Research Article

Transferring Instantly the State of Higher-Order Linear Descriptor (Regular) Differential Systems Using Impulsive Inputs

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In many applications, and generally speaking in many dynamical differential systems, the problem of transferring the initial state of the system to a desired state in (almost) zero-time time is desirable but difficult to achieve. Theoretically, this can be achieved by using a linear combination of Dirac δ -function and its derivatives. Obviously, such an input is physically unrealizable. However, we can think of it approximately as a combination of small pulses of very high magnitude and infinitely small duration. In this paper, the approximation process of the distributional behaviour of higher-order linear descriptor (regular) differential systems is presented. Thus, new analytical formulae based on linear algebra methods and generalized inverses theory are provided. Our approach is quite general and some significant conditions are derived. Finally, a numerical example is presented and discussed.

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1. Introduction

From the point of view of several important applications, in several fields of research, see for instance [1, 2], taking a given state of a linear system to a desired state in minimum time is very desirable, though it is a challenging problem in control and system theory.

Significant attention has been given to this problem in the case of linear systems; see [1–3]. Recently, Kalogeropoulos et al. [4] have further enriched these first approaches, as they have relaxed some of the rather restricted assumptions that [1, 2] are considered. Afterwards, the method has been also applied to the more general class of linear descriptor (regular) systems; see [5].

In this paper, a further extension of [5], in the class of linear descriptor (regular) equations, is provided. Comparing with the existing literature, see [1–5], we solve this problem

- (1) for higher-order linear differential descriptor (regular) systems (compare with [5]),
- (2) using higher-order consistent initial conditions (compare with [1–5]),

- (3) obtaining more analytical formulas, that is, see Appendices A and B, Theorems 20 and 23. (compare with [1–5]),
- (4) without using the controllability matrix (compare with [3]),
- (5) applying analytical methods for the exact determination of the generalized inverses of Vandermonde and relative to this matrices; see [6].

To summarize, in this paper, we investigate how we can transfer the initial state of an open loop, linear higher-order descriptor (regular) differential system in (practically speaking, almost) zero-time, that is,

$$F\dot{\underline{x}}^{(r)}(t) = G\underline{x}(t) + \underline{b}u(t) \quad (1)$$

with known initial conditions

$$\underline{x}(t_0), \underline{x}'(t_0), \dots, \underline{x}^{(r-1)}(t_0), \quad (2)$$

where $F, G \in \mathcal{M}(n \times n; \mathbb{F})$, and $\underline{b} \in \mathcal{M}(n \times 1; \mathbb{F})$ (i.e., $\mathcal{M}$ is the algebra of $n \times m$ matrices with elements in the field $\mathbb{F}$ with $\det M = 0$ (0 is the zero element of $\mathcal{M}(n = 1, \mathbb{F})$), $\underline{x}(t) \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(n \times 1; \mathbb{F}))$, and $u(t) \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(1; \mathbb{F}))$. For the sake of simplicity, we set in the sequel $\mathcal{M}_n = \mathcal{M}(n \times n; \mathbb{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathbb{F})$.

In order to solve this problem, the appropriate input vector has to be made up as a linear combination of the δ -function of Dirac and its derivatives, see [1, 2], and for more details consult [7]. Obviously, such an input is very hard to imagine physically. However, we can think of it approximately as a combination of small pulses of very high magnitude and infinitely small duration.

Linear descriptor (singular or regular) differential systems have been extensively used in control theory; see for instance [8–10], and for more details [11].

A brief outline of the paper is as follows. Section 2 provides the incentives and the typical modelling features of the problem. Moreover, a classical approximated expression for the controller, that is, a linear combination of the δ -function of Dirac and its derivatives that is based on the normal (Gaussian) probability density function, is used. Then, the need to determine the unknown coefficients is derived. Section 3 is divided into four extensive subsections. In Section 3.1, the reduction of the higher-order system to first-order is discussed. The first-order descriptor (regular) is divided into a fast and a slow subsystem, using the Weierstrass canonical form. Section 3.2 investigates and presents some analytical formulas based on the slow subsystem. In Section 3.3, the theory of the $\{1\}$ -generalized inverses is used. Finally, some significant conditions for the solution of the slow subsystem are presented in Section 3.4. In Section 4, a necessary condition based on the fast subsystem is discussed and obtained. Section 5 provides an interesting numerical application from physics and Section 6 concludes the paper. Two appendices for the analytical calculations of two important integrals are also available.

2. Preliminary Results—Matrix Pencil Framework

In this section, some preliminary results for matrix pencil and system theory are briefly presented. First, we assume that for an n th-order linear differential system, see (1), the input can be a linear combination of Dirac δ -function and its first $(n - 1)$ th-derivatives as follows:

$$\begin{aligned} u_o(t) &= a_0 \delta(t) + a_1 \delta^{(1)}(t) + a_2 \delta^{(2)}(t) + \cdots + a_{n-1} \delta^{(n-1)}(t) \\ &= \sum_{k=1}^{n-1} a_k \delta^{(k)}(t), \end{aligned} \quad (3)$$

where $\delta^{(k)}(t)$ or $d^k \delta(t)/dt^k$ is the k th-derivative of the Dirac δ -function, and a_i for $i = 0, 1, \dots, n - 1$ are the magnitudes of the delta function and its derivatives. Furthermore, we assume that the state of the system at time 0^- is

$$\underline{x}(0^-) = \underline{x}'(0^-) = \cdots = \underline{x}^{(r-1)}(0^-) = [0 \ 0 \ \cdots \ 0]^t, \quad (4)$$

and at time 0^+ , it achieves

$$\begin{aligned} \underline{x}(0^+) &= [x_1^0 \ x_2^0 \ \cdots \ x_n^0]^t, \quad \underline{x}'(0^+) = [x_1^1 \ x_2^1 \ \cdots \ x_n^1]^t \\ &\dots, \underline{x}^{(r-1)}(0^+) = [x_1^{r-1} \ x_2^{r-1} \ \cdots \ x_n^{r-1}]^t. \end{aligned} \quad (5)$$

With the following definitions, a brief presentation of the most important elements of matrix pencil theory is given.

Definition 1. Given $F, G \in \mathcal{M}_{nm}$ and an indeterminate $s \in \mathbb{F}$, the matrix pencil $sF - G$ is called regular when $m = n$ and $\det(sF - G) \neq 0$. In any other case, the pencil will be called singular.

Definition 2. The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}_m$ and $Q \in \mathcal{M}_n$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}. \quad (6)$$

In this paper, we consider the case that pencil is *regular*. Thus, the strict equivalence relation can be defined rigorously on the set of regular pencils as follows.

This is the set of *elementary divisors* (e.d.s) obtained by factorizing the invariant polynomials $f_i(s, \hat{s})$ into powers of homogeneous polynomials irreducible over field $\mathbb{F}$.

In the case where $sF - G$ is a regular, we have e.d. of the following types:

- (i) e.d. of the type s^p are called *zero finite elementary divisors* (**z.f.e.d.**),
- (ii) e.d. of the type $(s - a)^\pi$, $a \neq 0$ are called *nonzero finite elementary divisors* (**nz.f.e.d.**),
- (iii) e.d. of the type $\hat{s}^q$ are called *infinite elementary divisors* (**i.e.d.**).

Let $B_1, B_2, \dots, B_n$ be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus \cdots \oplus B_n$ is the *block diag* $\{B_1, B_2, \dots, B_n\}$.

Then, the complex Weierstrass form $sF_w - Q_w$ of the regular pencil $sF - G$ is defined by

$$sF_w - Q_w \triangleq sI_p - J_p \bigoplus sH_q - I_q. \quad (7)$$

Now, the Jordan type element, that is, J_p , is uniquely defined by the set of f.e.d.

$$(s - a_1)^{p_1}, \dots, (s - a_v)^{p_v}, \sum_{j=1}^v p_j = p, \quad (8)$$

of $sF - G$ and has the form

$$sI_p - J_p \triangleq sI_{p_1} - J_{p_1}(a_1) \bigoplus \cdots \bigoplus sI_{p_v} - J_{p_v}(a_v), \quad (9)$$

where

$$\begin{aligned} I_p &\triangleq I_{p_1} \bigoplus I_{p_2} \bigoplus \cdots \bigoplus I_{p_v}, \\ J_p &\triangleq J_{p_1}(a_1) \bigoplus J_{p_2}(a_2) \bigoplus \cdots \bigoplus J_{p_v}(a_v). \end{aligned} \quad (10)$$

The q blocks of the second, that is, $sH_q - I_q$, see (7), are uniquely defined by the set of i.e.d.

$$\hat{s}^{q_1}, \dots, \hat{s}^{q_\sigma}, \sum_{j=1}^{\sigma} q_j = q \quad (11)$$

of $sF - G$ and has the form

$$sH_q - I_q \triangleq sH_{q_1} - I_{q_1} \oplus \dots \oplus sH_{q_\sigma} - I_{q_\sigma}, \quad (12)$$

where

$$\begin{aligned} I_q &\triangleq I_{q_1} \oplus I_{q_2} \oplus \dots \oplus I_{q_\sigma}, \\ H_q &\triangleq H_{q_1} \oplus H_{q_2} \oplus \dots \oplus H_{q_\sigma}. \end{aligned} \quad (13)$$

Furthermore, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \dots, \sigma\}$, where

$$H_q^{\tilde{q}} = \mathbb{O}, \quad (14)$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$\begin{aligned} I_{p_j} &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathcal{M}_{p_j}, \\ J_{p_j}(a_j) &= \begin{bmatrix} a_j & 1 & 0 & \dots & 0 \\ 0 & a_j & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j}, \\ H_{q_j} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}. \end{aligned} \quad (15)$$

Moreover, for the matrices F and G , we have the parameterization

$$PFQ = F_w = I_p \oplus H_q, \quad (16)$$

$$PGQ = G_w = J_p \oplus I_q. \quad (17)$$

Since the state $\underline{x}(0^+)$ which we wish to reach is specified, we need to determine the n unknown coefficients a_i , for $i = 0, 1, \dots, n-1$.

In practice, we cannot create an exact impulse function nor its derivatives. However, if we use one of the approximations of Dirac δ -function, we will be able to change the state in some minimum practical time depending mainly upon

how well we generate the approximations. Let the Dirac δ -function be viewed as the limit of sequence function

$$\delta(t) = \lim_{a \rightarrow 0} \delta_a(t), \quad (18)$$

where $\delta_a(t)$ is called a *nascent* delta function. This limit is in the sense that

$$\lim_{a \rightarrow 0} \int_{-\infty}^{+\infty} \delta_a(t) f(t) dt = f(0). \quad (19)$$

Some wellknown and very useful in applications *nascent* delta functions are the Normal and Cauchy distributions, the rectangular function, the derivative of the sigmoid (or Fermi-Dirac) function, the Airy function, and so forth; see for instance [2, 5, 12–17]. The results given below are based on the normal function. Thus, by taking into consideration expression (18) and the normal (Gaussian) probability density distribution, we obtain

$$\delta(t) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma \sqrt{2\pi}} e^{-t^2/2\sigma^2} = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \Phi\left(\frac{t}{\sigma}\right), \quad (20)$$

where $\Phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.

So, the approximate expression for the controller (2) is given by

$$\begin{aligned} u(t) &= \frac{1}{\sigma} \Phi\left(\frac{t}{\sigma}\right) a_0 + \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{t}{\sigma}\right) a_1 \\ &\quad + \dots + \frac{1}{\sigma^n} \Phi^{(n-1)}\left(\frac{t}{\sigma}\right) a_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{t}{\sigma}\right) a_k. \end{aligned} \quad (21)$$

Then, we take the limit

$$u_o(t) = \lim_{\sigma \rightarrow 0} u(t). \quad (22)$$

In the next section, the main results are presented.

3. The Main Results

3.1. Order Reduction of System (1). The section begins with the following important lemma.

Lemma 3. *System (1) is divided into the following two subsystems:*

$$\underline{y}_{-p}^{(r)}(t) = J_p \underline{y}_{-p}(t) + \tilde{\underline{b}}_p u_o(t), \quad (23)$$

$$H_q \underline{y}_{-q}^{(r)}(t) = \underline{y}_{-q}(t) + \tilde{\underline{b}}_q u_o(t). \quad (24)$$

Proof. Consider the transformation

$$\underline{x}(t) = Q \underline{y}(t). \quad (25)$$

Substituting the previous expression into (1) and considering also (3), we obtain

$$FQ \underline{y}^{(r)}(t) = GQ \underline{y}(t) + \underline{b} u_o(t). \quad (26)$$

Multiplying by P , we arrive at

$$F_w \underline{y}^{(r)}(t) = G_w \underline{y}(t) + P \underline{b} u_o(t). \quad (27)$$

Now, we denote

$$\begin{aligned} \underline{y}(t) &= \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix}, \\ P \underline{b} &\triangleq \tilde{\underline{b}} = \begin{bmatrix} \tilde{\underline{b}}_p \\ \tilde{\underline{b}}_q \end{bmatrix}, \end{aligned} \quad (28)$$

where $p + q = n$.

Taking into account the following expressions, that is,

$$\begin{aligned} &\begin{bmatrix} I_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & H_q \end{bmatrix} \begin{bmatrix} \underline{y}_p^{(r)}(t) \\ \underline{y}_q^{(r)}(t) \end{bmatrix} \\ &= \begin{bmatrix} J_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & I_q \end{bmatrix} \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} + \begin{bmatrix} \tilde{\underline{b}}_p \\ \tilde{\underline{b}}_q \end{bmatrix} u_o(t), \end{aligned} \quad (29)$$

we arrive easily at (23) and (24). $\square$

System (23) is standard form of *nonhomogeneous higher-order linear differential equations of Apostol-Kolodner type*, which may be treated by methods that are more classical; see for instance [18] and references therein.

Thus, it is convenient to define new variables as

$$\begin{aligned} \underline{z}_1(t) &= \underline{y}_p(t), \\ \underline{z}_2(t) &= \underline{y}_p'(t), \\ &\vdots \\ \underline{z}_r(t) &= \underline{y}_p^{(r-1)}(t). \end{aligned} \quad (30)$$

Then, we have the following system of ordinary differential equations:

$$\begin{aligned} \underline{z}_1'(t) &= \underline{z}_2(t), \\ \underline{z}_2'(t) &= \underline{z}_3(t), \\ &\vdots \\ \underline{z}_r'(t) &= J_p \underline{z}_1(t) + \tilde{\underline{b}}_p u_o(t). \end{aligned} \quad (31)$$

Finally, (31) can be expressed by using vector-matrix equations

$$\underline{z}'(t) = \underline{R} \underline{z}(t) + \underline{L} u_o(t), \quad (32)$$

where $\underline{z}(t) = [\underline{z}_1^t(t) \ \underline{z}_2^t(t) \ \cdots \ \underline{z}_r^t(t)]^t$ ($(\cdot)^t$ is the transpose tensor) and the coefficient matrices $\underline{R}$ and $\underline{L}$ are given by

$$\begin{aligned} \underline{R} &= \begin{bmatrix} \mathbb{O}_p & I_p & \mathbb{O}_p & \cdots & \mathbb{O}_p \\ \mathbb{O}_p & \mathbb{O}_p & I_p & \cdots & \mathbb{O}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \cdots & I_p \\ J_p & \mathbb{O}_p & \mathbb{O}_p & \cdots & \mathbb{O}_p \end{bmatrix} \in \mathcal{M}_{rp}, \\ \underline{L} &= \begin{bmatrix} \underline{0}_p \\ \underline{0}_p \\ \vdots \\ \tilde{\underline{b}}_p \end{bmatrix} \in \mathcal{M}_{(rp)1}, \end{aligned} \quad (33)$$

with corresponding dimension of $\underline{z}(t)$, $rp \times 1$.

The state of system (1) at time 0^- is $\underline{x}(0^-) = [0 \ 0 \ \cdots \ 0]^t$ and at time 0^+ it achieves $\underline{x}(0^+) = [x_1 \ x_2 \ \cdots \ x_n]^t$.

