Research Article

The Best Achievable $\mathcal{H}_2$ Tracking Performances for SIMO Feedback Control Systems

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This paper is concerned with the inherent $\mathcal{H}_2$ tracking performance limitation of single-input and multiple-output (SIMO) linear time-invariant (LTI) feedback control systems. The performance is measured by the tracking error between a step reference input and the plant output with additional penalty on control input. We employ the plant augmentation strategy, which enables us to derive analytical closed-form expressions of the best achievable performance not only for discrete-time system, but also for continuous-time system by exploiting the delta domain version of the expressions.

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1. INTRODUCTION

Problems concerning the fundamental performance limitations and tradeoff in feedback control systems have been intensively studied for decades motivated by the work of Bode on logarithmic sensitivity integral [1]. There are two main research directions in the area. First direction lies in the extensions of Bode's integral theorem to assess design constraints and performance limitations via logarithmic type integrals (see, e.g., [2, 3]). Second direction focuses on the formulations of optimal control problems to quantify and characterize the fundamental performance limits in terms of plant parameters or properties.

Research in the latter direction relates to the plant/controller design integration, where the main attention is not to design a robust or optimal controller but to design a plant which is easily controllable in practice. Therefore, study on control performance limitations achievable by feedback control has been paid much attention in the recent years as seen in a special issue of the IEEE Transactions on Automatic Control in August 2003 and books by Seron et al. [4] and by Skogestad and Postletwaite [5].

Optimal tracking and regulation problems are two important research topics which follow the second direction. In the former problem, the tracking ability is measured by the minimal tracking error between output and reference input and is optimized over all possible stabilizing controllers. In

the latter problem, the control performance is measured by the control input energy against impulsive disturbance input.

In particular, the $\mathcal{H}_2$ tracking performance limitation achievable by feedback control has been intensively investigated, which has led to some complete results for single-input and single-output (SISO) continuous/discrete-time/sampled-data systems. Beyond the SISO case, existing results on the optimal tracking performance problem include the single-input and multiple-output (SIMO) [6, 7] and multiple-input and multiple-output (MIMO) [8–10] cases. Results on the $\mathcal{H}_2$ energy regulation problem are available for continuous-time systems [11] and discrete-time systems [12]. A recent result which is formulated in the state space approach and in both of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks can be found in [13]. It is known that the energy regulation problem relates closely to the problem of feedback stabilization over signal-to-noise ratio constrained channels [14, 15].

In general, the existing results demonstrate in a clear manner how the control performance level may be limited by plant unstable poles and nonminimum phase zeros, and additionally the directional aspects of such poles and zeros in the case of a multivariable system.

This paper focuses on the $\mathcal{H}_2$ optimal tracking problems with control input penalty for possibly unstable, nonminimum phase, SIMO LTI plants. The tracking performance is measured by the tracking error between the measurement

output and a step reference input with additional penalty on control input. We note that none of the results for the SIMO or MIMO cases [6, 7, 9, 10] except one in [8] are practically useful, since problems without control input constraint were only treated. Also [8] only considers the marginally stable case. Moreover, the result in [9] is only valid for the MIMO right invertible case, where the number of inputs is greater than or equal to that of outputs. In other words, the result cannot be applied to the SIMO case.

The problem formulation employed here is more realistic than the problem without penalty on the control input, since the controller cannot produce an input beyond the capability of the actuator in practice. The treatment of the SIMO case is also of practical significance, since in such a case, the plant to be controlled has only one actuator with two or more sensors, which commonly appears in real control applications to get better control performance by putting extra sensors. The classes of feedback systems investigated here cover continuous-time and discrete-time systems and thus are fairly wide. We will provide comprehensive results of the analytical closed-form expressions on the performance limitations by a unified approach (delta operator).

The main contribution of the present paper is to derive analytical closed-form expressions of the $\mathcal{H}_2$ optimal tracking performance under control input penalty for SIMO LTI discrete-time systems. The idea of the derivation is to implement the well-known plant augmentation strategy, which enables us to apply directly the result of the problem without control input penalty [6]. Corresponding expressions for continuous-time systems are derived by reformulating the problem in terms of the delta operator [16]. Then, parallel discussions with the continuous-time case can be carried out for the discrete-time case, where we strengthen the expression for unstable plants presented in [7].

In general, our results show that the plant gain as well as the plant's nonminimum phase zeros and unstable poles impose inevitable limitation on the tracking performance, and they are confirmed by a numerical example. We also apply the continuous-time results to an inverted pendulum system and provide the closed-form expression of the best achievable tracking performances in terms of plant parameters. This enables us to discuss the selection of the optimal pendulum length which gives the smallest possible value of tracking performance without solving the Riccati equation or the LMI problem corresponding to the $\mathcal{H}_2$ optimal control problem.

The remainder of this paper is organized as follows. In Section 2, we describe the problem formulation including the description of the standard unity feedback control system and a brief explanation about the plant augmentation strategy. Section 3 provides analytical closed-form expressions of the optimal performance in the discrete-time case. Section 4 is devoted to the continuous-time results, where we apply the results to an inverted pendulum system to show the effectiveness of our result for the plant design. We conclude the paper in Section 5 with remarks on the sampled-data feedback case.

The notation used throughout this paper is fairly standard. We denote the real set by $\mathbb{R}$ and the complex set by $\mathbb{C}$.

FIGURE 1: Unity feedback control system.

For any $z \in \mathbb{C}$, its complex conjugate is denoted by $\bar{z}$. For any vector v we will use v^T , v^H , and $\|v\|$ as its transpose, conjugate transpose, and Euclidean norm, respectively. For any matrix $A \in \mathbb{C}^{m \times n}$, we denote its conjugate transpose by A^H and its column space by $\mathbb{R}[A]$. Several subsets in the complex plane are defined as follows: $\mathbb{C}_- := \{s \in \mathbb{C} : \text{Re } s < 0\}$, $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re } s > 0\}$, $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} : \text{Re } s \geq 0\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geq 1\}$, $\overline{\mathbb{D}}^c := \{z \in \mathbb{C} : |z| > 1\}$. We denote by $\mathbb{R}\mathcal{H}_\infty$ the set of all rational matrix functions which are bounded and analytic in $\mathbb{D}^c$. We define by $\hat{x}(z)$ the $\mathcal{Z}$ -transform of sequence $x(k)$. The cardinality of a set S is denoted by $\#S$.

2. PROBLEM FORMULATION

2.1. Feedback control systems

We consider the LTI unity feedback control system depicted in Figure 1, where P denotes a SIMO LTI plant to be controlled and K is a stabilizing controller. The plant P can be written as

$$P = (P_1, P_2, \dots, P_m)^T, \quad (1)$$

where P_i ($i = 1, \dots, m$) are scalar transfer functions. The signals $r \in \mathbb{R}^m$, $u \in \mathbb{R}$, $y \in \mathbb{R}^m$, and $e := r - y \in \mathbb{R}^m$ are the reference input, the control input, the measurement output, and the error signals, respectively. Hereafter, it will be assumed that all the vectors and matrices involved in the sequel have compatible dimensions.

