az, Tx_{2n+1}) + \{d(Az, Bx_{2n+1})\}^2 \right. \right. \right. \\
 & \quad \left. \left. + d(Az, Sz) d(Az, Bx_{2n+1}) \right) \right. \\
 & \quad \left. \times (d(Az, Sz) \right. \\
 & \quad \left. + d(Az, Bx_{2n+1}) + d(Az, Tx_{2n+1}))^{-1} \right] \\
 & \quad \left. + \beta d(Az, Bx_{2n+1}) \right| + |d(Tx_{2n+1}, z)|. \tag{27}
 \end{aligned}$$

Letting  $n \rightarrow \infty$ , we have

$$\begin{aligned}
 & |d(Sz, z)| \\
 & \leq \left| \alpha \left[ \frac{d(z, Sz) d(z, z) + \{d(z, z)\}^2 + d(z, Sz) d(z, z)}{d(z, Sz) + d(Az, z) + d(z, z)} \right] \right. \\
 & \quad \left. + \beta d(z, z) \right| + |d(z, z)| = 0, \quad \text{which implies that} \\
 & Sz = z. \tag{28}
 \end{aligned}$$

Now, since  $SX \subseteq BX$ , there exists a point  $w$  in  $X$  such that  $Bw = z$ .

Thus, we have

$$\begin{aligned}
 & d(z, Tw) \\
 & = d(Sz, Tw) \\
 & \lesssim \alpha \left[ \left( d(Az, Sz) d(Az, Tw) + \{d(Az, Bw)\}^2 \right. \right. \\
 & \quad \left. \left. + d(Az, Sz) d(Az, Bw) \right) \right. \\
 & \quad \left. \times (d(Az, Sz) + d(Az, Bw) + d(Az, Tw))^{-1} \right] \\
 & \quad + \beta d(Az, Bw), \quad \text{which implies that}
 \end{aligned} \tag{29}$$

$$|d(z, Tw)| \leq \alpha \cdot 0 + \beta \cdot 0 = 0, \quad \text{that is, } Tw = z.$$

Since  $B$  and  $T$  are weakly commuting, so we have

$$\begin{aligned}
 & d(TBw, BTw) \lesssim d(Bw, Tw), \quad \text{that is,} \\
 & |d(TBw, BTw)| \leq |d(Bw, Tw)|, \quad \text{which implies that} \\
 & |d(Tz, Bz)| \leq |d(z, z)| = 0; \quad \text{that is, } Tz = Bz. \tag{30}
 \end{aligned}$$

Now, we will prove that  $Tz = z$ . Let, if possible,  $Tz \neq z$ .

From (3.3), we have

$$\begin{aligned}
 d(z, Tz) & = d(Sz, Tz) \\
 & \lesssim \alpha \left[ \left( d(Az, Sz) d(Az, Tz) + \{d(Az, Bz)\}^2 \right. \right. \\
 & \quad \left. \left. + d(Az, Sz) d(Az, Bz) \right) \right. \\
 & \quad \left. \times (d(Az, Sz) + d(Az, Bz) + d(Az, Tz))^{-1} \right] \\
 & \quad + \beta d(Az, Bz). \tag{31}
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 |d(z, Tz)| & \leq \left| \frac{\alpha}{2} + \beta \right| |d(z, Tz)|, \quad \text{which implies that} \\
 \left| \frac{\alpha}{2} + \beta \right| & \geq 1, \quad \text{a contradiction to } |\alpha + \beta| < 1. \tag{32}
 \end{aligned}$$

Hence,  $Tz = z = Bz$  and  $Sz = z = Az$ .

So,  $z$  is the common fixed point of  $A, B, S$ , and  $T$ .

Now, if one of the mappings  $B, S$ , or  $T$  is continuous instead of  $A$ , then one can show that  $A, B, S$ , and  $T$  have a common fixed point.

To show that  $z$  is unique, let  $u$  be another common fixed point of  $A$  and  $S$ .

From (3.3), we have

$$\begin{aligned}
 d(u, z) & = d(Su, Tz) \\
 & \lesssim \alpha \left[ \left( d(Au, Su) d(Au, Tz) + \{d(Au, Bz)\}^2 \right. \right. \\
 & \quad \left. \left. + d(Au, Su) d(Au, Bz) \right) \right. \\
 & \quad \left. \times (d(Au, Su) + d(Au, Bz) + d(Au, Tz))^{-1} \right] \\
 & \quad + \beta d(Au, Bz); \quad \text{that is,}
 \end{aligned}$$

$$\begin{aligned}
 |d(u, z)| & \leq \left| \alpha \left[ \left( d(Au, Su) d(Au, Tz) + \{d(Au, Bz)\}^2 \right. \right. \right. \\
 & \quad \left. \left. + d(Au, Su) d(Au, Bz) \right) \right. \\
 & \quad \left. \times (d(Au, Su) + d(Au, Bz) + d(Au, Tz))^{-1} \right] \right. \\
 & \quad \left. + \beta d(Au, Bz) \right|, \quad \text{which implies that}
 \end{aligned}$$

$$|d(u, z)| \leq \left| \frac{\alpha}{2} + \beta \right| |d(u, z)|, \quad \text{implies that,}$$

$$\left| \frac{\alpha}{2} + \beta \right| \geq 1, \quad \text{a contradiction to } |\alpha + \beta| < 1. \tag{33}$$

Thus, we have,  $u = z$  that is,  $A$  and  $S$  have a unique common fixed point.

In the same way, it can be shown that  $z$  is the unique common fixed point of  $B$  and  $T$ .  $\square$

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