Now, considering (25), at time 0^- we have

$$\begin{aligned} \underline{y}(0^-) &= Q^{-1} \underline{x}(0^-) = \underline{y}'(0^-) \\ &= Q^{-1} \underline{x}'(0^-) = \cdots = \underline{y}^{(r-1)}(0^-) \\ &= Q^{-1} \underline{x}^{(r-1)}(0^-) = \underline{0}^t, \end{aligned} \quad (34)$$

and at time 0^+ we obtain

$$\begin{aligned} \underline{y}(0^+) &= Q^{-1} \underline{x}(0^+) = [\chi_1^0 \ \chi_2^0 \ \cdots \ \chi_n^0]^t, \\ \underline{y}'(0^+) &= Q^{-1} \underline{x}'(0^+) = [\chi_1^1 \ \chi_2^1 \ \cdots \ \chi_n^1]^t, \dots, \\ \underline{y}^{(r-1)}(0^+) &= Q^{-1} \underline{x}^{(r-1)}(0^+) = [\chi_1^{r-1} \ \chi_2^{r-1} \ \cdots \ \chi_n^{r-1}]^t. \end{aligned} \quad (35)$$

Moreover,

$$\begin{aligned} \underline{y}_p^{(i)}(0^-) &= \underline{0}_p^t \in \mathcal{M}_p, \\ \underline{y}_p^{(i)}(0^+) &= [\chi_1^i \ \chi_2^i \ \cdots \ \chi_p^i]^t \in \mathcal{M}_p, \\ \underline{y}_q^{(i)}(0^-) &= \underline{0}_q^t \in \mathcal{M}_q, \\ \underline{y}_q^{(i)}(0^+) &= [\chi_{p+1}^i \ \chi_{p+2}^i \ \cdots \ \chi_n^i]^t \in \mathcal{M}_q, \end{aligned} \quad (36)$$

where $\chi_j^i \in \mathbb{F}$ for $i = 0, 1, 2, \dots, r-1$, and $j = 1, 2, \dots, n$.

Furthermore,

$$\begin{aligned}
 \underline{\mathbf{z}}(0^-) &= [\underline{\mathbf{z}}_1^t(0^-) \ \underline{\mathbf{z}}_2^t(0^-) \ \cdots \ \underline{\mathbf{z}}_r^t(0^-)]^t \\
 &= [\underline{\mathbf{y}}_p^t(0^-) \ \underline{\mathbf{y}}_p^{t'}(0^-) \ \cdots \ \underline{\mathbf{y}}_p^{(r-1)t}(0^-)]^t = \underline{\mathbf{0}}_{rp}^t, \\
 \underline{\mathbf{z}}(0^+) &= [\underline{\mathbf{z}}_1^t(0^+) \ \underline{\mathbf{z}}_2^t(0^+) \ \cdots \ \underline{\mathbf{z}}_r^t(0^+)]^t \\
 &= [\underline{\mathbf{y}}_p^t(0^+) \ \underline{\mathbf{y}}_p^{t'}(0^+) \ \cdots \ \underline{\mathbf{y}}_p^{(r-1)t}(0^+)]^t \\
 &= [\chi_1^0 \ \chi_2^0 \ \cdots \ \chi_p^0 \ \chi_1^1 \ \chi_2^1 \ \cdots \\
 &\quad \chi_p^1 \ \cdots \ \chi_1^{r-1} \ \chi_2^{r-1} \ \cdots \ \chi_p^{r-1}]^t.
 \end{aligned} \tag{37}$$

3.2. *The Solution of Subsystem (32).* In order to solve subsystem (32), the following definitions should be provided.

Definition 4. The characteristic polynomial of matrix $\mathbf{R}$ is given by

$$\varphi(\lambda) = \prod_{i=1}^{\kappa} (\lambda - \lambda_i)^{\tau_i}, \tag{38}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\sum_{i=1}^{\kappa} \tau_i = rp$. Without loss of generality, we assume that

$$\begin{aligned}
 d_1 &= \tau_1, \quad d_2 = \tau_2, \dots, \\
 d_l &= \tau_l, \quad \text{for } 1 < l < \kappa, \ d_{l+1} < \tau_{l+1}, \dots, d_{\kappa} < \tau_{\kappa},
 \end{aligned} \tag{39}$$

where $d_i, \tau_i, i = 1, 2, \dots, \kappa$ is the geometric and algebraic multiplicity of the given eigenvalues λ_i , respectively.

Generally speaking, the matrix $\mathbf{R}$ is not diagonalizable. However, we can generate rp linearly independent vectors $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_{rp}$ and $rp \times rp$ similarity transformation $\mathbf{C} = [\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_{rp}]$ that takes $\mathbf{R}$ into the Jordan canonical form, as the following definition clarifies.

Definition 5. There exists an invertible matrix $\mathbf{C} \in \mathcal{M}_{rp}$ such as $\mathbf{J} = \mathbf{C}^{-1}\mathbf{R}\mathbf{C}$, $\mathbf{J} \in \mathcal{M}_{rp}$; is it Jordan canonical form of matrix $\mathbf{R}$. Analytically,

$$\mathbf{J} = \mathbf{J}_o \oplus \mathbf{J}_{l+1} \oplus \mathbf{J}_{l+2} \oplus \cdots \oplus \mathbf{J}_{\kappa}. \tag{40}$$

(i) The block diagonal matrix $\mathbf{J}_o = \mathbf{J}_1 \oplus \mathbf{J}_2 \oplus \cdots \oplus \mathbf{J}_l$, where

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix} \in \mathcal{M}_{\tau_i}, \tag{41}$$

is also a diagonal matrix with diagonal elements the eigenvalue λ_i , for $i = 1, \dots, l$. Consequently, the dimension of $\mathbf{J}_o$ is $s \times s$, $s \triangleq \sum_{i=1}^l \tau_i$.

(ii) Also, the block matrix $\mathbf{J}_j = \mathbf{J}_{j,1} \oplus \mathbf{J}_{j,2} \oplus \cdots \oplus \mathbf{J}_{j,d_j} \in \mathcal{M}_{\tau_j}$, where

$$\mathbf{J}_{j,z_j} = \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & 1 & \\ & & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_j \end{bmatrix} \in \mathcal{M}_{z_j} \tag{42}$$

for $j = l+1, l+2, \dots, \kappa, z_j = 1, 2, \dots, d_j$.

According to the classical theory of ordinary differential equations, the solution of system (32) is given by the following lemma.

Lemma 6. The solution of subsystem (32) is given by

$$\underline{\mathbf{z}}(t) = \mathbf{C} \int_{-\infty}^t e^{\mathbf{J}(t-\tau)} \underline{\mathbf{b}} u_o(\tau) d\tau, \tag{43}$$

with initial condition $\underline{\mathbf{z}}(0^-) = \underline{\mathbf{0}}_{rp}^t$.

Proof. Consider the transformation

$$\underline{\mathbf{z}}(t) = \mathbf{C} \underline{\zeta}(t), \tag{44}$$

where nonsingular $\mathbf{C} \in \mathcal{M}_{rp}$, and $\underline{\zeta}(0^-) = \mathbf{C}^{-1} \underline{\mathbf{z}}(0^-) = \underline{\mathbf{0}}_{rp}^t$.

Substituting (44) into (43), we obtain

$$\begin{aligned}
 \mathbf{C} \underline{\zeta}'(t) &= \mathbf{R} \mathbf{C} \underline{\zeta}(t) + \underline{\mathbf{L}} u_o(t) \iff \underline{\zeta}'(t) \\
 &= \mathbf{C}^{-1} \mathbf{R} \mathbf{C} \underline{\zeta}(t) + \mathbf{C}^{-1} \underline{\mathbf{L}} u_o(t).
 \end{aligned} \tag{45}$$

Furthermore, we define $\underline{\mathbf{b}} \triangleq \mathbf{C}^{-1} \underline{\mathbf{L}}$, such as the last equation is transformed into

$$\underline{\zeta}'(t) = \mathbf{J} \underline{\zeta}(t) + \underline{\mathbf{b}} u_o(t). \tag{46}$$

Now, according to the relevant theory of first-order differential systems of (46) form, see for instant [16], using also (44), the solution is expressed by (43). $\square$

Definition 7. The exponential matrix $e^{\mathbf{J}(t-\tau)}$ is defined as

$$\begin{aligned}
 e^{\mathbf{J}(t-\tau)} &= e^{\mathbf{J}_o(t-\tau)} \oplus e^{\mathbf{J}_{l+1}(t-\tau)} \oplus e^{\mathbf{J}_{l+2}(t-\tau)} \oplus \cdots \oplus e^{\mathbf{J}_{\kappa}(t-\tau)} \in \mathcal{M}_{rp}, \\
 &\tag{47}
 \end{aligned}$$

where

$$\begin{aligned}
 e^{\mathbf{J}_o(t-\tau)} &= e^{\mathbf{J}_1(t-\tau)} \oplus e^{\mathbf{J}_2(t-\tau)} \oplus \cdots \oplus e^{\mathbf{J}_l(t-\tau)} \in \mathcal{M}_s, \\
 e^{\mathbf{J}_i(t-\tau)} &= e^{\lambda_i(t-\tau)} \oplus e^{\lambda_i(t-\tau)} \oplus \cdots \oplus e^{\lambda_i(t-\tau)} \in \mathcal{M}_{\tau_i},
 \end{aligned} \tag{48}$$

for $i = 1, \dots, l$.

Furthermore,

$$e^{\mathbf{J}_j(t-\tau)} = e^{\lambda_{j_1}(t-\tau)} \oplus e^{\lambda_{j_2}(t-\tau)} \oplus \cdots \oplus e^{\lambda_{j_{d_j}}(t-\tau)} \in \mathcal{M}_{\tau_j}, \tag{49}$$

where

$$e^{J_{jz_j}(t-\tau)} = e^{\lambda_j(t-\tau)} \begin{bmatrix} 1 & t-\tau & \cdots & \frac{(t-\tau)^{\mu_{z_j}-1}}{(\mu_{z_j}-1)!} \\ 0 & 1 & \cdots & \frac{(t-\tau)^{\mu_{z_j}-2}}{(\mu_{z_j}-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{\mu_{z_j}}, \quad (50)$$

where μ_{z_j} are the Weyr characteristics via Ferrer diagrams, for $j = l+1, l+2, \dots, k$ and $z_j = 1, 2, \dots, d_j$. Note that $\rho_j = \max_{z_j=1,2,\dots,d_j} \mu_{z_j}$ is the index of annihilation for the eigenvalue λ_j .

Now, taking into consideration (21) the solution (22) is transposed into

$$\underline{z}(t) = \lim_{\sigma \rightarrow 0} \left\{ C e^{Jt} \int_{-\infty}^t e^{-J\tau} \underline{b} u(\tau) d\tau \right\}, \quad (51)$$

or equivalently

$$\underline{z}(t) = C \begin{bmatrix} \lim_{\sigma \rightarrow 0} \left\{ e^{J_0 t} \int_{-\infty}^t e^{-J_0 \tau} \underline{b}_s u(\tau) d\tau \right\} \\ \lim_{\sigma \rightarrow 0} \left\{ e^{J_{l+1} t} \int_{-\infty}^t e^{-J_{l+1} \tau} \underline{b}_{\tau_{l+1}} u(\tau) d\tau \right\} \\ \vdots \\ \lim_{\sigma \rightarrow 0} \left\{ e^{J_k t} \int_{-\infty}^t e^{-J_k \tau} \underline{b}_{\tau_k} u(\tau) d\tau \right\} \end{bmatrix}, \quad (52)$$

where $\underline{b} = [\underline{b}_s^t \ \underline{b}_{\tau_{l+1}}^t \ \cdots \ \underline{b}_{\tau_k}^t]^t$.

Remark 1. [4, 5] Given the large number of terms involved, in order to make our calculations affordable, we consider the fact that $\Phi(t/\sigma)$ and its derivatives tend to zero very strongly with $t \rightarrow 0$ (note also that $\sigma \rightarrow 0$). Thus, by letting $t = K \cdot \sigma$, where K is chosen large enough (i.e., $K \rightarrow \infty$) the assumption as stated above is valid, that is,

$$\Phi\left(\frac{t}{\sigma}\right) \stackrel{t=K\sigma}{=} \Phi(K) \rightarrow 0, \quad (53)$$

and its derivatives $\Phi^{(i)}(t/\sigma) \stackrel{t=K\sigma}{=} \Phi^{(i)}(K) \stackrel{K \rightarrow \infty}{\rightarrow} 0$, for $i = 0, 1, 2, \dots, k-1$.

Now, using the Remark 1, we obtain that $\lim_{\sigma \rightarrow 0} e^{Jt} \stackrel{t=K\sigma}{=} I$, for $i = 1, 2, \dots, l, l+1, l+2, \dots, k$, and $\Phi^{-1}(K + \lambda_i \sigma) \rightarrow \Phi^{-1}(K)$, as well. Moreover, for the analytic determination of vector (3), we need to calculate the integral $\int_{-\infty}^t e^{-J\tau} \underline{b} u(\tau) d\tau$.

In this part of the paper, two subsystems are derived; see the following lemmas, and Appendices A and B.

Lemma 8. For the diagonal matrix J_0 , the following integral is given by (54):

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 \tau} \underline{b}_s u(\tau) d\tau = \Phi^{-1}(K) \begin{bmatrix} \underline{b}_{\tau_1} \sum_{k=0}^{n-1} \alpha_k \lambda_1^k \\ \vdots \\ \underline{b}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \\ \vdots \\ \underline{b}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \end{bmatrix} \in \mathcal{M}_{s1}. \quad (54)$$

Proof. See Appendix A. $\square$

Lemma 9. For the diagonal matrix $J_j = J_{j_1} \oplus J_{j_2} \oplus \cdots \oplus J_{j_{d_j}} \in \mathcal{M}_{\tau_j}$, the following integral is given by (55):

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_j \tau} \underline{b}_{\tau_j} u(\tau) d\tau = \Phi^{-1}(K) \begin{bmatrix} \underline{V}_{j_1} \\ \underline{V}_{j_2} \\ \vdots \\ \underline{V}_{j_{z_j}} \\ \vdots \\ \underline{V}_{j_{d_j}} \end{bmatrix} \cdot \underline{a}, \quad (55)$$

where

$$\underline{V}_{j_{z_j}} \triangleq \begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j^0}{d \lambda_j^{c-1}} & \sum_{c=1}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j}{d \lambda_j^{c-1}} & \cdots & \sum_{c=1}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j^{n-1}}{d \lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j^0}{d \lambda_j^{c-2}} & \sum_{c=2}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j}{d \lambda_j^{c-2}} & \cdots & \sum_{c=2}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j^{n-1}}{d \lambda_j^{c-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j^0}{d \lambda_j^{c-\mu_{z_j}}} & \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j}{d \lambda_j^{c-\mu_{z_j}}} & \cdots & \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \underline{b}_{j_{z_j}}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j^{n-1}}{d \lambda_j^{c-\mu_{z_j}}} \end{bmatrix}, \quad \underline{a} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}. \quad (56)$$

Proof. See Appendix B. $\square$

Now, we revisit (54), thus

$$\begin{aligned}
 & \Phi^{-1}(K) \begin{bmatrix} \underline{\mathbf{b}}_{\tau_1} \sum_{k=0}^{n-1} \alpha_k \lambda_1^k \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \end{bmatrix} \\
 &= \Phi^{-1}(K) \left(\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \cdots \oplus \underline{\mathbf{b}}_{\tau_l} \right) \begin{bmatrix} \sum_{k=0}^{n-1} \alpha_k \lambda_1^k \\ \sum_{k=0}^{n-1} \alpha_k \lambda_2^k \\ \vdots \\ \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \end{bmatrix} \\
 &= \Phi^{-1}(K) \left(\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \cdots \oplus \underline{\mathbf{b}}_{\tau_l} \right) \\
 & \quad \times \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_l & \lambda_l^2 & \cdots & \lambda_l^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} \\
 &= \Phi^{-1}(K) \left(\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \cdots \oplus \underline{\mathbf{b}}_{\tau_l} \right) \mathbf{V}_l \underline{\mathbf{a}}. \tag{57}
 \end{aligned}$$

Note that $\mathbf{V}_l$ is the well-known *Vandermonde* matrix.