The plant rational transfer function P admits right and left coprime factorizations

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad (2)$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$, and there exist $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$ that satisfy the double Bezout identity

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I. \quad (3)$$

The set of all stabilizing controllers K is then characterized by the Youla parameterization

$$\begin{aligned} \mathcal{K} &:= \{K : K = (Y - MQ)(NQ - X)^{-1} \\ &= (Q\tilde{N} - \tilde{X})^{-1}(\tilde{Y} - Q\tilde{M}); Q \in \mathbb{R}\mathcal{H}_\infty\}. \end{aligned} \quad (4)$$

We only focus on the discrete-time case in the remainder of this section and the next section, while the continuous-time case will be discussed in Section 4.

A number $\eta \in \mathbb{C}$ is said to be a zero of P if $P_i(\eta) = 0$ holds for some $i = 1, \dots, m$. In addition, if η lies in $\overline{\mathbb{D}}^c$, then η is

said to be a *nonminimum phase zero*. The plant P is said to be minimum phase if it has no nonminimum phase zeros; otherwise, it is said to be nonminimum phase. A number $\lambda \in \mathbb{C}$ is said to be a *pole* of P if $P(\lambda)$ is unbounded. If λ lies in $\mathbb{D}^c$, then λ is an *unstable pole* of P . We say P is stable if it has no unstable poles; otherwise, it is unstable. An equivalent statement for pole λ is that $\tilde{M}(\lambda)w = 0$ for some unitary vector w , and w is called a *pole direction* vector associated with λ . For technical reasons, it is assumed that the plant does not have nonminimum phase zeros and unstable poles at the same location.

A transfer function Θ , not necessarily square, is called an *inner* if Θ is in $\mathbb{RH}_\infty$ and $\Theta^*(z)\Theta(z) = I$ for all $z = e^{j\theta}$. Here we define $\Theta^*(z) := \Theta^T(z^{-1})$. A transfer function Φ is called an *outer* if Φ is in $\mathbb{RH}_\infty$ and has a right inverse which is analytic in $\mathbb{D}^c$. For $P \in \mathbb{RH}_\infty$,

$$P(z) = \Theta(z)\Phi(z), \quad (5)$$

where Θ is an inner and Φ is an outer, is called an *inner-outer factorization* of P . We call Θ the *inner factor* and Φ the *outer factor*.

2.2. $\mathcal{H}_2$ optimal tracking problem

The problem to be investigated in this paper is the standard $\mathcal{H}_2$ optimal tracking problem, where we consider a unit step function r as the reference input signal defined by

$$r(k) = \begin{cases} \nu, & k \geq 0 \\ 0, & k < 0 \end{cases}, \quad \hat{r}(z) = \frac{z\nu}{z-1} \quad (6)$$

with $\nu = (\nu_1, \nu_2, \dots, \nu_m)^T$ is a constant vector of unit length and specifies the direction of the reference input. An extension of this benchmark case can be undertaken by considering other typical classes of reference signals such as real sinusoids and ramp signals [10]. This result shows that nonminimum phase zeros and unstable poles maintain a significant role even in a more complex fashion and lead to a rather complicated expression of the optimal performance. Another extension may also dwell on the use of noncausal action for tracking such as preview control [17].

The performance index to be minimized in the present paper is given by

$$J_d := \sum_{k=0}^{\infty} (||e(k)||^2 + |u_w(k)|^2), \quad (7)$$

where u_w is the weighted control input, that is, $u_w(k) = \mathcal{Z}^{-1}\{W(z)\hat{u}(z)\}$, with proper, stable, and minimum phase weighting function $W(z)$. Note that, if $W = 0$, the problem then reduces to an $\mathcal{H}_2$ tracking error minimization problem (i.e., the $\mathcal{H}_2$ optimal tracking problem without control input penalty), which has been discussed in [6, 7].

It follows from the well-known Parseval's identity that (7) can be further written as

$$J_d = ||\hat{e}(z)||_2^2 + ||\hat{u}_w(z)||_2^2 \\ = ||S_o(z)\hat{r}(z)||_2^2 + ||W(z)K(z)S_o(z)\hat{r}(z)||_2^2, \quad (8)$$

where $S_o := (I + PK)^{-1}$ is the output sensitivity function. Using (2)–(4), the optimal performance can then be represented by

$$J_d^* = \inf_{Q \in \mathbb{RH}_\infty} \left\| \left\{ \begin{bmatrix} WY \\ X \end{bmatrix} - \begin{bmatrix} WM \\ N \end{bmatrix} Q \right\} \tilde{M}\hat{r} \right\|_2^2. \quad (9)$$

We make the following standard assumptions to guarantee the finiteness of J_d .

Assumption 1. $N(1) \neq 0$.

Assumption 2. For $r(k)$ in (6), $\nu \in \mathbb{R}[N(1)]$.

Assumption 3. $P(z)$ has a pole at $z = 1$.

In order for J_d to be finite, it is obvious that the output sensitivity function S_o must have a zero at $z = 1$ with input zero direction ν , that is, $S_o(1)\nu = 0$. The condition $N(1) \neq 0$ is then required to avoid any hidden pole-zero cancelation at $z = 1$ so that the open loop system has an integrator in the discrete-time sense. The condition $\nu \in \mathbb{R}[N(1)]$ requires that the input signal must enter from the direction lying in the column space of $N(1)$ and gives the condition of step reference signal r so that a nonright invertible plant P may track. In order to make the steady state error zero, the open-loop transfer function PK must contain an integrator. Consequently, plant P must have an integrator instead of compensator K , which should have no integrator to maintain a finite control energy cost. Assumption 3 is thus necessary.

2.3. Plant augmentation

To solve the tracking error problem under control penalty, we adopt the well-known plant augmentation approach. This key idea is initially implemented in solving the tracking performance problem in one of the authors' conference papers [11].

An augmented plant P_a is defined as

$$P_a := \begin{pmatrix} W \\ P \end{pmatrix}, \quad (10)$$

from which we obtain the corresponding step input signal $r_a := (0, r^T)^T$ with direction $\nu_a := (0, \nu^T)^T$ and the tracking measure

$$J_{da} := \sum_{k=0}^{\infty} ||e_a(k)||^2, \quad (11)$$

where

$$e_a := \begin{pmatrix} 0 \\ r \end{pmatrix} - \begin{pmatrix} u_w \\ y \end{pmatrix}. \quad (12)$$

One of the key points addressed by this strategy is that the tracking measure does not explicitly include the control input penalty u .

Furthermore, the corresponding right and left coprime factorizations of P_a are expressed as

$$P_a = N_a M_a^{-1} = \tilde{M}_a^{-1} \tilde{N}_a, \quad (13)$$

where

$$\begin{aligned} N_a &= \begin{pmatrix} WM \\ N \end{pmatrix}, & M_a &= M, \\ \tilde{M}_a &= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{M} \end{pmatrix}, & \tilde{N}_a &= \begin{pmatrix} W \\ \tilde{N} \end{pmatrix}, \end{aligned} \quad (14)$$

and the corresponding double Bezout identity is written as

$$\begin{pmatrix} \tilde{X}_a & -\tilde{Y}_a \\ -\tilde{N}_a & \tilde{M}_a \end{pmatrix} \begin{pmatrix} M_a & Y_a \\ N_a & X_a \end{pmatrix} = I, \quad (15)$$

where $Y_a = (0, Y)$, $\tilde{X}_a = \tilde{X}$, $\tilde{Y}_a = (0, \tilde{Y})$, and

$$X_a = \begin{pmatrix} 1 & WY \\ 0 & X \end{pmatrix}. \quad (16)$$

With a free parameter $Q_a = (Q_0, Q) \in \mathbb{R}\mathcal{H}_\infty$, where Q_0 is a new scalar free parameter, the optimal tracking performance J_{da}^* can be expressed as

$$J_{da}^* = \inf_{Q_a \in \mathbb{R}\mathcal{H}_\infty} \|(X_a - N_a Q_a) \tilde{M}_a \hat{r}_a\|_2^2, \quad (17)$$

and subsequently, we have

$$J_{da}^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left\{ \begin{bmatrix} WY \\ X \end{bmatrix} - \begin{bmatrix} WM \\ N \end{bmatrix} Q \right\} \tilde{M} \hat{r} \right\|_2^2. \quad (18)$$

The expression of J_{da}^* in (18) is exactly equivalent to that of J_d^* in (9) for the original plant P , that is, $J_{da}^* = J_d^*$ holds. By taking into account that there is no explicit penalty imposed on the control input for computing J_{da}^* in (18), we can immediately follow the approach of the tracking error problem in [6] to derive the analytical closed-form expression of J_d^* .

3. DISCRETE-TIME CASE

3.1. Closed-form expression

This section provides an analytical closed-form expression of the optimal tracking performance for the discrete-time case. The derivation is almost parallel to the continuous-time case [7, 11].

Note first that (17) can be expressed as

$$J_{da}^* = \inf_{Q_a \in \mathbb{R}\mathcal{H}_\infty} \left\| [I + N_a (\tilde{Y}_a - Q_a \tilde{M}_a)] \frac{\nu_a}{z-1} \right\|_2^2. \quad (19)$$

Write $N = (N_1, \dots, N_m)^T$ with N_i ($i = 0, 1, \dots, m$) being scalar transfer functions, and

$$N_a = (N_0, N)^T, \quad (20)$$

where $N_0 = WM$. We denote by $\lambda_k \in \mathbb{D}^c$ ($k = 1, \dots, n_\lambda$) the unstable poles of $P(z)$ and by $\eta_{ij} \in \mathbb{D}^c$ ($i = 1, \dots, m$, $j = 1, \dots, n_i$) the nonminimum phase zeros of $P_i(z)$.

We further define the following index sets:

$$\begin{aligned} \mathbb{J}_z &:= \{i : N_i(1) \neq 0\}, \\ \mathbb{J}_p &:= \{k : \tilde{M}(\lambda_k) \neq 0\}, \\ \mathbb{J}_{pi} &:= \{k : N_i(\lambda_k) = 0\} \quad (i = 0, 1, \dots, m). \end{aligned} \quad (21)$$

Note that $\mathbb{J}_p$ contains the index of unstable poles whose direction is coincident with that of the step input signal r . Due to the relation $N = PM$, $\mathbb{J}_{pi}$ contains the index of unstable poles of P but not those of P_i . The index set $\mathbb{J}_{pi}$ will play a key role of strengthening the expression for unstable plants in [7], where we explicitly identify the nonminimum phase zeros of N . To facilitate our derivation, we introduce the following inner-outer factorization of $N_a(z)$:

$$N_a(z) = \Theta(z)\Phi(z), \quad (22)$$

where Θ is an inner factor and Φ is an outer factor.

Theorem 1. Suppose that the SIMO plant $P(z)$ given in (1) has unstable poles λ_k ($k = 1, \dots, n_\lambda$) and $P_i(z)$ has nonminimum phase zeros η_{ij} ($i = 1, \dots, m$, $j = 1, \dots, n_i$). Then, under Assumptions 1–3, the optimal tracking performance J_d^* is given by

$$J_d^* = J_{ds} + J_{du}, \quad (23)$$

where

$$J_{ds} = J_{ds1} + J_{ds2} \quad (24)$$

with

$$\begin{aligned} J_{ds1} &:= \sum_{i \in \mathbb{J}_z} \nu_i^2 \sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2}, \\ J_{ds2} &:= \frac{1}{2\pi} \sum_{i \in \mathbb{J}_z} \nu_i^2 \int_0^\pi \log \left[\frac{|P_i(1)|^2}{\|P(1)\|^2} \times \frac{\|P(e^{j\theta})\|^2 + |W(e^{j\theta})|^2}{|P_i(e^{j\theta})|^2} \right] \\ &\quad \times \frac{d\theta}{1 - \cos \theta}, \\ J_{du} &= J_{du1} + J_{du2} \end{aligned} \quad (25)$$

with

$$\begin{aligned} J_{du1} &:= \sum_{i \in \mathbb{J}_z} \nu_i^2 \sum_{k \in \mathbb{J}_{pi}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2}, \\ J_{du2} &:= \sum_{k, \ell \in \mathbb{J}_p} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)}{\bar{h}_k h_\ell (\bar{\lambda}_k - 1)(\lambda_\ell - 1)(\bar{\lambda}_k \lambda_\ell - 1)} \\ &\quad \times (1 - \Theta^*(\bar{\lambda}_k)\Theta(1))(1 - \Theta^*(\lambda_\ell)\Theta(1)), \\ h_k &:= \begin{cases} 1; & \# \mathbb{J}_p = 1, \\ \prod_{\ell \in \mathbb{J}_p, \ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \bar{\lambda}_\ell \lambda_k}; & \# \mathbb{J}_p \geq 2. \end{cases} \end{aligned} \quad (26)$$

Proof. See the appendix for the proof. $\square$

Theorem 1 reveals that the optimal performance in tracking a step reference signal is explicitly characterized by the plant's nonminimum phase zeros η_{ij} and unstable poles λ_k , the plant direction [18] which is mostly determined by

the plant gain, and the reference input direction v . Furthermore, the problem of minimizing the tracking error under control input penalty generally provides additional limits imposed by W , which appears in the logarithmic term in J_{ds2} and the inner factor Θ in J_{du2} .

If we set $W = 0$ then we can easily obtain the nonpenalty result [6]. If the plant is marginally stable, we can see $J_d^* = J_{ds}$ (or, $J_{du} = 0$). In this theorem we also provide a stronger expression than that in [7] by examining explicitly additional effects caused by unstable poles λ_k in J_{cu1} . We defer the discussion of this part until the end of Section 4.2, while stating first a number of remarks and corollaries of Theorem 1.