Combining (52)–(57), we obtain the following system:

$$\begin{aligned}
 & C^{-1} \underline{\mathbf{z}}(0^+) \\
 &= \Phi^{-1}(K) \\
 & \quad \times \left(\underbrace{\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \cdots \oplus \underline{\mathbf{b}}_{\tau_l}}_l \oplus I_{\tau_{l+1}} \oplus \cdots \oplus I_{\tau_\kappa} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)_1} \\ \vdots \\ \mathbf{V}_{(l+1)_{d_{l+1}}} \\ \vdots \\ \mathbf{V}_{j_1} \\ \vdots \\ \mathbf{V}_{j_{d_j}} \\ \vdots \\ \mathbf{V}_{\kappa_1} \\ \vdots \\ \mathbf{V}_{\kappa_{d_\kappa}} \end{bmatrix} \underline{\mathbf{a}}, \tag{58}
 \end{aligned}$$

where $\mathbf{V}_l \in \mathcal{M}_{sn}$ and $\mathbf{V}_{j_{z_j}} \in \mathcal{M}_{\mu_{z_j}n}$ with $j = l+1, l+2, \dots, \kappa$ and $z_j = 1, 2, \dots, d_j$. Furthermore, we obtain the initial condition

$$\begin{aligned}
 & C^{-1} \underline{\mathbf{z}}(0^+) \\
 &= \begin{bmatrix} \underline{\mathbf{z}}_l^t(0^+) & \underline{\mathbf{z}}_{(l+1)_1}^t(0^+) & \cdots & \underline{\mathbf{z}}_{(l+1)_{d_{l+1}}}^t(0^+) & \cdots & \underline{\mathbf{z}}_{j_1}^t(0^+) \\ \cdots & \underline{\mathbf{z}}_{j_{d_j}}^t(0^+) & \cdots & \underline{\mathbf{z}}_{\kappa_1}^t(0^+) & \cdots & \underline{\mathbf{z}}_{\kappa_{d_\kappa}}^t(0^+) \end{bmatrix}. \tag{59}
 \end{aligned}$$

The system (58) can now be divided into two types of subsystems:

(S1)

$$\frac{1}{\Phi^{-1}(K)} \underline{\mathbf{z}}_l(0^+) = \left(\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \cdots \oplus \underline{\mathbf{b}}_{\tau_l} \right) \mathbf{V}_l \underline{\mathbf{a}}, \tag{60}$$

(S2)

$$\frac{1}{\Phi^{-1}(K)} \underline{\mathbf{z}}_{j_{z_j}}(0^+) = \mathbf{V}_{j_{z_j}} \underline{\mathbf{a}}, \tag{61}$$

for $j = l+1, l+2, \dots, \kappa$ and $z_j = 1, 2, \dots, d_j$.

Proposition 10. System (60) is solvable if

$$\frac{z_i^1}{b_i^1} = \frac{z_i^2}{b_i^2} = \cdots = \frac{z_i^{\tau_i}}{b_i^{\tau_i}}, \tag{62}$$

for every nonzero element of vectors $\underline{\mathbf{b}}_{\tau_1}, \underline{\mathbf{b}}_{\tau_2}, \dots, \underline{\mathbf{b}}_{\tau_l}$, where $\underline{\mathbf{z}}_i(0^+) = [z_i^1 \ z_i^2 \ \cdots \ z_i^{\tau_i}]^t$, for every $i = 1, 2, \dots, l$.

Moreover, if one of the elements of vectors $\underline{\mathbf{b}}_{\tau_1}, \underline{\mathbf{b}}_{\tau_2}, \dots, \underline{\mathbf{b}}_{\tau_l}$ is zero, then the relative element of the i th-row of the vector $\underline{\mathbf{z}}_{\tau_i}(0^+)$ should be also zero.

Proof. System (60) contains l -subsystems of the following type:

$$\begin{aligned} & (\underline{0}_{\tau_1} \oplus \cdots \oplus \underline{b}_{\tau_i} \oplus \cdots \oplus \underline{0}_{\tau_l}) \mathbf{V}_l \underline{\alpha} \\ &= \frac{1}{\Phi^{-1}(K)} (I_{\tau_1} \oplus \cdots \oplus I_{\tau_i} \oplus \cdots \oplus I_{\tau_l}) \mathbf{z}_i(0^+), \end{aligned} \quad (63)$$

or equivalently

$$\begin{bmatrix} b_i^1 & b_i^1 \lambda_i & \cdots & b_i^1 \lambda_i^{n-1} \\ b_i^2 & b_i^2 \lambda_i & \cdots & b_i^2 \lambda_i^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_i^{\tau_i} & b_i^{\tau_i} \lambda_i & \cdots & b_i^{\tau_i} \lambda_i^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} z_i^1(0^+) \\ z_i^2(0^+) \\ \vdots \\ z_i^{\tau_i}(0^+) \end{bmatrix}. \quad (64)$$

- (i) If one of the coefficients $b_i^\gamma = 0$, for $\gamma = 1, 2, \dots, \tau_i$, then the relative element of the i th row of the vector $\mathbf{z}_i(0^+)$ should be also zero (in order to obtain solution).
- (ii) If every of the coefficients $b_i^\gamma \neq 0$, for $\gamma = 1, 2, \dots, \tau_i$, then we have

$$\left. \begin{aligned} b_i^1 (\alpha_0 + \alpha_1 \lambda_i + \cdots + \alpha_{n-1} \lambda_i^{n-1}) &= z_i^1 \frac{1}{\Phi^{-1}(K)} \\ b_i^2 (\alpha_0 + \alpha_1 \lambda_i + \cdots + \alpha_{n-1} \lambda_i^{n-1}) &= z_i^2 \frac{1}{\Phi^{-1}(K)} \\ &\vdots \\ b_i^{\tau_i} (\alpha_0 + \alpha_1 \lambda_i + \cdots + \alpha_{n-1} \lambda_i^{n-1}) &= z_i^{\tau_i} \frac{1}{\Phi^{-1}(K)} \end{aligned} \right\} \Leftrightarrow$$

$$\left. \begin{aligned} \alpha_0 + \alpha_1 \lambda_i + \cdots + \alpha_{n-1} \lambda_i^{n-1} &= \frac{z_i^1}{b_i^1} \frac{1}{\Phi^{-1}(K)} \\ &\vdots \\ \alpha_0 + \alpha_1 \lambda_i + \cdots + \alpha_{n-1} \lambda_i^{n-1} &= \frac{z_i^{\tau_i}}{b_i^{\tau_i}} \frac{1}{\Phi^{-1}(K)} \end{aligned} \right\}$$

$$\Rightarrow \frac{z_i^1}{b_i^1} = \frac{z_i^2}{b_i^2} = \cdots = \frac{z_i^{\tau_i}}{b_i^{\tau_i}} \quad (65)$$

for every $i = 1, 2, \dots, l$.

Consequently, system (60) is solvable if (62) is satisfied for every $i = 1, 2, \dots, l$. $\square$

Now, we will work with subsystems (61) which can be written as follows:

$$\begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \frac{b_{jz_j}^c}{(c-1)!} \frac{d^{c-1} \lambda_j^0}{d \lambda_j^{c-1}} & \sum_{c=1}^{\mu_{z_j}} \frac{b_{jz_j}^c}{(c-1)!} \frac{d^{c-1} \lambda_j}{d \lambda_j^{c-1}} & \sum_{c=1}^{\mu_{z_j}} \frac{b_{jz_j}^c}{(c-1)!} \frac{d^{c-1} \lambda_j^2}{d \lambda_j^{c-1}} & \cdots & \sum_{c=1}^{\mu_{z_j}} \frac{b_{jz_j}^c}{(c-1)!} \frac{d^{c-1} \lambda_j^{n-1}}{d \lambda_j^{c-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{jz_j}^{\mu_{z_j}-1} & b_{jz_j}^{\mu_{z_j}-1} \lambda_j + b_{jz_j}^{\mu_{z_j}} & b_{jz_j}^{\mu_{z_j}-1} \lambda_j^2 + 2 \lambda_j b_{jz_j}^{\mu_{z_j}} & \cdots & b_{jz_j}^{\mu_{z_j}-1} \lambda_j^{n-1} + (n-1) \lambda_j^{n-2} b_{jz_j}^{\mu_{z_j}} \\ b_{jz_j}^{\mu_{z_j}} & b_{jz_j}^{\mu_{z_j}} \lambda_j, & b_{jz_j}^{\mu_{z_j}} \lambda_j^2 & \cdots & b_{jz_j}^{\mu_{z_j}} \lambda_j^{n-1} \end{bmatrix} \underline{\alpha} = \frac{1}{\Phi^{-1}(K)} \mathbf{z}_{jz_j}(0^+). \quad (66)$$

The coefficient matrix $\mathbf{V}_{jz_j}$ can be transposed to the following equivalent matrix; see (67). Note that $\mathbf{V}_{jz_j}$ is a $\mu_{z_j} \times n$ matrix. Moreover, from the μ_{z_j} elements $b_{jz_j}^c$, for

$c = 1, 2, \dots, \mu_{z_j}$, we can assume that $b_{jz_j}^{\mu_{z_j}} = b_{jz_j}^{\mu_{z_j}-1} = \cdots = b_{jz_j}^{\mu_{z_j}-\mu} = 0$. Thus, we obtain

$$\mathbf{V}_{jz_j}^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & \frac{1}{(\mu_{z_j} - \mu)!} \frac{d^{\mu_{z_j} - \mu} \lambda_j^{\mu_{z_j} + 1 - \mu}}{d \lambda_j^{\mu_{z_j} - \mu}} & \cdots & \frac{1}{(\mu_{z_j} - \mu)!} \frac{d^{\mu_{z_j} - \mu} \lambda_j^{n-1}}{d \lambda_j^{\mu_{z_j} - \mu}} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & \vdots & \vdots & \cdots & \frac{1}{2!} \frac{d^2 \lambda_j^{n-1}}{d \lambda_j^2} \\ 0 & 1 & 2 \lambda_j & \cdots & \vdots & \vdots & \cdots & (n-1) \lambda_j^{n-1} \\ 1 & \lambda_j & \lambda_j^2 & \cdots & \vdots & \vdots & \cdots & \lambda_j^{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (67)$$

In order to be able to understand (67) better, the following example is helpful.

Example 1. We take the coefficient matrix $\mathbf{V}$

$$\begin{bmatrix} b_3 & b_3\lambda + b_4 & b_3\lambda^2 + 2\lambda b_4 + b_5 & b_3\lambda^3 + 3\lambda^2 b_4 + 3\lambda b_5 + b_6 & b_3\lambda^4 + 4\lambda^3 b_4 + 6\lambda^2 b_5 + 2\lambda b_6 \\ b_4 & b_4\lambda + b_5 & b_4\lambda^2 + 2\lambda b_5 + b_6 & b_4\lambda^3 + 3\lambda^2 b_5 + 3\lambda b_6 & b_4\lambda^4 + 4\lambda^3 b_5 + 6\lambda^2 b_6 \\ b_5 & b_5\lambda + b_6 & b_5\lambda^2 + 2\lambda b_6 & b_5\lambda^3 + 3\lambda^2 b_6 & b_5\lambda^4 + 4\lambda^3 b_6 \\ b_6 & b_6\lambda & b_6\lambda^2 & b_6\lambda^3 & b_6\lambda^4 \end{bmatrix}. \quad (68)$$

We assume that $b_6 = 0$ and $b_5 \neq 0$. Since $b_5 \neq 0$, we are not interested in b_3, b_4 .

Thus, under our assumption, we obtain

$$\begin{bmatrix} b_3 & b_3\lambda + b_4 & b_3\lambda^2 + 2\lambda b_4 + b_5 & b_3\lambda^3 + 3\lambda^2 b_4 + 3\lambda b_5 & b_3\lambda^4 + 4\lambda^3 b_4 + 6\lambda^2 b_5 \\ b_4 & b_4\lambda + b_5 & b_4\lambda^2 + 2\lambda b_5 & b_4\lambda^3 + 3\lambda^2 b_5 & b_4\lambda^4 + 4\lambda^3 b_5 \\ b_5 & b_5\lambda & b_5\lambda^2 & b_5\lambda^3 & b_5\lambda^4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (69)$$

Afterwards, we multiply the 3rd row with $-b_4/b_5$ and it is added to the 2nd row. Moreover, we multiply the 3rd row with $-b_3/b_5$ and it is added with the 1st row. Then, the following equivalent matrix is derived:

$$\begin{bmatrix} 0 & b_4 & 2\lambda b_4 + b_5 & 3\lambda^2 b_4 + 3\lambda b_5 & 4\lambda^3 b_4 + 6\lambda^2 b_5 \\ 0 & b_5 & 2\lambda b_5 & 3\lambda^2 b_5 & 4\lambda^3 b_5 \\ b_5 & b_5\lambda & b_5\lambda^2 & b_5\lambda^3 & b_5\lambda^4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (70)$$

Now, we multiply the 2nd row with $-b_4/b_5$ and it is added with the 1st row. Thus, we obtain

$$\begin{bmatrix} 0 & 0 & b_5 & 3\lambda b_5 & 6\lambda^2 b_5 \\ 0 & b_5 & 2\lambda b_5 & 3\lambda^2 b_5 & 4\lambda^3 b_5 \\ b_5 & b_5\lambda & b_5\lambda^2 & b_5\lambda^3 & b_5\lambda^4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (71)$$

Finally, we multiply the matrix with the element $1/b_5$ and we conclude to obtain

$$\begin{bmatrix} 0 & 0 & 1 & 3\lambda & 6\lambda^2 \\ 0 & 1 & 2\lambda & 3\lambda^2 & 4\lambda^3 \\ 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{(1)''}{2!} & \frac{(\lambda)''}{2!} & \frac{(\lambda^2)''}{2!} & \frac{(\lambda^3)''}{2!} & \frac{(\lambda^4)''}{2!} \\ \frac{(1)'}{1!} & \frac{(\lambda)'}{1!} & \frac{(\lambda^2)'}{1!} & \frac{(\lambda^3)'}{1!} & \frac{(\lambda^4)'}{1!} \\ 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (72)$$

Remark 2. Considering the results already presented, it is clear that there exists a nonsingular matrix $L_{jz_j}, \mu_{z_j} \times \mu_{z_j}$ such that

$$L_{jz_j} \mathbf{V}_{jz_j} = \mathbf{V}_{jz_j}^*. \quad (73)$$

Thus, system (61) can be transformed into

$$\frac{1}{\Phi^{-1}(K)} L_{jz_j} \mathbf{z}_{jz_j} (0^+) = L_{jz_j} \mathbf{V}_{jz_j} \mathbf{a} \Leftrightarrow \mathbf{V}_{jz_j}^* \mathbf{a} = \mathbf{z}_{jz_j}^* (0^+), \quad (74)$$

where

$$\begin{aligned} \mathbf{z}_{jz_j}^* (0^+) &= \frac{1}{\Phi^{-1}(K)} L_{jz_j} \mathbf{z}_{jz_j} (0^+), \\ \forall j &= l+1, l+2, \dots, \kappa, \\ \forall z_j &= 1, 2, \dots, d_j. \end{aligned} \quad (75)$$

Proposition 11. Subsystems (74) are solvable when the elements of vectors

$$\mathbf{z}_{jz_j}^* (0^+), \quad \text{for every } z_j = 1, 2, \dots, d_j, \quad j = l+1, l+2, \dots, \kappa \quad (76)$$

are included into the vector with the greatest dimension, that is, $\rho_j \times 1$ where

$$\rho_j = \max \mu_{z_j}, \quad z_j = 1, 2, \dots, d_j. \quad (77)$$

Equivalently, if one assumes that

$$\mathbf{z}_{j_1}^* (0^+) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{\rho_j} \end{bmatrix}, \quad (78)$$

then each of the following vectors $\mathbf{z}_{j_2}^* (0^+), \mathbf{z}_{j_3}^* (0^+), \dots, \mathbf{z}_{j_{d_j}}^* (0^+)$

should be vectors of $\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_\xi \end{bmatrix}$ type with $\xi \leq \rho_j$.

Proof. We take the d_j subsystems; as follows:

$$\begin{aligned} \mathbf{V}_{j_1}^* \underline{\mathbf{a}} &= \underline{\mathbf{z}}_{j_1}^* (0^+), \\ &\dots \end{aligned} \quad (79)$$

$$\mathbf{V}_{j_{d_j}}^* \underline{\mathbf{a}} = \underline{\mathbf{z}}_{j_{d_j}}^* (0^+).$$

The matrix

$$\begin{aligned} \mathbf{V}_{j_1}^* &\text{ has } \mu_1 \text{ rows,} \\ \mathbf{V}_{j_2}^* &\text{ has } \mu_2 \text{ rows,} \\ &\vdots \\ \mathbf{V}_{j_{d_j}}^* &\text{ has } \mu_{d_j} \text{ rows.} \end{aligned}$$

Without loss of generality, we assume that $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{d_j} = \rho_j$.