We may also derive the optimal controller within this approach. For (marginally) stable case, for instance, we can make the term (A.9) in the appendix zero by selecting the optimal free parameter Q_a^* as

$$Q_a^*(z) = \left[\frac{\Theta^{\sim}(1)}{\Phi(z)} + \tilde{Y}_a(z) \right] \tilde{M}_a^{-1}(z). \quad (27)$$

The expression of the optimal controller can then be obtained by following the Youla parameterization (4). For unstable case, we need additional effort to simplify the expression of Q_a^* .

3.2. Remarks and corollaries

We here note that J_{ds1} and J_{du1} in Theorem 1 reflect the effects contributed by nonminimum phase zeros and unstable poles. Such zeros or poles close to the unit circle may or may not have a significant effect depending on the input direction v_i and those farther outside the unit circle provide less effect. An interesting fact may also be drawn concerning with the zero at infinity. Immediately we have $(|\eta_{ij}|^2 - 1)/|\eta_{ij} - 1|^2 \rightarrow 1$ as $\eta_{ij} \rightarrow \infty$, which reveals that in discrete-time system one zero at infinity contribute one penalty to the performance level. Therefore without loss of generality the effect of time delays can be treated as a number of zeros at infinity, instead of directly handled as a time delay system as in [8, 10], which provides a consistent result. The second term J_{ds2} concerns the effect due to the changes in the plant direction with frequency, (see [7, 18]).

Now we make a couple of remarks on the last term J_{du2} , in particular, on the existence of the inner factor $\Theta(z)$.

- (1) The expression in the theorem is complete for SIMO marginally stable plants in the sense that the best achievable tracking performance with control input penalty is characterized by nonminimum phase zeros and the gain of the plant without using any inner outer factors or solving any Riccati equation. (See Corollary 1 for the SISO case).
- (2) The expression for the general unstable case is not complete in that it includes an inner factor $\Theta(z)$ in the last term J_{du2} . We can only obtain a closed-form expression of $\Theta(z)$ for the SISO without control input penalty case. (see Corollary 2).
- (3) However, fortunately, there exists a special case where the term J_{du2} caused by unstable poles is zero even if the plant is unstable. (see Corollary 3).

- (4) We can also show that $J_{du} = J_{du1} + J_{du2}$ is zero when the sets of all unstable poles of $P_i(z)$ ($i = 1, \dots, m$) are completely the same as seen in Corollary 4. The case often happens for practical applications where we have only one actuator but we may add one or more extra sensors. The extra sensor can dramatically improve the tracking performance for unstable and nonminimum phase plants as seen in the example of an inverted pendulum in Section 4.3.

We now consider four specific cases for illustrating the implication of Theorem 1. The first case is the simplest case, where we consider a (marginally) stable scalar system.

Corollary 1. *Suppose that the plant $P(z)$ is SISO marginally stable and has nonminimum phase zeros η_i ($i = 1, \dots, n_\eta$). Under Assumptions 1 and 3, one has*

$$J_d^* = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \frac{1}{2\pi} \int_0^\pi \frac{\log[1 + |W(e^{j\theta})|^2 / |P(e^{j\theta})|^2]}{1 - \cos \theta} d\theta. \quad (28)$$

By Corollary 1, the effect of constant weighting function W can be understood easily, where a bigger W makes the control effort more costly and thus worsen the performance level. The performance can also be constrained by minimum phase zeros close to the unit circle, since in the vicinity of such zeros $|P(e^{j\theta})|$ is rather small. In addition, in the low-frequency range such zeros may have more negative effect due to the terms $1/(1 - \cos \theta)$. In some limited and simple cases, we can obtain the closed expression of the integral term. For example, if $P(z) = (z - \eta)/z(z - 1)$ ($0 < \eta < 1$) and $W = 1$, then

$$J_d^* = 1 + \frac{\eta - \sqrt{\eta^2 + 2\eta + 5}}{2(\eta - 1)}, \quad (29)$$

where the first term of the right-hand side is caused by a zero at infinity and the second one corresponds to the integral term. We can readily see from the second term that the minimum phase zero η also affects the performance limit and that it tends to infinity as η goes to one.

The second corollary is for the SISO without control input penalty case, that is, $W(z) = 0$. Suppose the plant has nonminimum phase zeros η_i ($i = 1, \dots, n_\eta$). Then the inner factor in (22) can, without loss of generality, be fixed as

$$\Theta(z) = \prod_{i=1}^{n_\eta} \frac{z - \eta_i}{1 - \bar{\eta}_i z}, \quad (30)$$

from which we get $\Theta(1) = 1$. Let $\phi(z) := \Theta^{\sim}(z)\Theta(1)$, that is,

$$\phi(z) = \prod_{i=1}^{n_\eta} \frac{1 - \eta_i z}{z - \bar{\eta}_i}. \quad (31)$$

Then the achievable performance for SISO without control input penalty case can be stated as follows.

Corollary 2. Consider the nonpenalty case, that is, $W(z) = 0$, for the SISO plant $P(z)$ which has nonminimum phase zeros η_i ($i = 1, \dots, n_\eta$) and unstable poles λ_k ($k = 1, \dots, n_\lambda$). Then, one has

$$J_d^* = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \sum_{k, \ell \in \mathbb{J}_p} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)}{\bar{h}_k h_\ell (\bar{\lambda}_k - 1)(\lambda_\ell - 1)(\bar{\lambda}_k \lambda_\ell - 1)} \times (1 - \phi(\bar{\lambda}_k))(1 - \phi(\lambda_\ell)). \quad (32)$$

The third corollary is for a special class of SIMO unstable systems, where the tracking performance limit is explicitly given in terms of plant characteristics.

Corollary 3. Let the SIMO plant P satisfy $P(1) = [P_1(1), 0, \dots, 0]^T$, and let the input signal r be given by (6) with $v = (1, 0, \dots, 0)^T$. Suppose that $P_1(z)$ is (marginally) stable and has nonminimum phase zeros η_{1j} ($j = 1, \dots, n_1$) and P has unstable poles λ_k ($k = 1, \dots, n_\lambda$). Then, one has

$$J_d^* = \sum_{j=1}^{n_1} \frac{|\eta_{1j}|^2 - 1}{|\eta_{1j} - 1|^2} + \sum_{k \in \mathbb{J}_{p1}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + \frac{1}{2\pi} \int_0^\pi \frac{\log[(|P(e^{j\theta})|^2 + |W(e^{j\theta})|^2) / |P_1(e^{j\theta})|^2]}{1 - \cos \theta} d\theta. \quad (33)$$

Proof. Since $v = (1, 0, \dots, 0)^T$, only the nonminimum phase zeros and the plant gain of $P_1(z)$ give effects. The unstable poles of $P(z)$ contributed by $P_i(z)$ ($i \geq 2$), if any, will not give effects through J_{du2} , that is, $J_{du2} = 0$, since the pole directions do not coincide with that of the input signal, that is $\tilde{M}(\lambda_k)v \neq 0$, but through J_{du1} as the nonminimum phase zeros of $N_1(z)$. $\square$

The last corollary deals with SIMO plant $P(z)$, in which the set of unstable poles of $P_i(z)$ ($i = 1, \dots, m$) are completely the same.

Corollary 4. Consider the SIMO plant $P(z)$ given in (1). Suppose that $P_i(z)$ has nonminimum phase zeros η_{ij} ($i = 1, \dots, m$, $j = 1, \dots, n_i$) and has unstable poles λ_k ($k = 1, \dots, n_\lambda$) for all $i = 1, \dots, m$, that is, the set of unstable poles of $P_i(z)$ are the same. Then, one has

$$J_d^* = J_{ds1} + J_{ds2}. \quad (34)$$

Proof. If the set of unstable poles of $P_i(z)$ ($i = 1, \dots, m$) are completely the same, then it is not difficult to verify that $\mathbb{J}_{pi}$ is empty for $i = 1, \dots, m$. Note that $\mathbb{J}_{p0}$ may not be empty, but this will not give any effect since the first element of v_a is zero. Furthermore, we know that $\mathbb{J}_p$ is also empty for the case, since $\tilde{M}(\lambda_k)v \neq 0$ for all $k = 1, \dots, n_\lambda$. These two facts make $J_{du1} = 0$ and $J_{du2} = 0$. Hence, $J_{du} = 0$. $\square$

FIGURE 2: J_d^* with respect to k and W .

3.3. Numerical example

Consider a single-input two-output discrete-time plant $P(z)$ given by

$$P(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \end{bmatrix} = \begin{bmatrix} \frac{z - \eta}{(z - 1)(1 - 2z)} \\ \frac{k(z - 1)}{z(z - \lambda)} \end{bmatrix}, \quad (35)$$

where $|\lambda| > 1$ and $k > 0$. Thus, $P_1(z)$ is (marginally) stable and has nonminimum phase zeros at $z = \infty$ and $z = \eta$ provided $|\eta| > 1$, and $P_2(z)$ has one nonminimum phase zero at $z = \infty$ and one unstable pole at $z = \lambda$.

We can easily see that coprime factors of $P(z)$ can be written as

$$N(z) = \begin{bmatrix} \frac{(z - \eta)(z - \lambda)}{z(1 - 2z)(1 - \lambda z)} \\ \frac{k(z - 1)^2}{z^2(1 - \lambda z)} \end{bmatrix}, \quad M(z) = \frac{(z - 1)(z - \lambda)}{z(1 - \lambda z)},$$

$$\tilde{M}(z) = \begin{bmatrix} \frac{z - 1}{z} & 0 \\ 0 & \frac{z - \lambda}{1 - \lambda z} \end{bmatrix}, \quad \tilde{N}(z) = \begin{bmatrix} \frac{z - \eta}{z(1 - 2z)} \\ \frac{k(z - 1)}{z(1 - \lambda z)} \end{bmatrix}. \quad (36)$$

We choose $v = (1, 0)^T \in \mathbb{R}[N(1)]$, which gives $\tilde{M}(\lambda)v \neq 0$. Therefore, this example is in the case of Corollary 3. We calculate the optimal tracking performance J_d^* based on the expression in the corollary. Note that the first element of $N(z)$, $N_1(z)$, includes a nonminimum phase zero at $z = \lambda$ which is an unstable pole of $P_2(s)$ in addition to the original zeros at $z = \infty$ and $z = \eta$. This extra zero in $N_1(z)$ yields the second term of J_d^* . For example, Figure 2 depicts the optimal tracking performance for $\eta = 2$ and $\lambda = 3$ with respect to

FIGURE 3: J_d^* with respect to η and W .

$k = \{1, 2, 4\}$ and W from 0 to 1.5. In this case, the optimal performance is given by

$$J_d^* = 6 + \frac{1}{2\pi} \int_0^\pi \frac{\log[1 + (|P_2(e^{j\theta})|^2 + W^2)/|P_1(e^{j\theta})|^2]}{1 - \cos\theta} d\theta, \quad (37)$$

where the first term of the right-hand side is calculated as

$$6 = \lim_{\eta_\infty \rightarrow \infty} \frac{\eta_\infty^2 - 1}{(\eta_\infty - 1)^2} + \frac{\eta^2 - 1}{(\eta - 1)^2} + \frac{\lambda^2 - 1}{(\lambda - 1)^2} \quad (38)$$

with $\eta = 2$ and $\lambda = 3$.

We can see from Figure 2 that bigger k increases the gain of $P_2(z)$ while larger W makes the control effort more expensive. Furthermore, Figure 3 plots the achievable performance for $k = 1$, $W = \{1, 2, 3\}$, and $\lambda = 3$ when η varies from 0.6 to 1.5. It is shown that the performance tends to infinity as η tends to 1 from either side. A minimum phase zero close to 1 makes $|P_1(e^{j\theta})|$ small, thus rendering the integral term large, and a nonminimum phase zero close to 1 blows the first term up.

4. CONTINUOUS-TIME CASE

4.1. Approach using δ -operator

This section is devoted to the continuous-time case and we derive an analytical closed-form expression of the minimal value of the performance index

$$J_c := \int_0^\infty (||e(t)||^2 + |u_w(t)|^2) dt \quad (39)$$

for the continuous-time plant $P(s)$, where $u_w(t) = \mathcal{L}^{-1}\{W(s)\hat{u}(s)\}$. To this end, we invoke the delta domain

expression by reformulating the discrete-time problem in terms of the delta operator [16], and we show that we can readily derive the continuous-time counterpart based on the discrete-time result obtained in the previous section.

For any sequence $x(k)$, $k = 1, 2, \dots$, the delta operator δ is defined by

$$\delta x(k) = \frac{x(k+1) - x(k)}{T}, \quad (40)$$

where $T > 0$ is the sampling time. By taking the $\mathcal{Z}$ -transform of the above equation we obtain

$$\delta \hat{x}(z) = \frac{z-1}{T} \hat{x}(z). \quad (41)$$

Hence, we obtain the relationship between the $\mathcal{Z}$ -transform variable z and the delta operator δ as

$$\delta = \frac{z-1}{T} \quad \text{or} \quad z = T\delta + 1. \quad (42)$$

Furthermore, for any sequence $x(k)$ we define its delta transform by

$$\mathcal{D}\{x(k)\} = \hat{x}_T(\delta) := T \sum_{k=0}^{\infty} x(k)(T\delta + 1)^{-k}, \quad (43)$$

or equivalently, $\hat{x}_T(\delta) = T\hat{x}(z)|_{z=T\delta+1}$.