Looking carefully at the type of matrices

$$\mathbf{V}_{j_1}^*, \mathbf{V}_{j_2}^*, \dots, \mathbf{V}_{j_{d_j}}^*, \quad (80)$$

we can easily verify that the μ_1 -first rows of them are identically same. Thus, it is necessary the relevant μ_1 -first rows of $\underline{\mathbf{z}}_{j_1}^* (0^+), \underline{\mathbf{z}}_{j_2}^* (0^+), \dots, \underline{\mathbf{z}}_{j_{d_j}}^* (0^+)$ to be also identically the same. Analogously, the μ_2 -first rows of $\mathbf{V}_{j_1}^*, \mathbf{V}_{j_2}^*, \dots, \mathbf{V}_{j_{d_j}}^*$ should be identically the same with the relevant μ_2 -first rows of $\underline{\mathbf{z}}_{j_1}^* (0^+), \underline{\mathbf{z}}_{j_2}^* (0^+), \dots, \underline{\mathbf{z}}_{j_{d_j}}^* (0^+)$. And so on until the $\mu_{d_j} = \rho_j$ row. $\square$

$$\begin{aligned} &\Phi^{-1}(K) \left(\underbrace{\underline{\mathbf{b}}_{\tau_1} \oplus \underline{\mathbf{b}}_{\tau_2} \oplus \dots \oplus \underline{\mathbf{b}}_{\tau_l}}_l \right. \\ &\quad \left. \oplus \frac{1}{\Phi^{-1}(K)} [I_{\tau_{l+1}} \oplus \dots \oplus I_{\tau_k}] \right) \\ &\quad \times \begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)_1}^* \\ \vdots \\ \mathbf{V}_{(l+1)_{d_{l+1}}}^* \\ \vdots \\ \mathbf{V}_{j_1}^* \\ \vdots \\ \mathbf{V}_{j_{d_j}}^* \\ \vdots \\ \mathbf{V}_{\kappa_1}^* \\ \vdots \\ \mathbf{V}_{\kappa_{d_k}}^* \end{bmatrix} \underline{\mathbf{a}} = \begin{bmatrix} \underline{\mathbf{z}}_l(0^+) \\ \underline{\mathbf{z}}_{(l+1)_1}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{(l+1)_{d_{l+1}}}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{j_1}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{j_{d_j}}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{\kappa_1}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{\kappa_{d_k}}^* (0^+) \end{bmatrix}. \end{aligned} \quad (81)$$

Consequently, it is time to use the results of Propositions 10 and 11. Thus, system (81) is solvable if we obtain the following.

(i) The first nonzero elements of vector coefficient $\underline{\mathbf{b}}_{\tau_1}, \underline{\mathbf{b}}_{\tau_2}, \dots, \underline{\mathbf{b}}_{\tau_l}$ (practically speaking, without loss of generality, we assume that the first elements are nonzero).

(ii) The matrix $\mathbf{V}_{(q+1)_{z_{q+1}}}^*$ with dimension $\rho_{l+1} \times n$, where $\rho_{l+1} = \max \mu_{z_{l+1}}, z_{l+1} = 1, 2, \dots, d_{l+1}$.

(iii) The matrix $\mathbf{V}_{j_{z_j}}^*$ with dimension $\rho_j \times n$, where $\rho_j = \max \mu_{z_j}, z_j = 1, 2, \dots, d_j$.

(iv) The matrix $\mathbf{V}_{\kappa_{z_k}}^*$ with dimension $\rho_k \times n$, where $\rho_k = \max \mu_{z_k}, z_k = 1, 2, \dots, d_k$.

Consequently, system (81) is transposed to the solvable system (83)

$$\begin{aligned} &\Phi^{-1}(K) \left(\underbrace{b_{\tau_1} \oplus b_{\tau_2} \oplus \dots \oplus b_{\tau_l}}_l \right. \\ &\quad \left. \oplus \frac{1}{\Phi^{-1}(K)} [I_{\tau_{l+1}} \oplus \dots \oplus I_{\tau_k}] \right) \end{aligned} \quad (82)$$

$$\times \begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)_{z_{l+1}}}^* \\ \mathbf{V}_{(l+2)_{z_{l+2}}}^* \\ \vdots \\ \mathbf{V}_{\kappa_{z_k}}^* \end{bmatrix} \underline{\mathbf{a}} = \begin{bmatrix} \underline{\mathbf{z}}_l(0^+) \\ \underline{\mathbf{z}}_{(l+1)_{z_{l+1}}}^* (0^+) \\ \underline{\mathbf{z}}_{(l+2)_{z_{l+2}}}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{\kappa_{z_k}}^* (0^+) \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)_{z_{l+1}}}^* \\ \mathbf{V}_{(l+2)_{z_{l+2}}}^* \\ \vdots \\ \mathbf{V}_{\kappa_{z_k}}^* \end{bmatrix} \underline{\mathbf{a}} = \begin{bmatrix} \underline{\mathbf{z}}_l(0^+) \\ \underline{\mathbf{z}}_{(l+1)_{z_{l+1}}}^* (0^+) \\ \underline{\mathbf{z}}_{(l+2)_{z_{l+2}}}^* (0^+) \\ \vdots \\ \underline{\mathbf{z}}_{\kappa_{z_k}}^* (0^+) \end{bmatrix}, \quad (83)$$

where $\underline{\mathbf{z}}_l^* (0^+) = (1/\Phi^{-1}(K)) (1/b_{\tau_1} \oplus 1/b_{\tau_2} \oplus \dots \oplus 1/b_{\tau_l}) \underline{\mathbf{z}}_l(0^+)$, since $(b_{\tau_1} \oplus b_{\tau_2} \oplus \dots \oplus b_{\tau_l})$ is nonsingular.

Remark 3. The matrices given by (67) with some row transformations can be transformed to the following:

$$\begin{bmatrix} 1 & \lambda_j & \lambda_j^2 & \lambda_j^3 & \dots & \dots & \lambda_j^{n-1} \\ 0 & 1 & 2\lambda_j & 3\lambda_j^2 & \dots & \dots & (n-1)\lambda_j^{n-2} \\ 0 & 0 & 1 & 3\lambda_j & \dots & \dots & \frac{(n-1)(n-2)}{2!}\lambda_j^{n-3} \\ \vdots & & & & & & \\ 0 & & 1 & \frac{1}{(\mu_{z_j}-\mu)!} \frac{d^{\mu_{z_j}-\mu} \lambda_j^{\mu_{z_j}-\mu+1}}{d\lambda_j^{\mu_{z_j}-\mu}} & \dots & \frac{1}{(\mu_{z_j}-\mu)!} \frac{d^{\mu_{z_j}-\mu} \lambda_j^{n-1}}{d\lambda_j^{\mu_{z_j}-\mu}} \\ \hline 0 & 0 & & & & & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & & & & & 0 \end{bmatrix}. \quad (84)$$

The matrix (84) is denoted by $\mathbf{V}_{jz_j}^{**}$. Thus, there is a nonsingular matrix Λ_{jz_j} such as

$$\Lambda_{jz_j} \mathbf{V}_{jz_j}^* = \mathbf{V}_{jz_j}^{**}. \quad (85)$$

Then, subsystems (74) are transformed to

$$\Lambda_{jz_j} \mathbf{V}_{jz_j}^* \mathbf{a} = \Lambda_{jz_j} \mathbf{z}_{jz_j}^* (0^+) \iff \mathbf{V}_{jz_j}^{**} \mathbf{a} = \mathbf{z}_{jz_j}^{**} (0^+), \quad (86)$$

where $\Lambda_{jz_j} \mathbf{z}_{jz_j}^* (0^+) = \mathbf{z}_{jz_j}^{**} (0^+)$ for every $z_j = 1, 2, \dots, d_j$, and $j = l+1, l+2, \dots, \kappa$. Consequently, system (83) is transformed to

$$\begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)z_{l+1}}^{**} \\ \mathbf{V}_{(l+2)z_{l+2}}^{**} \\ \vdots \\ \mathbf{V}_{\kappa d_\kappa}^{**} \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{z}_l^* (0^+) \\ \mathbf{z}_{(l+1)z_{l+1}}^{**} (0^+) \\ \mathbf{z}_{(l+2)z_{l+2}}^{**} (0^+) \\ \vdots \\ \mathbf{z}_{\kappa d_\kappa}^{**} (0^+) \end{bmatrix}. \quad (87)$$

Comment 1. Without loss of generality, we can assume that the matrices $\mathbf{V}_{jd_j}^{**}$ do not contain zero rows. On contrary, if there exist some zero rows, the solvability of system (87) is

not affected; only the number of nonzero rows which are included in (87) is changed.

Remark 4. System (87) can be further transposed to a more convenient system. Analytically, if we multiply the 1st row of Vandermonde matrix $\mathbf{V}_l$, that is, $[1 \ \lambda_1 \ \lambda_1^2 \ \dots \ \lambda_1^{n-1}]$ with the number (-1) and it is added to the 1st row of each of $\mathbf{V}_{(l+1)z_{l+1}}^{**}, \mathbf{V}_{(l+2)z_{l+2}}^{**}, \dots, \mathbf{V}_{\kappa d_\kappa}^{**}$, then $\mathbf{V}_{jz_j}^{**}$ is given by

$$\begin{bmatrix} 0 & \lambda_j - \lambda_1 & \lambda_j^2 - \lambda_1^2 & \dots & \lambda_j^{n-1} - \lambda_1^{n-1} \\ 0 & 1 & 2\lambda_j & \dots & (n-1)\lambda_j^{n-2} \\ \vdots & & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1, \frac{1}{(\rho_j-1)!} \frac{d^{\rho_j-1} \lambda_j^{\rho_j+1}}{d\lambda_j^{\rho_j-1}}, \dots, \frac{1}{(\rho_j-1)!} \frac{d^{\rho_j-1} \lambda_j^{n-1}}{d\lambda_j^{\rho_j-1}} \end{bmatrix}. \quad (88)$$

(Note that we have assumed that the matrices $\mathbf{V}_{jz_j}^{**}$, for $j = l+1, l+2, \dots, \kappa$, do not contain zero rows; see also *Comment*)

We can easily see that the 1st row of matrix (88) can be rewritten as what follows.

$$\text{The element } \lambda_j^\kappa + \lambda_j^{\kappa-1} \lambda_1 + \dots + \lambda_j \lambda_1^{\kappa-1} + \lambda_1^\kappa = \sum_{m_1, m_2=0, \sum_{i=1}^2 m_i=\kappa} \lambda_j^{m_1} \lambda_1^{m_2}.$$

Thus, the first row is presented as

$$(\lambda_j - \lambda_1) \begin{bmatrix} 0 & 1 & \sum_{\substack{m_1, m_2=0 \\ \sum_{i=1}^2 m_i=1}}^1 \lambda_j^{m_1} \lambda_1^{m_2} & \sum_{\substack{m_1, m_2=0 \\ \sum_{i=1}^2 m_i=2}}^2 \lambda_j^{m_1} \lambda_1^{m_2} \\ \vdots & \vdots & \sum_{\substack{m_1, m_2=0 \\ \sum_{i=1}^2 m_i=n-2}}^{n-2} \lambda_j^{m_1} \lambda_1^{m_2} \end{bmatrix}. \quad (89)$$

Since, the element $\lambda_j - \lambda_1 \neq 0$, we can multiply by left the equation (88) with a properly chosen transformation matrix, so as to obtain

$$\begin{bmatrix} 0 & 1 & \sum_{\substack{m_1, m_2=0 \\ \sum_{i=1}^2 m_i=1}}^1 \lambda_j^{m_1} \lambda_1^{m_2} & \dots & \sum_{\substack{m_1, m_2=0 \\ \sum_{i=1}^2 m_i=n-2}}^{n-2} \lambda_j^{m_1} \lambda_1^{m_2} \\ 0 & 1 & 2\lambda_j & \dots & (n-1)\lambda_j^{n-2} \\ & & 0 & & \\ 0 & \dots & 1 & \frac{1}{(\rho_j-1)!} \frac{d^{\rho_j-1} \lambda_j^{\rho_j+1}}{d\lambda_j^{\rho_j-1}} & \dots & \frac{1}{(\rho_j-1)!} \frac{d^{\rho_j-1} \lambda_j^{n-1}}{d\lambda_j^{\rho_j-1}} \end{bmatrix}. \quad (90)$$

Finally, this subsection is concluded by considering system (91), where the matrices $\mathbf{A}_{jz_j}$ for $j = l+1, l+2, \dots, \kappa$ are derived by taking into account a properly chosen transformation left-matrix Z as follows:

$$Z \begin{bmatrix} \mathbf{V}_l \\ \mathbf{V}_{(l+1)z_{l+1}}^{**} \\ \mathbf{V}_{(l+2)z_{l+2}}^{**} \\ \vdots \\ \mathbf{V}_{\kappa z_\kappa}^{**} \end{bmatrix} \mathbf{a} = Z \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{z}_{(l+1)z_{l+1}}^{**}(0^+) \\ \mathbf{z}_{(l+2)z_{l+2}}^{**}(0^+) \\ \vdots \\ \mathbf{z}_{\kappa z_\kappa}^{**}(0^+) \end{bmatrix} \iff \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix}, \quad (91)$$

where

$$\begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix} = Z \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{z}_{(l+1)z_{l+1}}^{**}(0^+) \\ \mathbf{z}_{(l+2)z_{l+2}}^{**}(0^+) \\ \vdots \\ \mathbf{z}_{\kappa z_\kappa}^{**}(0^+) \end{bmatrix}, \quad (92)$$

with

$$\begin{aligned} \mathbf{V}_l &\in \mathcal{M}_{ln}, & \mathbf{V}_j &\in \mathcal{M}_{jn} \\ \text{for } j &= l+1, l+2, \dots, \kappa, \\ \mathbf{z}_l^*(0^+) &\in \mathcal{M}_{l1}, & \mathbf{d}_{l+2}(0^+) &\in \mathcal{M}_{j1}, \\ \text{for } j &= l+1, l+2, \dots, \kappa. \end{aligned} \quad (93)$$

$\rho_j = \max_{z_i=1,2,\dots,d_j} \mu_{z_j}$ is the index of annihilation for the eigenvalue λ_j .

Consequently, the system (91) contains the following subsystems

$$\left. \begin{aligned} \mathbf{V}_l \mathbf{a} &= \mathbf{z}_l^*(0^+), \\ \mathbf{A}_{l+1} \mathbf{a} &= \mathbf{d}_{l+1}(0^+), \\ &\vdots \\ \mathbf{A}_\kappa \mathbf{a} &= \mathbf{d}_\kappa(0^+) \end{aligned} \right\}, \quad (94)$$

where $\mathbf{V}_l, \mathbf{A}_j$ for $j = l+1, l+2, \dots, \kappa$ are nonsquare matrices.

In order to solve system (91), we should use the relevant theory of generalized inverses. Some basic elements are briefly presented in the next subsection.

3.3. Solving Subsystem (91). This subsection is beginning with the brief presentation of the most basic results of $\{1\}$ -generalized inverses.

Definition 12. For a matrix $A \in \mathcal{M}_{pq}$, the $X \in \mathcal{M}_{qp}$ is called $\{1\}$ -inverses of matrix A , if

$$AXA = A. \quad (95)$$

The following proposition provides the means for finding $\{1\}$ -inverses; see [19, 20] for proofs and details.

Proposition 13. Let $A \in \mathcal{M}_{pq}$, with $\text{rank } A = r < \min\{p, q\}$ and $P \in \mathcal{M}_p, Q \in \mathcal{M}_q$, invertible such that

$$PAQ = I_r \oplus \mathbb{O}_{p-r, q-r}. \quad (96)$$

Then, every $X, \{1\}$ -inverse of matrix A , is written as follows:

$$X = Q(I_r \oplus \mathbb{O}_{p-r, q-r})P, \quad (97)$$

for arbitrary $L \in \mathcal{M}_{(q-r)(p-r)}$.

Definition 14. We define any $\{1\}$ -inverse of matrix A by $A^{(1)}$.

The following result is due to [6].

Theorem 15. Let $A \in \mathcal{M}_{pq}, B \in \mathcal{M}_{dv}, D \in \mathcal{M}_{py}$. Then the matrix equation

$$AXB = D, \quad (98)$$

is consistent if and only if for some $A^{(1)}, B^{(1)}$

$$AA^{(1)}DB^{(1)}B = D \quad (99)$$

and the general solution is given by

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \quad (100)$$

for arbitrary $Y \in \mathcal{M}_{pd}$.

The following characterization of the set of $A^{(1)}$ is due to [20].

Corollary 16. Let $A \in \mathcal{M}_{pq}$ and $A^{(1)} \in \mathcal{M}_{qp}$, then any other $\{1\}$ -inverse $\tilde{A}^{(1)} \in \mathcal{M}_{qp}$ of matrix A is given by

$$\tilde{A}^{(1)} = A^{(1)} + Z - A^{(1)}AZAA^{(1)} \quad (101)$$

for some $Z \in \mathcal{M}_{qp}$.

A special case of $\{1\}$ -inverses is the left/right inverses of a full column/row rank matrix, respectively. The proofs and further details are presented extensively by [19, 20].

Definition 17. Consider the following matrices.

(a) Let $P_i(\alpha)$ be a $m \times m$ -matrix

$$P_i(\alpha) = \begin{bmatrix} 1 & & & & \mathbb{O} \\ & 1 & & & \\ & & \ddots & & \\ & & & \alpha & \\ & & & & 1 \\ & & & & & \ddots \\ \mathbb{O} & & & & & & 1 \end{bmatrix}. \quad (102)$$

(The nonzero element a is in the i th row and the i th column.)

Thus, whenever a matrix A is multiplied from the left by $P_i(\alpha)$, then its i th row is multiplied by the nonzero number a .

(b) Let $P_i(j, a)$ be an $m \times m$ -matrix

$$P_i(j, a) = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \cdots & a \\ & & & & \ddots & \vdots \\ & & & & & 1 \\ 0 & & & & & & 1 \end{bmatrix}. \quad (103)$$

(The nonzero element a is in the i th row and the j th column.)

Thus, whenever a matrix A is multiplied from the left by $P_i(j, a)$, then its j th row is multiplied by the nonzero number a and it is added to the j th row of A .

(c) Let $Q_i(a)$ be an $n \times n$ -matrix

$$Q_i(a) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}. \quad (104)$$

(The nonzero element a is in the i th-row and the i th-column.)

Thus, whenever a matrix A is multiplied from the right by $Q_i(a)$, then its i th column is multiplied by the nonzero number a .

(d) Let $Q_i(j, a)$ be an $n \times n$ -matrix

$$Q_i(j, a) = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & a \cdots 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix}. \quad (105)$$

(The nonzero element a is in the j th row and the i th column.)

Thus, whenever a matrix A is multiplied from the right by $Q_i(j, a)$, then its i th column is multiplied by the nonzero number a and it is added to the j th column of A .

Now, in order to calculate the generalized inverses of Vandermonde and the relative to it matrices, we use the recent results of paper [6]. In this context, the following propositions are relevant.

Proposition 18. The $\{1\}$ -inverse of the Vandermonde matrix is given by

$$\mathbf{V}_l^{(1)} = \mathbf{Q} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \mathbf{P}, \quad (106)$$

where

$$\mathbf{P} = \prod_{s=1}^l \prod_{j=l}^{s+1} P_j \left(\frac{1}{\lambda_j - \lambda_s} \right) P_j(s, -1) \in \mathcal{M}_l, \quad (107)$$

(the multiplication counts backwards, starting from $l \rightarrow l-1 \rightarrow \dots$ etc.)

$$\mathbf{Q} = \prod_{s=1}^l \prod_{r=s+1}^n Q_r \left(s, - \sum_{\substack{m_1, \dots, m_s=0 \\ m_1 + \dots + m_s = r-s}}^{r-s} \prod_{\zeta=1}^s \lambda_{\zeta}^{m_{\zeta}} \right) \in \mathcal{M}_n, \quad (108)$$

and the general solution of system $\mathbf{V}_l \mathbf{a} = \mathbf{z}_l^*(0^+)$ is given by

$$\mathbf{a} = \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + (\mathbf{I}_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}, \quad (109)$$

for an arbitrarily chosen vector $\underline{\psi} \in \mathcal{M}_{n-l}$.

Proof. The proof is a straightforward application of Theorem 15. Analytically,

$$\mathbf{P} \mathbf{V}_l \mathbf{Q} = \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \Leftrightarrow \mathbf{V}_l = \mathbf{P}^{-1} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1}. \quad (110)$$

In order for the matrix $\mathbf{Q} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \mathbf{P}$ to be the $\{1\}$ -inverse of $\mathbf{V}_l$, we should prove the following equality

$$\mathbf{V}_l \mathbf{Q} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \mathbf{P} \mathbf{V}_l = \mathbf{V}_l. \quad (111)$$

Thus,

$$\begin{aligned} \mathbf{V}_l \mathbf{Q} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \mathbf{P} \mathbf{V}_l &= \mathbf{P}^{-1} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1} \mathbf{Q} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \mathbf{P} \mathbf{P}^{-1} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1} \\ &= \mathbf{P}^{-1} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \begin{bmatrix} I_l \\ \mathbb{O}_{n-l, l} \end{bmatrix} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1} \\ &= \mathbf{P}^{-1} (I_l \cdot I_l + \mathbb{O}_{l, n-l} \cdot \mathbb{O}_{n-l, l}) \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1} \\ &= \mathbf{P}^{-1} \begin{bmatrix} I_l & \mathbb{O}_{l, n-l} \end{bmatrix} \mathbf{Q}^{-1} = \mathbf{V}_l. \end{aligned} \quad (112)$$

Hence, the solution of system $\mathbf{V}_l \mathbf{a} = \mathbf{z}_l^*(0^+)$ is given by (113). $\square$

Proposition 19. The $\{1\}$ -inverse of the matrix $\mathbf{A}_l$ for $j = l + 1, l + 2, \dots, \kappa$ is given by

$$\mathbf{A}_j^{(1)} = \mathbf{Q}_j \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \mathbf{P}_j, \quad (113)$$

where

$$\begin{aligned} \mathbf{P}_j &= \prod_{s=2}^{p_j} P_{p_j-s+2} \left(\frac{1}{\lambda_j - \lambda_1} \right) P_{p_j-s+2} (p_j - s + 1, -1), \\ \mathbf{Q}_j &= \prod_{s=2}^{p_j+1} \prod_{m=s+1}^n Q_m \left(s, -\frac{1}{(s-2)!} \frac{d^{s-2}}{d\lambda_j^{s-2}} \left(\sum_{\substack{m_1, m_2=0 \\ m_1+m_2=m-2}}^{m-2} \lambda_1^{m_1} \lambda_j^{m_2} \right) \right). \end{aligned} \quad (114)$$

for $j = l + 1, l + 2, \dots, \kappa$, and the general solution of systems $\mathbf{A}_j \mathbf{a} = \mathbf{d}_j(0^+)$ is given by

$$\mathbf{a} = \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (\mathbf{I}_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}_j \quad (115)$$

for an arbitrarily chosen vector $\underline{\psi}_j \in \mathcal{M}_{n1}$ for $j = l + 1, l + 2, \dots, \kappa$.

Proof. The proof is also a straightforward application of Theorem 15. Analytically,

$$\begin{aligned} \mathbf{P}_j \mathbf{A}_j \mathbf{Q}_j &= \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \\ \iff \mathbf{A}_j &= \mathbf{P}_j^{-1} \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1}. \end{aligned} \quad (116)$$

In order for the matrix

$$\mathbf{Q}_j \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \mathbf{P}_j \quad (117)$$

to be the $\{1\}$ -inverse of $\mathbf{A}_j$, we should prove the following equality:

$$\mathbf{A}_j \mathbf{Q}_j \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \mathbf{P}_j \mathbf{A}_j = \mathbf{A}_j. \quad (118)$$

Thus,

$$\begin{aligned} \mathbf{A}_j \mathbf{Q}_j \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \mathbf{P}_j \mathbf{A}_j &= \mathbf{P}_j^{-1} \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1} \mathbf{Q}_j \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \mathbf{P}_j \mathbf{P}_j^{-1} \\ &\times \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1} \\ &= \mathbf{P}_j^{-1} \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \begin{bmatrix} \underline{0}_{p_j}^t \\ I_{p_j} \\ \mathbb{O}_{n-p_j-1, p_j} \end{bmatrix} \\ &\times \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1} \\ &= \mathbf{P}_j^{-1} (\underline{0}_{p_j} \cdot \underline{0}_{p_j}^t + I_{p_j} \cdot I_{p_j} + \mathbb{O}_{p_j, n-p_j-1} \cdot \mathbb{O}_{n-p_j-1, p_j}) \\ &\times \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1} \\ &= \mathbf{P}_j^{-1} \begin{bmatrix} \underline{0}_{p_j} & I_{p_j} & \mathbb{O}_{p_j, n-p_j-1} \end{bmatrix} \mathbf{Q}_j^{-1} = \mathbf{A}_j. \end{aligned} \quad (119)$$

Hence, the solution of system $\mathbf{A}_j \mathbf{a} = \mathbf{d}_j(0^+)$ is given by (115). $\square$

3.4. The Solution of System (91). This subsection concludes the whole discussion. The following theorem holds.

Theorem 20. System (91) is solvable, that is,

$$\mathbf{A} \mathbf{a} = \mathbf{b}, \quad (120)$$

where

$$\mathbf{A} \triangleq \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix}, \quad \mathbf{b} \triangleq \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix}, \quad (121)$$

if the following conditions are satisfied:

$$\mathbf{V}_l [\mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (\mathbf{I}_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}_j] = \mathbf{d}_l, \quad (122)$$

$$\mathbf{A}_j [\mathbf{V}_l^{(1)} \mathbf{z}_l(0^+) + (\mathbf{I}_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}_j] = \mathbf{d}_j, \quad (123)$$

$$\mathbf{A}_i [\mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (\mathbf{I}_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}_j] = \mathbf{d}_i \quad (124)$$

for $i \neq j$ and $i, j = l + 1, l + 2, \dots, \kappa$.

Proof. Following Propositions 18 and 19, we have

- (i) for system $\mathbf{V}_l \mathbf{a} = \mathbf{z}_l^*(0^+)$, the solution is given by $\mathbf{a} = \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}$,
- (ii) for systems $\mathbf{A}_j \mathbf{a} = \mathbf{d}_j(0^+)$, the solution is given by $\mathbf{a} = \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}$ for arbitrarily chosen vector $\underline{\psi} \in \mathcal{M}_{n1}$ for $j = l+1, l+2, \dots, \kappa$.

There are infinitely many solutions, since the vector $\underline{\psi} \in \mathcal{M}_{n1}$ is arbitrary. Moreover, each of the above subsystems also contains the solution of system $\mathbf{Aa} = \mathbf{b}$. Thus, we assume that the desired solution is given by $\mathbf{a}$, then it should also satisfy the first system, that is, $\mathbf{V}_l \mathbf{a} = \mathbf{z}_l^*(0^+)$. Thus, we take

$$\begin{aligned}
 \mathbf{a} &= \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}, \\
 \mathbf{Aa} = \mathbf{b} &\Leftrightarrow \mathbf{A}(\mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}) = \mathbf{b} \\
 &\Leftrightarrow \mathbf{AV}_l^{(1)} \mathbf{z}_l^*(0^+) + \mathbf{A}(I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi} = \mathbf{b} \\
 &\Leftrightarrow \mathbf{AV}_l^{(1)} \mathbf{z}_l^*(0^+) + (\mathbf{A} - \mathbf{AV}_l^{(1)} \mathbf{V}_l) \underline{\psi} = \mathbf{b} \\
 &\Leftrightarrow \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + \begin{pmatrix} \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} - \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} \mathbf{V}_l^{(1)} \mathbf{V}_l \end{pmatrix} \underline{\psi} \\
 &\triangleq \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{A}_{l+1} \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) \\ \mathbf{A}_{l+2} \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) \\ \vdots \\ \mathbf{A}_\kappa \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) \end{bmatrix} \\
 &+ \begin{bmatrix} \mathbb{O}_l \\ \mathbf{A}_{l+1} (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \\ \mathbf{A}_{l+2} (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \\ \vdots \\ \mathbf{A}_\kappa (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \end{bmatrix} \underline{\psi} = \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix} \\
 &\Leftrightarrow \mathbf{A}_j \mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + \mathbf{A}_j (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi} = \mathbf{d}_j(0^+) \\
 &\Leftrightarrow \mathbf{A}_j [\mathbf{V}_l^{(1)} \mathbf{z}_l^*(0^+) + (I_n - \mathbf{V}_l^{(1)} \mathbf{V}_l) \underline{\psi}] = \mathbf{d}_j(0^+) \tag{125}
 \end{aligned}$$

for every $j = l+1, l+2, \dots, \kappa$. So (123) has been derived. Analogously, the system $\mathbf{Aa} = \mathbf{b}$ should also satisfy every subsystem $\mathbf{A}_j \mathbf{a} = \mathbf{d}_j(0^+)$ for $j = l+1, l+2, \dots, \kappa$.

Consequently, we have that

$$\begin{aligned}
 \mathbf{a} &= \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}, \\
 &\text{for every } j = l+1, l+2, \dots, \kappa, \\
 \mathbf{Aa} = \mathbf{b} &\Leftrightarrow \mathbf{A}(\mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi}) = \mathbf{b} \\
 &\Leftrightarrow \mathbf{AA}_j^{(1)} \mathbf{d}_j(0^+) + \mathbf{A}(I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \underline{\psi} = \mathbf{b} \\
 &\Leftrightarrow \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) + \begin{pmatrix} \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} - \begin{bmatrix} \mathbf{V}_l \\ \mathbf{A}_{l+1} \\ \mathbf{A}_{l+2} \\ \vdots \\ \mathbf{A}_\kappa \end{bmatrix} \mathbf{A}_j^{(1)} \mathbf{A}_j \end{pmatrix} \underline{\psi} \\
 &\triangleq \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \mathbf{V}_l \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) \\ \mathbf{A}_{l+1} \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) \\ \mathbf{A}_{l+2} \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) \\ \vdots \\ \mathbf{A}_\kappa \mathbf{A}_j^{(1)} \mathbf{d}_j(0^+) \end{bmatrix} \\
 &+ \begin{bmatrix} \mathbf{V}_l (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \\ \mathbf{A}_{l+1} (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \\ \mathbf{A}_{l+2} (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \\ \vdots \\ \mathbf{A}_\kappa (I_n - \mathbf{A}_j^{(1)} \mathbf{A}_j) \end{bmatrix} \underline{\psi} = \begin{bmatrix} \mathbf{z}_l^*(0^+) \\ \mathbf{d}_{l+1}(0^+) \\ \mathbf{d}_{l+2}(0^+) \\ \vdots \\ \mathbf{d}_\kappa(0^+) \end{bmatrix}. \tag{126}
 \end{aligned}$$

From the above equalities, the expressions (122)–(124) are obtained. $\square$

With this important theorem, the discussion of system (32) has finished. In the next section, we present the solution of system (24).

4. The Solution of Subsystem (24)

The differential system of (15) is given by

$$H_q \mathbf{y}_p^{(r)}(t) = \mathbf{y}_q(t) + \tilde{\mathbf{b}}_q u_o(t). \tag{127}$$

We start this subsection by observing that—as is well known, see Section 2—there exists a $q^* \in \mathbb{N}$ such that $H_q^{q^*} = \mathbb{O}$, that is, q^* is the annihilation index of H_q . We obtain

$$\begin{aligned}
 H_q \mathbf{y}_p^{(r)}(t) &= \mathbf{y}_q(t) + \tilde{\mathbf{b}}_q u_o(t) \Rightarrow H_q^{q^*} \mathbf{y}_p^{(r)}(t) \\
 &= H_q^{q^*-1} \mathbf{y}_q(t) + H_q^{q^*-1} \tilde{\mathbf{b}}_q u_o(t) \tag{128}
 \end{aligned}$$

or equivalently

$$H_q^{q^*-1} \underline{y}_q(t) + H_q^{q^*-1} \tilde{\underline{b}}_q u_o(t) = \underline{0}_q. \quad (129)$$

Afterwards, we present a condition, in order to accept the already known solution $\underline{a}$ (see Section 3).