The optimal performance J_c^* is then recovered by letting the sampling time T approach zero in the following performance index:

$$J_T := T \sum_{k=0}^{\infty} (||e(k)||^2 + |u_w(k)|^2), \quad (44)$$

where $u_w(k) = \mathcal{D}^{-1}\{W(\delta)\hat{u}_T(\delta)\}$. We here omit the derivation process since it can be done in a straightforward manner. For instance, it is immediate to derive the delta domain counterparts of Lemmas 1 and 2 in the appendix. The complete derivation is given in [19].

The delta domain expression, however, helps us to understand the dynamics of the solution with respect to the sampling time T . In addition, making T tend to zero will recover the continuous-time results in [7], which is rewritten in the following section.

4.2. Closed-form expression

For the finiteness of J_c we impose the following assumptions for $P(s)$ with coprime factorizations (2).

Assumption 4. $N(0) \neq 0$.

Assumption 5. $v \in \mathbb{R}[N(0)]$.

Assumption 6. $P(s)$ has a pole at $s = 0$.

We denote by $p_k \in \mathbb{C}_+$ ($k = 1, \dots, n_p$) the unstable poles of $P(s)$ and by $z_{ij} \in \mathbb{C}_+$ ($i = 1, \dots, m$, $j = 1, \dots, n_i$) the

nonminimum phase zeros of $P_i(s)$. We further introduce the following index sets:

$$\begin{aligned}\mathbb{I}_z &:= \{i : N_i(0) \neq 0\}, \\ \mathbb{I}_p &:= \{k : \widetilde{M}(p_k) \nu = 0\}, \\ \mathbb{I}_{pi} &:= \{k : N_i(p_k) = 0\} \quad (i = 0, 1, \dots, m).\end{aligned}\quad (45)$$

Define an inner-outer factorization of $N_a(s)$ such that

$$N_a(s) = \Theta(s)\Phi(s), \quad (46)$$

and subsequently define $\Theta^\sim(s) := \Theta^T(-s)$.

Theorem 2. Suppose that the SIMO plant $P(s)$ given in (1) has unstable poles p_k ($k = 1, \dots, n_p$) and $P_i(s)$ has nonminimum phase zeros z_{ij} ($i = 1, \dots, m$, $j = 1, \dots, n_i$). Then, under Assumptions 4–6, the optimal tracking performance J_c^* is given by

$$J_c^* = J_{cs} + J_{cu}, \quad (47)$$

where

$$J_{cs} = J_{cs1} + J_{cs2} \quad (48)$$

with

$$\begin{aligned}J_{cs1} &:= \sum_{i \in \mathbb{I}_z} \nu_i^2 \sum_{j=1}^{n_i} \frac{2 \operatorname{Re} z_{ij}}{|z_{ij}|^2}, \\ J_{cs2} &:= \frac{1}{\pi} \sum_{i \in \mathbb{I}_z} \nu_i^2 \int_0^\infty \log \left[\frac{|P_i(0)|^2}{|P(0)|^2} \times \frac{\|P(j\omega)\|^2 + |W(j\omega)|^2}{|P_i(j\omega)|^2} \right] \\ &\quad \times \frac{d\omega}{\omega^2}, \\ J_{cu} &= J_{cu1} + J_{cu2}\end{aligned}\quad (49)$$

with

$$\begin{aligned}J_{cu1} &:= \sum_{i \in \mathbb{I}_z} \nu_i^2 \sum_{k \in \mathbb{I}_{pi}} \frac{2 \operatorname{Re} p_k}{|p_k|^2}, \\ J_{cu2} &:= \sum_{k, \ell \in \mathbb{I}_p} \frac{4 \operatorname{Re} p_k \operatorname{Re} p_\ell}{(\bar{p}_k + p_\ell) \bar{p}_k p_\ell \bar{\sigma}_k \sigma_\ell} \\ &\quad \times (1 - \Theta^\sim(\bar{p}_k) \Theta(0)) (1 - \Theta^\sim(p_\ell) \Theta(0)), \\ \sigma_k &:= \begin{cases} 1; & \# \mathbb{I}_p = 1, \\ \prod_{\ell \in \mathbb{I}_p, \ell \neq k} \frac{p_\ell - p_k}{\bar{p}_\ell + p_k}; & \# \mathbb{I}_p \geq 2. \end{cases}\end{aligned}\quad (50)$$

In the case of $W = 0$, that is, the problem without control input penalty, the expressions in Theorem 2 validate those in [7, Theorem 3.3] in a more explicit form. In the above theorem we explicitly spell out the effect of the nonminimum phase zeros of N into two terms J_{cs1} and J_{cu1} , while [7, Theorem 3.3] writes it out as a single term J_s^* , the optimal performance with respect to the stable part of the plant P_s . In our

FIGURE 4: The inverted pendulum system.

version, J_{cs1} accounts for the effects contributed by nonminimum phase zeros z_{ij} ($i \in \mathbb{I}_z$) of the plant P and J_{cu1} identifies those affected by unstable poles p_k ($k \in \mathbb{I}_{pi}$). Then it is revealed that unstable poles may also deteriorate the achievable performance in the same way as nonminimum phase zeros. This fact, however, is not elucidated in [7].

4.3. Application to inverted pendulum system

As an illustration of our results, we consider the inverted pendulum system shown in Figure 4, where an inverted pendulum is mounted on a motor-driven cart. We assume that the pendulum moves only in the vertical plane, that is, a two dimensional control problem is considered. Let M , m , and $2l$ denote the mass of the cart, the mass of the pendulum, and the length of the pendulum, respectively. Also we assume that the friction between the track and the cart is μ_t and that between the pendulum and the cart is μ_p . We consider a uniform pendulum so that its inertia is given by $I = (1/3)ml^2$.

The equations of motion between the control input u , which is the force to the cart, the position of the cart x , the angle of the pendulum θ are represented by

$$\begin{aligned}\frac{4}{3}ml^2\ddot{\theta} + \mu_p\dot{\theta} - mgl\theta &= -ml\ddot{x}, \\ (M + m)\ddot{x} + \mu_t\dot{x} - ml\ddot{\theta} &= u,\end{aligned}\quad (51)$$

under the assumption that the angle θ is small. Then, the transfer functions from u to x (denoted by P_x) and from u to θ (denoted by P_θ) are, respectively, given by

$$P_x(s) = \frac{(4/3)ml^2s^2 + \mu_p s - mgl}{sD(s)}, \quad P_\theta(s) = \frac{-mls}{D(s)}, \quad (52)$$

where $D(s) := a_3s^3 + a_2s^2 + a_1s + a_0$ with $a_3 := (1/3)(4M + m)ml^2$, $a_2 := (M + m)\mu_p + (4/3)\mu_t ml^2$, $a_1 := -(M + m)mgl + \mu_p\mu_t$, and $a_0 := -\mu_t mgl$.

Note that both of the transfer functions have a common unstable pole p and $P_x(s)$ has one nonminimum phase zero at

$$z = \frac{-3\mu_p + \sqrt{9\mu_p^2 + 48m^2g l^3}}{8ml^2}. \quad (53)$$

Now we derive analytical closed-form expressions for the best achievable tracking error performance J_c^* for two different cases, namely, the SISO case (Case 1): $P(s) = P_x(s)$, and the SITO case (Case 2): $P(s) = [P_x(s), P_\theta(s)]^T$. We assume $W(s) = 1$ for simplicity and define the following functions:

$$\begin{aligned} f_1(\omega) &:= \omega^2[(a_0 - a_2\omega^2)^2 + (a_1\omega - a_3\omega^3)^2], \\ f_2(\omega) &:= m^2 l^2 \left(\frac{4}{3} l \omega^2 + g \right)^2 + \mu_p^2 \omega^2. \end{aligned} \quad (54)$$

Case 1. The control objective in this case is to stabilize the system by measuring the cart position x only. The optimal tracking performance is given by

$$\begin{aligned} J_{c1}^* &= J_z + \frac{1}{\pi} \int_0^\infty \log \left[1 + \frac{f_1(\omega)}{f_2(\omega)} \right] \frac{d\omega}{\omega^2} \\ &\quad + \frac{2[1 - \Theta^*(p)\Theta(0)]^2}{p}, \end{aligned} \quad (55)$$

where p is the unstable pole of $P_x(s)$ and

$$J_z = \frac{16ml^2}{-3\mu_p + \sqrt{9\mu_p^2 + 48m^2gl^3}}. \quad (56)$$

Note that, in (55), $\Theta(s)$ is determined from the following inner-outer factorization:

$$\begin{bmatrix} M_x(s) \\ N_x(s) \end{bmatrix} = \Theta(s)\Phi(s), \quad (57)$$

where $P_x(s) = N_x(s)M_x(s)^{-1}$.

Figure 5 illustrates the situation in Case 1. Here we assume that the ratio between the mass and the length of the pendulum is constant, that is, $m/l = 2.145\pi$ [kg/m] and $M = 2$ [kg]. We also assume that $\mu_f = 1/4$ and $\mu_p = 1/3$. We compute the optimal tracking performance for l from 1 to 4 [m] by using (55), that is, Theorem 2. We can see from the figure that the values of J_{c1}^* are fairly large which are caused by the third term of J_{c1}^* . Although the optimal length of the pendulum which gives the lowest possible value of J_{c1}^* is around 2.8 [m], it is implied that the stabilization by a single sensor of the cart position is very hard in practice.

One of the reasonable practical solutions is to add an extra sensor which measures the pendulum angle θ . Indeed, the continuous-time counterpart of Corollary 4 can be applied to the case and a reasonable result can be obtained as seen below.

Case 2. We will control both the cart position x and the pendulum angle θ by measuring those two quantities. Note that $P_\theta(0) = 0$, which leads to the step input direction $v = (1, 0)^T$. Thanks to the shared unstable pole in both $P_x(s)$ and $P_\theta(s)$, the optimal tracking error performance is simply represented by

$$J_{c2}^* = J_z + \frac{1}{\pi} \int_0^\infty \log \left[1 + \frac{f_1(\omega) + m^2 l^2 \omega^4}{f_2(\omega)} \right] \frac{d\omega}{\omega^2}. \quad (58)$$

The values of optimal costs J_{c2}^* for Case 2 are quite small in comparison with J_{c1}^* as seen in Figure 6. The main reason

FIGURE 5: The optimal tracking performance of Case 1.

FIGURE 6: The optimal tracking performance of Case 2.

is that the unstable pole p does not give any effect, that is, $J_{cu}^* = 0$, since p is a common pole of $P_x(s)$ and $P_\theta(s)$, and its direction is not coincident with that of step input signal, that is, $\tilde{M}(p)v \neq 0$.

Although the expression (58) is rather complicated, it only includes the plant parameters. Hence, we can directly see the relation between the best achievable control performance and the plant parameters without solving the corresponding Riccati equation or the LMI problem. This is one of the advantages of deriving the closed-form expression. For example, we can see that J_z is a monotone increasing function of the length l , while the second term in (58) tends to infinity when the length l goes to zero. These clearly indicate that there exists an optimal length l which minimizes the $\mathcal{H}_2$ cost (58). In fact, Figure 6 shows that the best length of the pendulum is about 0.2 [m] for this case.

FIGURE 7: Sampled-data feedback control system.

FIGURE 8: Approximation of the sampled-data feedback control system.

5. CONCLUSION

We have examined the $\mathcal{H}_2$ tracking performance problem for SIMO LTI feedback control systems, where the tracking performance is quantified by the error response with control input penalty. We first provided analytical closed-form expressions of the best achievable tracking performance for discrete-time systems by introducing the idea of plant augmentation. We have addressed several cases where we can have complete solutions for the $\mathcal{H}_2$ performance limits. We next discussed the continuous-time case by exploiting the delta domain expression based on the discrete-time result and derived the continuous-time counterpart. We then applied the result to an inverted pendulum system to confirm the effectiveness of the closed-form expression in plant design.

Before closing the paper, it should be noted that the idea of plant augmentation proposed in this paper can be applied to the performance limitation for a single-input single-output sampled-data feedback control system depicted in Figure 7, where $P_c(s)$ represents the continuous-time plant and $K_d(z)$ the discrete-time stabilizing controller. Note that e_k and u_k represent digital signals related to $e(t)$ and $u(t)$ by the sampler S and the zero-order hold H with sampling time T . We here evaluate the tracking measure in the continuous-time setting rather than the discrete-time setting. In other words, we take the inter-sample behavior into account to evaluate the tracking performance. The study on performance limitations has been done for years, for example, see [20, 21].

Under the fast sampling procedure, a fast sampler with sampling time T/N is embedded at the reference input and the plant output, from which we subdivide the k th sampling interval $[kT, (k+1)T)$ into N subintervals $[kT + (i/N)T, kT + (i+1/N)T)$, $i = 0, 1, \dots, N-1$. Then the feedback control setup of Figure 7 can be approximated by that of Figure 8. Then, we can use the idea of plant augmentation and derive the corresponding expression to approximate the optimal tracking performance of the sampled-data system. (see [19] for the closed-form expression of the approximated performance limit and a numerical example).

APPENDIX

PROOF OF THEOREM 1

We introduce two lemmas which play important roles in the proof of Theorem 1. These lemmas serve as the discrete-time counterparts of [7, Lemmas 1 and 2]. The proofs are immediate by application of bilinear transformation. Consider the class of functions in

$$\mathbb{F} := \left\{ f : \lim_{R \rightarrow \infty} \max_{\theta \in [-\pi/2, \pi/2]} \frac{|f(\operatorname{Re}^{j\theta})|}{R} = 0 \right\}. \quad (\text{A.1})$$

The above class consists of functions with restricted behavior at infinity. By this, we intend to deal with integration over a contour that becomes arbitrarily long. Generally speaking, if f is analytic and bounded magnitude in $\mathbb{D}^c$, then f is of class $\mathbb{F}$.