We know that the nilpotent block matrix H_q is given by

$$H_{q_1} \oplus H_{q_2} \oplus \cdots \oplus H_{q_\mu}. \quad (130)$$

Now, taking into consideration that $H_q^{q^*} = \mathbb{O}$, it follows that for each

$$H_q^{q^*-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (131)$$

$$q^* = \max\{q_j : j = 1, 2, \dots, \mu\}.$$

Lemma 21 (see [5]). *All the elements of the block diagonal matrix $H_q^{q^*-1}$ are zero, that is, $H_{ij}^{q^*-1} = 0$ apart from the ξ -elements at positions $(1, q^*)$, $(q^* + 1, 2q^*)$, ..., $((\xi - 1)q^* + 1, \xi q^*)$ which are one. Note that ξ is the number of maximum block matrices with the same dimension, that is, $q^* \times q^*$, where $q^* = \max\{q_j : j = 1, 2, \dots, r\}$.*

Theorem 22. *The condition which makes the system (24) solvable and provides an acceptable choice of vector $\underline{a}$ is given by*

$$H_q^{q^*-1} \underline{y}_q(0^+) = \underline{0}_q. \quad (132)$$

Proof. First, we obtain system (129), that is, $H_q^{q^*-1} \underline{y}_q(t) + H_q^{q^*-1} \tilde{\underline{b}}_q u_o(t) = \underline{0}_q$. Then, for the solution of system (129), we take into account that

$$u_o(t) = \lim_{\sigma \rightarrow 0} \left(\alpha_0 \frac{1}{\sigma} \Phi\left(\frac{t}{\sigma}\right) + \alpha_1 \frac{1}{\sigma^2} \Phi\left(\frac{t}{\sigma}\right) + \cdots + \alpha_{n-1} \frac{1}{\sigma^n} \Phi^{(n-1)}\left(\frac{t}{\sigma}\right) \right). \quad (133)$$

Following Remark 1, we can assume that $t = K\sigma$ with K big enough so that

$$\Phi^{(k)}(K) \rightarrow 0, \quad \text{for every } k = 1, 2, \dots, n. \quad (134)$$

Moreover, we have $K\sigma \rightarrow 0^+$ and $t \rightarrow 0^+$. Then, we take

$$\begin{aligned} \lim_{t \rightarrow 0^+} \left(H_q^{q^*-1} \underline{y}_q(t) + H_q^{q^*-1} \tilde{\underline{b}}_q u_o(t) \right) \\ = \underline{0}_q \Leftrightarrow H_q^{q^*-1} \lim_{t \rightarrow 0^+} \underline{y}_q(t) + H_q^{q^*-1} \tilde{\underline{b}}_q \lim_{\sigma \rightarrow 0} u(t). \end{aligned} \quad (135)$$

However,

$$\begin{aligned} \lim_{\sigma \rightarrow 0} u(t) &= \lim_{\sigma \rightarrow 0} \left(\alpha_0 \frac{1}{\sigma} \Phi\left(\frac{t}{\sigma}\right) + \cdots + \alpha_{n-1} \frac{1}{\sigma^n} \Phi^{(n-1)}\left(\frac{t}{\sigma}\right) \right) \\ &= \lim_{\sigma \rightarrow 0} \left(\alpha_0 \frac{1}{\sigma} \Phi(K) + \cdots + \frac{1}{\sigma^n} \Phi^{(n-1)}(K) \right) = 0, \\ \lim_{t \rightarrow 0^+} \underline{y}_q(t) &= \underline{y}_q(0^+). \end{aligned} \quad (136)$$

Thus, (132) is derived. $\square$

An equivalent proof to [5] for Theorem 23 is provided.

Theorem 23. *A necessary condition for obtaining the solution of system (24) is given by*

$$\underline{x}(0^+) = (I_n - G^{(1)}G)X, \quad (137)$$

where

$$\begin{aligned} G &= \begin{bmatrix} \tilde{q}_{p+q^*,1} & \tilde{q}_{p+q^*,2} & \cdots & \tilde{q}_{p+q^*,n} \\ \tilde{q}_{p+2q^*,1} & \tilde{q}_{p+2q^*,2} & \cdots & \tilde{q}_{p+2q^*,n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{q}_{p+\xi q^*,1} & \tilde{q}_{p+\xi q^*,2} & \cdots & \tilde{q}_{p+\xi q^*,n} \end{bmatrix} \\ &\in \mathcal{M}_{\xi n}; \left(\tilde{q}_{ij} \right)_{i=p+q^*, \dots, p+\xi q^*} \\ &\quad j=1, 2, \dots, n} \end{aligned} \quad (138)$$

are elements of the matrix $\tilde{Q}$, the $G^{(1)}$ is $\{1\}$ -inverse of matrix G , and for an arbitrarily chosen $X \in \mathcal{M}_n$.

Proof. Considering the transformation (25), we obtain

$$\underline{y}(0^+) = Q^{-1} \underline{x}(0^+) \quad (139)$$

or equivalently by defining $\tilde{Q} \triangleq Q^{-1}$

$$\begin{aligned} \begin{bmatrix} \underline{y}_p(0^+) \\ \underline{y}_q(0^+) \end{bmatrix} &= \left(\tilde{q}_{ij} \right)_{i,j=1,2,\dots,n} \begin{bmatrix} \underline{x}_p(0^+) \\ \underline{x}_q(0^+) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \underline{y}_p(0^+) \\ \underline{y}_q(0^+) \end{bmatrix} \\ &= \left(\sum_{j=1}^n \tilde{q}_{ij} x_j(0^+) \right)_{i=1,2,\dots,n} \end{aligned} \quad (140)$$

and finally

$$H_q^{q^*-1} \underline{y}_q(0^+) = \underline{0}_q \Rightarrow \begin{cases} \sum_{j=1}^n \tilde{q}_{p+q^*,j} x_j(0^+) = 0, \\ \sum_{j=1}^n \tilde{q}_{p+2q^*,j} x_j(0^+) = 0, \\ \vdots \\ \sum_{j=1}^n \tilde{q}_{p+\xi q^*,j} x_j(0^+) = 0. \end{cases} \quad (141)$$

System (141) can be expressed as follows:

$$G \underline{x}(0^+) = \underline{0}, \quad (142)$$

where

$$G = \begin{bmatrix} \tilde{q}_{p+q^*,1} & \tilde{q}_{p+q^*,1} & \cdots & \tilde{q}_{p+q^*,n} \\ \tilde{q}_{p+2q^*,1} & \tilde{q}_{p+2q^*,2} & \cdots & \tilde{q}_{p+2q^*,n} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{q}_{p+\xi q^*,1} & \tilde{q}_{p+\xi q^*,2} & \cdots & \tilde{q}_{p+\xi q^*,n} \end{bmatrix} \in \mathcal{M}_{\xi n}. \quad (143)$$

The proof is concluded by using the results of Theorem 22. $\square$

5. An Illustrative Numerical Application

In this section, a numerical example of the above method is presented. We consider the second-order system

$$F\ddot{\underline{x}}(t) = G\underline{x}(t) + \underline{b}u(t), \quad (144)$$

where the matrix coefficients are given by

$$F = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -6 & 3 & -2 & 1 & -1 & -2 \\ 1 & -1 & 1 & -1 & 0 & -1 & -1 \\ 1 & -3 & 2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -3 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}, \quad (145)$$

$$G = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & -1 & -1 \\ -2 & -2 & -2 & 1 & 1 & 3 & 4 \\ -1 & 2 & 0 & 1 & -2 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 1 & 1 \\ -2 & -1 & -2 & 1 & 1 & 2 & 3 \end{bmatrix},$$

and vector coefficient $\underline{b} = [1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1]^t$.

Let us suppose that the system at time 0^- is

$$\underline{x}(0^-) = \underline{x}'(0^-) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^t, \quad (146)$$

and at time 0^+ it achieves

$$\underline{x}(0^+) = \underline{x}'(0^+) = [-3 \ -2 \ 2 \ 3 \ 1 \ 2 \ 1]^t. \quad (147)$$

From the regularity of $sF - G$, there exist nonsingular matrices in $\mathbb{R}^{7 \times 7}$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (148)$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

such that

$$\begin{aligned} PFQ &= F_w = \begin{bmatrix} I_2 & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & H_5 \end{bmatrix}, \\ PGQ &= G_w = \begin{bmatrix} J_2(1) & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & I_5 \end{bmatrix}, \end{aligned} \quad (149)$$

where I_2 , $J_2(1)$, H_5 are given by (17).

Consequently, system (144) is divided into the following subsystems:

$$\begin{aligned} \underline{y}_2''(t) &= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \underline{y}_2(t) + \underline{b}_2 u_o(t), \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & 0 & 1 \\ & 0 & 0 \end{bmatrix} \underline{y}_5''(t) &= \underline{y}_5(t) + \underline{b}_5 u_o(t). \end{aligned} \quad (150)$$

Moreover, we obtain

$$u_o(t) = a_0 \delta(t) + a_1 \delta^{(1)}(t) + \cdots + a_5 \delta^{(5)}(t) + a_6 \delta^{(6)}(t). \quad (151)$$

Before we go further with the analytic determination of the unknown coefficients a_i , we should verify the important condition (132). First, we calculate the inverse matrix of Q ,

$$Q^{-1} = (\tilde{q}_{ij})_{i,j=1,2,\dots,n} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix},$$

$$\underline{y}(0^+) = Q^{-1}\underline{x}(0^+) = \begin{bmatrix} y_2(0^+) \\ y_5(0^+) \end{bmatrix} = \begin{bmatrix} -5 & -2 & -3 & 1 & 0 & 5 & -1 \end{bmatrix}^t. \quad (152)$$

Consequently, it is not difficult to verify that condition (132) holds, that is, $H_5^2 y_5(0^+) = \underline{0}_5$.

Furthermore, the $\tilde{\underline{b}} \triangleq P\underline{b} = [0 \ 1 \ 2 \ 2 \ 1 \ 0 \ 1]^T$ should be calculated. Thus

$$\tilde{\underline{b}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{\underline{b}}_5 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad (153)$$

Now, the approximate expression for (151) is given by

$$u(t) = \frac{a_0}{\sigma} \Phi\left(\frac{t}{\sigma}\right) + \frac{a_1}{\sigma^2} \Phi^{(1)}\left(\frac{t}{\sigma}\right) + \dots + \frac{a_5}{\sigma^5} \Phi^{(5)}\left(\frac{t}{\sigma}\right) + \frac{a_6}{\sigma^6} \Phi^{(6)}\left(\frac{t}{\sigma}\right). \quad (154)$$

System (150) is a standard form of *nonhomogeneous second-order linear differential equations of Apostol-Kolodner type*. Thus, using the results of Section 3.1, we obtain

$$\underline{z}'(t) = \underline{R}\underline{z}(t) + \underline{L}u_o(t), \quad (155)$$

where $\underline{z}(t) = [\underline{z}_1^t(t) \ \underline{z}_2^t(t)]^t$ and the coefficient matrices $\underline{R}$ and $\underline{L}$ are given by

$$\underline{R} = \begin{bmatrix} \mathbb{O}_2 & I_2 \\ J_2(1) & \mathbb{O}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (156)$$

$$\underline{L} = \begin{bmatrix} \underline{0}_2 \\ \tilde{\underline{b}}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Considering (25), at time 0^+ we obtain

$$\underline{y}(0^+) = Q^{-1}\underline{x}(0^+) = \underline{y}'(0^+) = Q^{-1}\underline{x}'(0^+). \quad (157)$$

Moreover,

$$\underline{y}_2(0^+) = \begin{bmatrix} -5 \\ -2 \end{bmatrix}, \quad \underline{y}_2'(0^+) = \begin{bmatrix} -5 \\ -2 \end{bmatrix}. \quad (158)$$

Furthermore,

$$\underline{z}(0^+) = \begin{bmatrix} \underline{y}_2^t(0^+) & \underline{y}_2'^t(0^+) \end{bmatrix}^t = \begin{bmatrix} -5 \\ -2 \\ -5 \\ -2 \end{bmatrix}. \quad (159)$$

There exists an invertible matrix

$$C = \begin{bmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ 0 & -1/2 & 0 & 1/2 \\ -1/4 & 0 & 1/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \quad (160)$$

such that $\underline{J} = C^{-1}\underline{R}C$ is Jordan canonical form of matrix $\underline{R}$. Analytically,

$$\underline{J} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J_2(-1) \oplus J_2(1). \quad (161)$$

Now, taking into consideration (52) and Remark 1 the solution is given by

$$C^{-1}\underline{z}(t) = \begin{bmatrix} \lim_{\sigma \rightarrow 0} \left\{ \int_{-\infty}^t e^{-J_2(-1)\tau} \underline{b}_2 u(\tau) d\tau \right\} \\ \lim_{\sigma \rightarrow 0} \left\{ \int_{-\infty}^t e^{-J_2(1)\tau} \underline{b}_2 u(\tau) d\tau \right\} \end{bmatrix}, \quad (162)$$

where

$$\underline{b} \triangleq C^{-1}\underline{L} = \begin{bmatrix} \underline{b}_2 \\ \underline{b}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad (163)$$

Using (55), we obtain

$$\begin{aligned}
 C^{-1}\mathbf{z}(t) &= \begin{bmatrix} -\int_{-\infty}^t \tau e^{\tau} u(\tau) d\tau \\ \int_{-\infty}^t e^{\tau} u(\tau) d\tau \\ -\int_{-\infty}^t e^{-\tau} \tau u(\tau) d\tau \\ \int_{-\infty}^t e^{-\tau} u(\tau) d\tau \end{bmatrix} = \Phi^{-1}(K) \begin{bmatrix} \sum_{k=0}^6 \alpha_k \frac{d\lambda_1^k}{d\lambda_1} \\ \sum_{k=0}^6 \alpha_k \lambda_1^k \\ \sum_{k=0}^6 \alpha_k \frac{d\lambda_2^k}{d\lambda_2} \\ \sum_{k=0}^6 \alpha_k \lambda_2^k \end{bmatrix} \\
 &= \Phi^{-1}(K) \begin{bmatrix} 0 & 1 & -2 & 3 & -4 & 5 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{a}, \\
 \mathbf{A} &= \begin{bmatrix} \mathbf{A}_{\tau_1=2} \\ \mathbf{A}_{\tau_2=2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \quad (164)
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 \mathbf{A}_{\tau_1=2} &= \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 \end{bmatrix}, \\
 \mathbf{A}_{\tau_2=2} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}. \quad (165)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 P_1(2,1)\mathbf{A}_{\tau_1=2} &= \begin{bmatrix} 1 & 0 & -1 & 2 & -3 & 4 & -5 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 \end{bmatrix} = \tilde{\mathbf{A}}_{\tau_1=2}, \\
 \tilde{\mathbf{A}}_{\tau_1=2} Q_1(3,1) Q_2(3,2) Q_1(4,2) Q_2(4,-3) Q_1(5,3) \\
 \times Q_2(5,4) Q_1(6,4) Q_2(6,-5) Q_1(7,5) Q_2(7,6) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (166)
 \end{aligned}$$

where

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} 1 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 1 & 2 & -3 & 4 & -5 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (167)$$

Moreover,

$$\begin{aligned}
 P_1(2,-1)\mathbf{A}_{\tau_2=2} &= \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} = \tilde{\mathbf{A}}_{\tau_2=2}, \\
 \tilde{\mathbf{A}}_{\tau_1=2} Q_1(3,1) Q_2(3,2) Q_1(4,2) Q_2(4,-3) Q_1(5,3) \\
 \times Q_2(5,-4) Q_1(6,4) Q_2(6,-5) Q_1(7,5) Q_2(7,-6) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (168)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{P}_2 &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \\
 \mathbf{Q}_2 &= \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -2 & -3 & -4 & -5 & -6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (169)
 \end{aligned}$$

Thus, considering the result of Section 3, the $\{1\}$ -inverse of matrices $\mathbf{A}_{\tau_1=2}$ and $\mathbf{A}_{\tau_2=2}$ are given by,

$$Q_1 \begin{bmatrix} I_2 \\ \mathbb{O}_{5,2} \end{bmatrix} \mathbf{P}_1, \quad Q_2 \begin{bmatrix} I_2 \\ \mathbb{O}_{5,2} \end{bmatrix} \mathbf{P}_2, \text{ respectively,} \quad (170)$$

Finally, the general solution is given by the expression

$$\mathbf{a} = \frac{1}{\Phi^{-1}(K)} \mathbf{A}_{\tau_1=2}^{(1)} \mathbf{d}_{\tau_1=2}(0^+) + (I_7 - \mathbf{A}_{\tau_1=2}^{(1)} \mathbf{A}_{\tau_1=2}) \underline{\psi}, \quad (171)$$

where the following conditions should also be satisfied:

$$\begin{aligned}
 \mathbf{A}_{\tau_1=2} \left[\frac{1}{\Phi^{-1}(K)} \mathbf{A}_{\tau_2=2}^{(1)} \mathbf{d}_{\tau_2=2}(0^+) + (I_7 - \mathbf{A}_{\tau_2=2}^{(1)} \mathbf{A}_{\tau_2=2}) \underline{\psi} \right] &= \mathbf{d}_{\tau_1=2}, \\
 \mathbf{A}_{\tau_2=2} \left[\frac{1}{\Phi^{-1}(K)} \mathbf{A}_{\tau_1=2}^{(1)} \mathbf{d}_{\tau_1=2}(0^+) + (I_7 - \mathbf{A}_{\tau_1=2}^{(1)} \mathbf{A}_{\tau_1=2}) \underline{\psi} \right] &= \mathbf{d}_{\tau_2=2}. \quad (172)
 \end{aligned}$$