Lemma 1. Let $f(z) \in \mathbb{F}$ and analytic in $\mathbb{D}^c$. Denote that $f(e^{j\theta}) = f_1(\theta) + j f_2(\theta)$. Suppose that $f(z)$ is conjugate symmetric, that is, $f(z) = \overline{f(\bar{z})}$. Then

$$f'(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(\theta) - f_1(0)}{1 - \cos \theta} d\theta. \quad (\text{A.2})$$

Lemma 2. Let $f(z)$ be a meromorphic function in $\mathbb{D}^c$ and has no zero or pole on unit circle. Suppose that $f(z)$ is conjugate symmetric and $\log f(z) \in \mathbb{F}$. Also, suppose that $\eta_i \in \mathbb{D}^c$ ($i = 1, \dots, n_\eta$) and $\lambda_k \in \mathbb{D}^c$ ($k = 1, \dots, n_\lambda$) are, respectively, the zeros and the poles of $f(z)$, all counting multiplicities. Provided that $f(1) \neq 0$, then

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{f(e^{j\theta})}{f(1)} \right| \frac{d\theta}{1 - \cos \theta} \\ &= \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} - \sum_{k=1}^{n_\lambda} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + \frac{f'(1)}{f(1)}. \end{aligned} \quad (\text{A.3})$$

We are now ready to prove Theorem 1. Define

$$\Psi(z) := \begin{bmatrix} \Theta^\sim(z) \\ I - \Theta(z)\Theta^\sim(z) \end{bmatrix}. \quad (\text{A.4})$$

It is easy to show that $\Psi(z)$ is a norm-preserving function, that is, $\Psi^\sim(e^{j\theta})\Psi(e^{j\theta}) = I$. By premultiplying Ψ to (19), we get

$$\begin{aligned} J_d^* &:= \inf_{Q_a \in \mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} \Theta^\sim + \Phi(\tilde{Y}_a - Q_a \tilde{M}_a) \\ z - 1 \end{bmatrix} v_a \right\|_2^2 \\ &+ \left\| \frac{(I - \Theta\Theta^\sim)v_a}{z - 1} \right\|_2^2. \end{aligned} \quad (\text{A.5})$$

Noting that

$$\frac{(\Theta^\sim - \Theta^\sim(1))v_a}{z - 1} \in \mathcal{H}_2^\perp, \quad (\text{A.6})$$

we may select $Q_a \in \mathbb{R}\mathcal{H}_\infty$ such that $\Theta^\sim(1) - \Phi(1)(\tilde{Y}_a(1) + Q_a(1)\tilde{M}_a(1)) = 0$, and therefore

$$\frac{[\Theta^\sim(1) + \Phi(\tilde{Y}_a - Q_a \tilde{M}_a)]v_a}{z - 1} \in \mathcal{H}_2. \quad (\text{A.7})$$

As a result, we can write $J_d^* = J_1 + J_2$, where

$$J_1 := \left\| \frac{(\Theta^\sim - \Theta^\sim(1))\nu_a}{z-1} \right\|_2^2 + \left\| \frac{(I - \Theta\Theta^\sim)\nu_a}{z-1} \right\|_2^2, \quad (\text{A.8})$$

$$J_2 := \inf_{Q_a \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{[\Theta^\sim(1) + \Phi(\tilde{Y}_a - Q_a\tilde{M}_a)]\nu_a}{z-1} \right\|_2^2. \quad (\text{A.9})$$

Next we will show that $J_1 = J_{ds1}^* + J_{ds2}^* + J_{du1}$ and $J_2 = J_{du2}$. A direct calculation leads to

$$J_1 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\text{Re}(\nu_a^H \Theta(e^{j\theta}) \Theta^\sim(1) \nu_a) - 1}{1 - \cos \theta} d\theta. \quad (\text{A.10})$$

Let $f(z) := \nu_a^H \Theta(z) \Theta^\sim(1) \nu_a$. Under Assumption 2, we obtain $f(1) = 1$. Applying Lemma 1 yields $J_1 = -f'(1) = -\nu_a^H \Theta(1) \Theta^\sim(1) \nu_a$. Denote the inner factor $\Theta(z)$ in (22) as $\Theta(z) = [w_0(z), w_1(z), \dots, w_m(z)]^T$. According to Assumption 2, we may select $\nu_a = \Theta(1)$ without loss of generality. Noting that the first element of ν_a is zero, we have

$$J_1 = -\sum_{i=0}^m w_i(1)w_i'(1) = -\sum_{i \in \mathbb{J}_z} \nu_i^2 \frac{w_i'(1)}{w_i(1)}. \quad (\text{A.11})$$

Condition $i \in \mathbb{J}_z$ guarantees that $w_i(1) \neq 0$. Since $w_i(z)$ is an element of the inner factor $\Theta(z)$, it has the same set of non-minimum phase zeros as $N_i(z)$, which includes the set of unstable poles of P but not those of P_i as well as the set of non-minimum phase zeros of $P_i(z)$. Hence, by invoking Lemma 2, we have¹

$$\begin{aligned} \frac{w_i'(1)}{w_i(1)} &= -\sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2} - \sum_{k \in \mathbb{J}_{pi}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} \\ &\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{w_i(e^{j\theta})}{w_i(1)} \right| \frac{d\theta}{1 - \cos \theta}. \end{aligned} \quad (\text{A.12})$$

Noting here that $|w_i(e^{j\theta})| = |P_i(e^{j\theta})|/\|P_a(e^{j\theta})\|$, we obtain

$$\log \left| \frac{w_i(e^{j\theta})}{w_i(1)} \right| = -\frac{1}{2} \log \left[\frac{|P_i(1)|^2 \|P_a(e^{j\theta})\|^2}{\|P_a(1)\|^2 |P_i(e^{j\theta})|^2} \right]. \quad (\text{A.13})$$

We can see from Assumption 3 that $|P_i(1)|$ and $\|P(1)\|$ are infinite but $|W(1)|$ is finite, and therefore

$$\frac{|P_i(1)|^2}{\|P_a(1)\|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2 + |W(1)|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2} \quad (\text{A.14})$$

holds. Also note that

$$\frac{\|P_a(e^{j\theta})\|^2}{|P_i(e^{j\theta})|^2} = \frac{\|P(e^{j\theta})\|^2 + |W(e^{j\theta})|^2}{|P_i(e^{j\theta})|^2}. \quad (\text{A.15})$$

This completes the proof of $J_1 = J_{ds1}^* + J_{ds2}^* + J_{du1}$. Next, by factorizing $\tilde{M}_a(z)\nu_a = g_m(z)h(z)$, where $g_m(z)$ is left invertible in $\mathbb{R}\mathcal{H}_\infty$ and $h(z)$ is defined by

$$h(z) = \prod_{k \in \mathbb{J}_p} \frac{z - \lambda_k}{1 - \bar{\lambda}_k z}, \quad (\text{A.16})$$

we can show that $J_2 = J_{du2}$ by following the standard partial fraction expansion using in the proof of [7, Theorem 3.3].

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¹ The second term in the right-hand side is not explicitly examined in the expression in [7].

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