Analytically,

$$\underline{\mathbf{d}}(0^+) = C^{-1}\underline{\mathbf{z}}(0^+) = \begin{bmatrix} \underline{\mathbf{d}}_{r_1=2}(0^+) \\ \underline{\mathbf{d}}_{r_2=2}(0^+) \end{bmatrix} = \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \\ -22 \\ -4 \end{bmatrix},$$

$$\underline{\mathbf{a}} = \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 2 & -3 & 4 & -5 & 6 \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{bmatrix}, \quad (173)$$

where

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 \end{bmatrix}$$

$$\times \left[\frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 2 & -3 & 4 & -5 & 6 \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{bmatrix} \right]$$

$$= \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\psi_3 + \psi_4 - \psi_5 + \psi_6 - \psi_7 \\ 2\psi_3 - 3\psi_4 + 4\psi_5 - 5\psi_6 + 6\psi_7 \end{bmatrix}$$

$$= \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$\times \left[\frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -18 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & -3 & -4 & -5 & -6 \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \end{bmatrix} \right]$$

$$= \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -22 \\ -4 \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -22 \\ -4 \end{bmatrix} - \begin{bmatrix} \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7 \\ 2\psi_3 + 3\psi_4 + 4\psi_5 + 5\psi_6 + 6\psi_7 \end{bmatrix}$$

$$= \frac{1}{\Phi^{-1}(K)} \begin{bmatrix} -22 \\ -4 \end{bmatrix}. \quad (174)$$

Following the above conditions, we can see that $\psi_4 = \psi_6 = 0$, ψ_3, ψ_5 , and ψ_7 are arbitrary. Thus, we conclude this numerical example with the determination of the unknown coefficients a_i , for $i = 0, 1, \dots, 6$

$$\underline{\mathbf{a}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\Phi^{-1}(K)} + \psi_3 + 3\psi_5 + 5\psi_7 \\ 2\psi_3 + 4\psi_5 + 6\psi_7 \\ \psi_3 \\ 0 \\ \psi_5 \\ 0 \\ \psi_7 \end{bmatrix} \quad (175)$$

for arbitrary $\psi_i \in \mathbb{R}$, for $i = 3, 5, 7$ and $\Phi^{-1}(K) = (1/\sqrt{2\pi}) \int_{-\infty}^K e^{-x^2/2} dx$; K is large enough (i.e., $K \rightarrow \infty$) that the assumption $\lim_{x \rightarrow K} (1/\sqrt{2\pi}) e^{-x^2/2} = 0$ is valid.

6. Conclusions

In this paper, an analytical method and several sufficient conditions were proposed and discussed for the changing of the state of a higher-order linear descriptor (regular) differential system in (almost) zero time by using the impulse function and its derivatives. The input vector has to be made up a linear combination of the δ -function of Dirac and its derivatives approximated using the normal (Gaussian) probability density function. Using the tools of linear algebra and generalized inverses, exact calculations of the relevant matrices are obtained.

To date, no other approximating function has been used in the analysis of the problem studied in this paper. It will be of interest to use some other approximations and obtain comparative results. In [14, 18, 21], there are only some hints about the minimum time. However, there is no known formula which can calculate the minimum time considering some other significant problem parameters, such as the volatility σ , K and so forth.

Since we have applied an approximation for the impulse input, we transfer the initial state ε -close to a desired state. Consequently, a natural question is to ask how close we can get, and how we can calculate that distance, and if it is possible to minimize it.

Appendices

A. The Analytic Calculation of Integral

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 t} \underline{b}_s u(\tau) d\tau$$

Consider that

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 t} \underline{b}_s u(\tau) d\tau = \int_{-\infty}^t \begin{bmatrix} e^{-J_1 \tau} & & \\ & \ddots & \\ & & e^{-J_l \tau} \end{bmatrix} \begin{bmatrix} \underline{b}_{\tau_1} \\ \underline{b}_{\tau_2} \\ \vdots \\ \underline{b}_{\tau_l} \end{bmatrix} u(\tau) d\tau, \quad (\text{A.1})$$

and finally, we obtain

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 t} \underline{b}_s u(\tau) d\tau = \begin{bmatrix} \int_{-\infty}^t e^{-J_1 \tau} \underline{b}_{\tau_1} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-J_i \tau} \underline{b}_{\tau_i} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-J_l \tau} \underline{b}_{\tau_l} u(\tau) d\tau \end{bmatrix} \in \mathcal{M}_{s1} \quad (\text{A.2})$$

with $i = 1, 2, \dots, l$ and $\underline{b}_{\tau_i} \in \mathcal{M}_{(\tau_i)1}$. Moreover, we take for every $i = 1, 2, \dots, l$,

$$\begin{aligned} \int_{-\infty}^t e^{-J_i \tau} \underline{b}_{\tau_i} u(\tau) d\tau &= \int_{-\infty}^t e^{-\lambda_i \tau} \underline{b}_{\tau_i} u(\tau) d\tau \\ &= \underline{b}_{\tau_i} \int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau. \end{aligned} \quad (\text{A.3})$$

Afterwards, we calculate the integral $\int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau$.

Analytically, considering also (21), we obtain

$$\int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau = \int_{-\infty}^t e^{-\lambda_i \tau} \sum_{k=0}^{n-1} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) a_k d\tau, \quad (\text{A.4})$$

for every $i = 1, 2, \dots, q$. Thus,

$$\int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau = \sum_{k=0}^{n-1} a_k \int_{-\infty}^t e^{-\lambda_i \tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau, \quad (\text{A.5})$$

where $(1/\sigma)\Phi^0(\tau/\sigma) = (1/\sigma)\Phi(\tau/\sigma)$ and $\Phi^{(m)}(\tau/\sigma) = d^m \Phi(\tau/\sigma)/d(\tau/\sigma)^m$.

Remark 5. It is well-known that $\int_{-\infty}^t (1/\sqrt{2\pi}) e^{-\omega^2/2} d\omega = \Phi^{-1}(t)$. Consequently,

$$\int_{-\infty}^t \Phi(\omega) d\omega = \Phi^{-1}(t), \quad (\text{A.6})$$

where $\Phi^{-1}(t)$ is the inverse function of normal probability distribution which is also a continuous one.

Lemma 24. One has

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda \tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \\ = e^{-\lambda t} \left(\sum_{\ell=0}^{k-1} \lambda^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda^k e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda \sigma\right). \end{aligned} \quad (\text{A.7})$$

Proof. First, we calculate the integral

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda \tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \\ = e^{(\lambda^2 \sigma^2)/2} \cdot \frac{1}{\sigma} \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-(\tau/\sigma + \lambda \sigma)^2/2} d\tau. \end{aligned} \quad (\text{A.8})$$

Afterwards, we denote $(\tau/\sigma) + \lambda \sigma = \omega$. Then, $d\tau = \sigma \cdot d\omega$.

Consequently, we obtain

$$e^{\lambda^2 \sigma^2/2} \frac{1}{\sigma} \int_{-\infty}^{t/\sigma + \lambda \sigma} \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} \sigma d\omega = e^{\lambda^2 \sigma^2/2} \int_{-\infty}^{t/\sigma + \lambda \sigma} \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} d\omega. \quad (\text{A.9})$$

Then,

$$\int_{-\infty}^t e^{-\lambda \tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau = e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda \sigma\right). \quad (\text{A.10})$$

Continuing our calculations, we take

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda \tau} \frac{\Phi^{(1)}(\tau/\sigma)}{\sigma^2} d\tau \\ = \frac{1}{\sigma} \left[e^{-\lambda \tau} \Phi\left(\frac{\tau}{\sigma}\right) \right]_{-\infty}^t + \lambda \frac{1}{\sigma} \int_{-\infty}^t e^{-\lambda \tau} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \\ \left(\text{But } \lim_{\tau \rightarrow -\infty} e^{-\lambda \tau} \Phi\left(\frac{\tau}{\sigma}\right) = 0 \right). \end{aligned} \quad (\text{A.11})$$

Moreover, considering also (A.10), we obtain

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda\tau} \frac{\Phi^{(1)}(\tau/\sigma)}{\sigma^2} d\tau \\ = \frac{1}{\sigma} e^{-\lambda t} \Phi\left(\frac{t}{\sigma}\right) + \lambda e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.12})$$

Continuing the above procedure, we have to calculate the integral $\int_{-\infty}^t e^{-\lambda\tau} (\Phi^{(2)}(\tau/\sigma)/\sigma^3) d\tau$. Thus,

$$\int_{-\infty}^t e^{-\lambda\tau} \frac{\Phi^{(2)}(\tau/\sigma)}{\sigma^3} d\tau = \quad (\text{A.13})$$

(we denote $\tau/\sigma = z \Leftrightarrow \tau = \sigma z$, then $d\tau = \sigma dz$ and $-\infty/\sigma < z < t/\sigma$. Note that $-\infty/\sigma = -\infty$ because $\sigma > 0$ —by definition of the normal probability distribution)

$$= \int_{-\infty}^{t/\sigma} e^{-\lambda\sigma z} \frac{\Phi^{(2)}(z)}{\sigma^3} \cdot \sigma dz \quad (\text{A.14})$$

$$= \frac{1}{\sigma^2} e^{-\lambda t} \Phi^{(1)}\left(\frac{t}{\sigma}\right) + \frac{1}{\sigma^2} \lambda \sigma \int_{-\infty}^{t/\sigma} e^{-\lambda\sigma z} \Phi^{(1)}(z) dz =$$

(we reuse the transformation $z = \tau/\sigma$)

$$\begin{aligned} &= \frac{1}{\sigma^2} e^{-\lambda t} \Phi^{(1)}\left(\frac{t}{\sigma}\right) \\ &+ \lambda \left[\frac{1}{\sigma} e^{-\lambda t} \Phi\left(\frac{t}{\sigma}\right) + \lambda e^{(\lambda^2 \sigma^2)/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right) \right]. \end{aligned} \quad (\text{A.15})$$

Consequently, we obtain

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda\tau} \frac{\Phi^{(2)}(\tau/\sigma)}{\sigma^3} d\tau &= \frac{1}{\sigma^2} e^{-\lambda t} \Phi^{(1)}\left(\frac{t}{\sigma}\right) + \lambda \frac{1}{\sigma} e^{-\lambda t} \Phi\left(\frac{t}{\sigma}\right) \\ &+ \lambda^2 e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.16})$$

Following the previous expressions, we assume that

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \\ = e^{-\lambda t} \left(\sum_{\ell=0}^{k-1} \lambda^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda^k e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.17})$$

Now, in order to conclude the proof, the recursive formula should be proved, that is,

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda\tau} \frac{1}{\sigma^{k+2}} \Phi^{(k+1)}\left(\frac{\tau}{\sigma}\right) d\tau \\ = e^{-\lambda t} \left(\sum_{\ell=0}^k \lambda^{k-\ell} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda^{k+1} e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.18})$$

Thus, we have

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda\tau} \frac{1}{\sigma^{k+2}} \Phi^{(k+1)}\left(\frac{\tau}{\sigma}\right) d\tau \quad \left(\text{denote } \frac{\tau}{\sigma} = z\right) \\ = \frac{1}{\sigma^{k+1}} e^{-\lambda t} \Phi^{(k)}\left(\frac{t}{\sigma}\right) + \lambda \frac{1}{\sigma^{k+1}} \int_{-\infty}^{t/\sigma} e^{-\lambda\sigma z} \Phi^{(k)}(z) \sigma dz \end{aligned} \quad (\text{A.19})$$

(reusing the transformation $z = \tau/\sigma$)

$$\begin{aligned} &\stackrel{(\text{A.16})}{=} \frac{1}{\sigma^{k+1}} e^{-\lambda t} \Phi^{(k)}\left(\frac{t}{\sigma}\right) + e^{-\lambda t} \sum_{\ell=0}^{k-1} \lambda^{k-\ell} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \\ &+ \lambda^{k+1} e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.20})$$

However, the

$$\begin{aligned} &\frac{1}{\sigma^{k+1}} e^{-\lambda t} \Phi^{(k)}\left(\frac{t}{\sigma}\right) + e^{-\lambda t} \sum_{\ell=0}^{k-1} \lambda^{k-\ell} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \\ &= e^{-\lambda t} \sum_{\ell=0}^k \lambda^{k-\ell} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}}. \end{aligned} \quad (\text{A.21})$$

Consequently, we have proved that

$$\begin{aligned} \int_{-\infty}^t \sigma^{1/(k+2)} e^{-\lambda\tau} \Phi^{(k+1)}\left(\frac{\tau}{\sigma}\right) d\tau \\ = e^{-\lambda t} \left(\sum_{\ell=0}^k \lambda^{k-\ell} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda^{k+1} e^{\lambda^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right). \end{aligned} \quad (\text{A.22})$$

And the proof has been concluded. $\square$

Lemma 25. One has

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau \\ = \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_i t} \left(\sum_{\ell=0}^{k-1} \lambda_i^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) \right. \\ \left. + \lambda_i^k e^{\lambda_i^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda_i \sigma\right) \right]. \end{aligned} \quad (\text{A.23})$$

Proof. We have seen that

$$\begin{aligned} \int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau \\ \stackrel{(\text{A.17})}{=} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_i t} \left(\sum_{\ell=0}^{k-1} \lambda_i^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) \right. \\ \left. + \lambda_i^k e^{\lambda_i^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda_i \sigma\right) \right]. \end{aligned} \quad (\text{A.24})$$

$\square$

Substituting (A.23) into (A.3), we obtain

$$\begin{aligned} \underline{\mathbf{b}}_{\tau_i} \int_{-\infty}^t e^{-\lambda_i \tau} u(\tau) d\tau \\ = \underline{\mathbf{b}}_{\tau_i} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_i t} \left(\sum_{\ell=0}^{k-1} \lambda_i^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) \right. \\ \left. + \lambda_i^k e^{\lambda_i^2 \sigma^2/2} \Phi^{-1}\left(\frac{t}{\sigma} + \lambda_i \sigma\right) \right]. \end{aligned} \quad (\text{A.25})$$

Thus, combining (A.2) and (A.25), we obtain

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 t} \underline{\mathbf{b}}_s u(\tau) d\tau = \begin{bmatrix} \underline{\mathbf{b}}_{\tau_1} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_1 t} \left(\sum_{\ell=0}^{k-1} \lambda_1^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda_1^k e^{\lambda_1^2 \sigma^2/2} \Phi^{-1} \left(\frac{t}{\sigma} + \lambda_1 \sigma \right) \right] \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_l t} \left(\sum_{\ell=0}^{k-1} \lambda_l^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda_l^k e^{\lambda_l^2 \sigma^2/2} \Phi^{-1} \left(\frac{t}{\sigma} + \lambda_l \sigma \right) \right] \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_l t} \left(\sum_{\ell=0}^{k-1} \lambda_l^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda_l^k e^{\lambda_l^2 \sigma^2/2} \Phi^{-1} \left(\frac{t}{\sigma} + \lambda_l \sigma \right) \right] \end{bmatrix} \in \mathcal{M}_{s1}. \quad (\text{A.26})$$

Lemma 26. One has

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \sum_{k=0}^{n-1} \alpha_k \left[e^{-\lambda_i t} \left(\sum_{\ell=0}^{k-1} \lambda_i^{k-\ell-1} \frac{\Phi^{(\ell)}(t/\sigma)}{\sigma^{\ell+1}} \right) + \lambda_i^k e^{\lambda_i^2 \sigma^2/2} \Phi^{-1} \left(\frac{t}{\sigma} + \lambda_i \sigma \right) \right] \\ \approx \Phi^{-1}(K) \sum_{k=0}^{n-1} \alpha_k \lambda_i^k \quad \text{for } i = 1, 2, \dots, l. \end{aligned} \quad (\text{A.27})$$

Proof. Considering here, Remark 2.1, and denoting $t = K \sigma$ with K big enough such as

$$\frac{\Phi^{(k)}(K)}{\sigma^{k+1}} \approx 0 \quad \text{for every } k = 0, 1, 2, \dots, n-1, \quad (\text{A.28})$$

then we obtain (A.27). $\square$

This appendix is finished with (A.29). This equation is a consequence of (A.26) and (A.27)

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_0 t} \underline{\mathbf{b}}_s u(\tau) d\tau = \Phi^{-1}(K) \begin{bmatrix} \underline{\mathbf{b}}_{\tau_1} \sum_{k=0}^{n-1} \alpha_k \lambda_1^k \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \\ \vdots \\ \underline{\mathbf{b}}_{\tau_l} \sum_{k=0}^{n-1} \alpha_k \lambda_l^k \end{bmatrix} \in \mathcal{M}_{s1}. \quad (\text{A.29})$$

B. The Analytic Calculation of Integral

$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_i t} \underline{\mathbf{b}}_{\tau_i} u(\tau) d\tau$ for $i = l+1, l+2, \dots, k$

Consider that

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_j t} \underline{\mathbf{b}}_{\tau_j} u(\tau) d\tau \\ = \int_{-\infty}^t \begin{bmatrix} e^{-J_{j1} \tau} \\ e^{-J_{j2} \tau} \\ \vdots \\ e^{-J_{jd_j} \tau} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{b}}_{j1} \\ \underline{\mathbf{b}}_{j2} \\ \vdots \\ \underline{\mathbf{b}}_{jd_j} \end{bmatrix} u(\tau) d\tau \end{aligned}$$

$$= \begin{bmatrix} \int_{-\infty}^t e^{-J_{j1} \tau} \mathbf{T}_{j1} u(\tau) d\tau \\ \int_{-\infty}^t e^{-J_{j2} \tau} \mathbf{T}_{j2} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-J_{jd_j} \tau} \mathbf{T}_{jd_j} u(\tau) d\tau \end{bmatrix} \quad (\text{B.1})$$

with $z_j = 1, 2, \dots, d_j$ and the matrix $\underline{\mathbf{b}}_{jz_j} \in \mathcal{M}_{(jz_j)1}$.

Thus,

$$\begin{aligned} \int_{-\infty}^t e^{-J_{z_j} \tau} \underline{\mathbf{b}}_{j_{z_j}} u(\tau) d\tau &= \int_{-\infty}^t \begin{bmatrix} e^{-\lambda_j \tau} & (-1)\tau e^{-\lambda_j \tau} & \dots & (-1)^{(\mu_{z_j}-1)} \frac{\tau^{(\mu_{z_j}-1)}}{(\mu_{z_j}-1)!} e^{-\lambda_j \tau} \\ 0 & e^{-\lambda_j \tau} & \dots & (-1)^{(\mu_{z_j}-2)} \frac{\tau^{(\mu_{z_j}-2)}}{(\mu_{z_j}-2)!} e^{-\lambda_j \tau} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_j \tau} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{j_{z_j}}^1 \\ \mathbf{b}_{j_{z_j}}^2 \\ \vdots \\ \mathbf{b}_{j_{z_j}}^{\mu_j} \end{bmatrix} u(\tau) d\tau \\ &= \begin{bmatrix} \int_{-\infty}^t e^{-\lambda_j \tau} \mathbf{b}_{j_{z_j}}^1 u(\tau) d\tau + \int_{-\infty}^t (-1)\tau e^{-\lambda_j \tau} \mathbf{b}_{j_{z_j}}^2 u(\tau) d\tau + \dots + \int_{-\infty}^t (-1)^{(\mu_{z_j}-1)} \frac{\tau^{(\mu_{z_j}-1)}}{(\mu_{z_j}-1)!} e^{-\lambda_j \tau} \mathbf{b}_{j_{z_j}}^{\mu_j} u(\tau) d\tau \\ \int_{-\infty}^t e^{-\lambda_j \tau} \mathbf{b}_{j_{z_j}}^2 u(\tau) d\tau + \dots + \int_{-\infty}^t (-1)^{(\mu_{z_j}-2)} \frac{\tau^{(\mu_{z_j}-2)}}{(\mu_{z_j}-2)!} \mathbf{b}_{j_{z_j}}^{\mu_j} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-\lambda_j \tau} \mathbf{b}_{j_{z_j}}^{\mu_j} u(\tau) d\tau \end{bmatrix}. \end{aligned} \quad (\text{B.2})$$

Remark 6. According to (B.2), in order to be able to calculate the integral $\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_{z_j} \tau} \underline{\mathbf{b}}_{j_{z_j}} u(\tau) d\tau$, we should calculate the $\lim_{\sigma \rightarrow 0} \int_{-\infty}^t (\tau^\rho / \rho!) e^{-\lambda \tau} u(\tau) d\tau$.

Lemma 27. One has

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda \tau} u(\tau) d\tau = (-1)^\rho \left(\sum_{k=0}^{n-1} \alpha_k \frac{d^\rho \lambda^k}{d\lambda^\rho} \frac{1}{\rho!} \right) \Phi^{-1}(K). \quad (\text{B.3})$$

However, before we prove Lemma 27, the following lemmas should be considered firstly.

Lemma 28. One has

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} u(\tau) d\tau = - \sum_{k=0}^{n-1} \alpha_k \frac{d\lambda^k}{d\lambda} \Phi^{-1}(K). \quad (\text{B.4})$$

Proof. After some basic calculations, we obtain

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} u(\tau) d\tau = \sum_{k=0}^{n-1} \alpha_k \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau. \quad (\text{B.5})$$

First, we calculate

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \\ = \lim_{\sigma \rightarrow 0} \left(\int_{-\infty}^t \tau \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-((\tau/\sigma)+\lambda\sigma)^2/2} d\tau \cdot e^{(\lambda^2\sigma^2)/2} \right). \end{aligned} \quad (\text{B.6})$$

Then $\lim_{\sigma \rightarrow 0} e^{\lambda^2\sigma^2/2} = 1$ and we obtain $\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau (1/\sqrt{2\pi}) e^{-((\tau/\sigma)+\lambda\sigma)^2/2} d\tau$.

Afterwards, we denote $(\tau/\sigma) + \lambda\sigma = z \Leftrightarrow \tau = \sigma z - \lambda\sigma^2$ and $d\tau = \sigma dz$.

And, we take

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{-\infty}^{(t/\sigma)+\lambda\sigma} (\sigma z - \lambda\sigma^2) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-z^2/2} \sigma dz \\ = \lim_{\sigma \rightarrow 0} \left[-\frac{\sigma}{\sqrt{2\pi}} e^{-((t/\sigma)+\lambda\sigma)^2/2} - \sigma^2 \lambda \Phi^{-1}\left(\frac{t}{\sigma} + \lambda\sigma\right) \right]. \end{aligned} \quad (\text{B.7})$$

In this part, we use also Remark 2.1, with as $t = K\sigma$ with $\sigma \rightarrow 0$ and $K \gg 0$. Thus

$$\lim_{\sigma \rightarrow 0} \left[-\frac{\sigma}{\sqrt{2\pi}} e^{-(K+\lambda\sigma)^2/2} - \sigma^2 \lambda \Phi^{-1}(K + \lambda\sigma) \right] = 0. \quad (\text{B.8})$$

Moreover,

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau = 0. \quad (\text{B.9})$$

Now, we calculate

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda \tau} \frac{\Phi^{(1)}(\tau/\sigma)}{\sigma^2} d\tau \left(\text{denote } \frac{\tau}{\sigma} = z. \text{ Then } d\tau = \sigma dz \right) \\ = \lim_{\sigma \rightarrow 0} \left[\left[z e^{-\lambda \sigma z} \Phi(z) \right]_{-\infty}^{t/\sigma} - \int_{-\infty}^{t/\sigma} (e^{-\lambda \sigma z} - \lambda \sigma z e^{-\lambda \sigma z}) \Phi(z) dz \right]. \end{aligned} \quad (\text{B.10})$$

The $\lim_{\sigma \rightarrow 0} [ze^{-\lambda\sigma z}\Phi(z)]_{-\infty}^{t/\sigma} = \lim_{\sigma \rightarrow 0} t/\sigma e^{-\lambda t}\Phi(t/\sigma) = 0$, as we have seen (using Remark 2.1).

Then, we obtain

$$\begin{aligned} & -\lim_{\sigma \rightarrow 0} \int_{-\infty}^{t/\sigma} e^{-\lambda\sigma z}\Phi(z)dz + \lambda \lim_{\sigma \rightarrow 0} \int_{-\infty}^{t/\sigma} \sigma z e^{-\lambda\sigma z}\Phi(z)dz \\ & = -\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \\ & \quad + \lambda \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau = \end{aligned} \quad (\text{B.11})$$

(using (B.9) and (A.10)) $= -\Phi^{-1}(K) + \lambda \cdot 0 = -\Phi^{-1}(K)$.

Thus, we conclude

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{\Phi^{(1)}(t/\sigma)}{\sigma^2} d\tau = -\Phi^{-1}(K). \quad (\text{B.12})$$

Now, we calculate

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{\Phi^{(2)}(\tau/\sigma)}{\sigma^3} d\tau \\ & = \lim_{\sigma \rightarrow 0} \left(\frac{1}{\sigma} \left[ze^{-\lambda\sigma z}\Phi^{(1)}(z) \right]_{-\infty}^{t/\sigma} \right. \\ & \quad \left. - \frac{1}{\sigma} \int_{-\infty}^{t/\sigma} (e^{-\lambda\sigma z} - \lambda\sigma z e^{-\lambda\sigma z}) \Phi^{(1)}(z) dz \right) \end{aligned} \quad (\text{B.13})$$

(since $\sigma \rightarrow 0$ and $t = K\sigma$, then $[ze^{-\lambda\sigma z}\Phi^{(1)}(z)]_{-\infty}^{t/\sigma} \rightarrow 0$). Thus, we obtain

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \left(-\frac{1}{\sigma} \int_{-\infty}^{t/\sigma} e^{-\lambda\sigma z}\Phi^{(1)}(z) dz + \lambda \frac{1}{\sigma} \int_{-\infty}^{t/\sigma} \sigma z e^{-\lambda\sigma z}\Phi^{(1)}(z) dz \right) \\ & = \lim_{\sigma \rightarrow 0} \left(-\int_{-\infty}^t e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{\tau}{\sigma}\right) d\tau + \lambda \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{\tau}{\sigma}\right) d\tau \right) \end{aligned} \quad (\text{B.14})$$

(using (B.9) and (A.12)) $= -\lambda\Phi^{-1}(K) - \lambda\Phi^{-1}(K) = -2\lambda\Phi^{-1}(K)$.

Consequently,

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{\Phi^{(2)}(\tau/\sigma)}{\sigma^3} d\tau = -2\lambda\Phi^{-1}(K). \quad (\text{B.15})$$

Generalized the above expression, we take

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{\Phi^{(k)}(\tau/\sigma)}{\sigma^{k+1}} d\tau = -\frac{d\lambda^k}{d\lambda} \Phi^{-1}(K). \quad (\text{B.16})$$

Now, it should be proved the recursive (B.17)

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \tau e^{-\lambda\tau} \frac{\Phi^{(k+1)}(\tau/\sigma)}{\sigma^{k+2}} d\tau = -\frac{d\lambda^{k+1}}{d\lambda} \Phi^{-1}(K). \quad (\text{B.17})$$

Thus, we obtain

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^{t/\sigma} \sigma z e^{-\lambda\sigma z} \frac{\Phi^{(k+1)}(z)}{\sigma^{k+2}} \sigma dz \\ & = \lim_{\sigma \rightarrow 0} \left(-\frac{1}{\sigma^k} \int_{-\infty}^t e^{-\lambda\tau} \frac{\Phi^{(k)}(\tau/\sigma)}{\sigma} d\tau \right. \\ & \quad \left. + \lambda \int_{-\infty}^t \tau \frac{1}{\sigma^{k+1}} \Phi^{(k)}(\tau/\sigma) d\tau \right) = \end{aligned} \quad (\text{B.18})$$

(using (B.16) and (A.16)) $= -(\frac{d\lambda^{k+1}}{d\lambda})\Phi^{-1}(K)$. The above expression concludes the recursive proof. $\square$

Lemma 29. One has

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} u(\tau) d\tau = (-1)^2 \frac{1}{2!} \left(\sum_{k=0}^{n-1} \alpha_k \frac{d^2\lambda^k}{d\lambda^2} \right) \Phi^{-1}(K). \quad (\text{B.19})$$

Proof. Consider that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} u(\tau) d\tau \\ & = \sum_{k=0}^{n-1} \alpha_k \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau. \end{aligned} \quad (\text{B.20})$$

First, we calculate

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \\ & = \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-((\tau/\sigma)+\lambda\sigma)^2/2} d\tau = \\ & \quad \left(\text{Denote } \frac{\tau}{\sigma} + \lambda\sigma = z \Leftrightarrow \tau = \sigma z - \lambda\sigma^2, d\tau = \sigma dz \right) \\ & = \lim_{\sigma \rightarrow 0} \int_{-\infty}^{t/\sigma+\lambda\sigma} \frac{1}{2!} (\sigma z - \lambda\sigma^2)^2 \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma dz \\ & = \lim_{\sigma \rightarrow 0} \left(\sigma^2 \frac{1}{2!} \int_{-\infty}^{(t/\sigma)+\lambda\sigma} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - \frac{2\lambda\sigma^3}{2!} \int_{-\infty}^{(t/\sigma)+\lambda\sigma} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \right. \\ & \quad \left. + \frac{\lambda^2\sigma^4}{2!} \int_{-\infty}^{(t/\sigma)+\lambda\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \right). \end{aligned} \quad (\text{B.21})$$

Following Remark 2.1, we assume that $\sigma \rightarrow 0$ and $t = K\sigma$ with $K \gg 0$. Thus,

$$\begin{aligned} & \int_{-\infty}^{(t/\sigma)+\lambda\sigma} z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \approx 0, \\ & \int_{-\infty}^{(t/\sigma)+\lambda\sigma} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \approx 0, \end{aligned} \quad (\text{B.22})$$

$$\int_{-\infty}^{(t/\sigma)+\lambda\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \approx \Phi^{-1}(K).$$

Moreover, we take $\lim_{\sigma \rightarrow 0} (\lambda^2 \sigma^4 / 2!) \Phi^{-1}(K) = 0$. And

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau = 0 \cdot \Phi^{-1}(K) = 0. \quad (\text{B.23})$$

Then, we calculate that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{\tau}{\sigma}\right) d\tau \\ &= \lim_{\sigma \rightarrow 0} \int_{-\infty}^{t/\sigma} \frac{\sigma^2 z^2}{2!} e^{-\lambda\sigma z} \frac{1}{\sigma^2} \Phi^{(1)}(z) \sigma dz \\ &= \lim_{\sigma \rightarrow 0} \left(-\int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau + \lambda \int_{-\infty}^t \tau^2 e^{-\lambda\tau} \frac{1}{\sigma} \Phi\left(\frac{\tau}{\sigma}\right) d\tau \right) \end{aligned} \quad (\text{B.24})$$

(using (B.9) and (B.23)) = 0.

Thus, we obtain

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{\tau}{\sigma}\right) d\tau = 0. \quad (\text{B.25})$$

Afterwards, we calculate

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^3} \Phi^{(2)}\left(\frac{\tau}{\sigma}\right) d\tau \\ &= \lim_{\sigma \rightarrow 0} \left(-\int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}\left(\frac{\tau}{\sigma}\right) d\tau \right. \\ & \quad \left. + \frac{1}{2} \lambda \int_{-\infty}^t \tau^2 e^{-\lambda\tau} \frac{1}{\sigma^2} \Phi^{(1)}(\tau) d\tau \right) = \end{aligned} \quad (\text{B.26})$$

(using (B.12) and (B.25)) = $-(-\Phi^{-1}(K) + 0) = \Phi^{-1}(K)$.

Hence, we conclude that

$$\int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^3} \Phi^{(2)}\left(\frac{\tau}{\sigma}\right) d\tau = \Phi^{-1}(K). \quad (\text{B.27})$$

Now, we calculate that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^4} \Phi^{(3)}\left(\frac{\tau}{\sigma}\right) d\tau \\ &= \lim_{\sigma \rightarrow 0} \left(-\int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma^3} \Phi^{(2)}\left(\frac{\tau}{\sigma}\right) d\tau \right. \\ & \quad \left. + \frac{1}{2} \lambda \int_{-\infty}^t \tau^2 e^{-\lambda\tau} \frac{1}{\sigma^3} \Phi^{(2)}\left(\frac{\tau}{\sigma}\right) d\tau \right) = \end{aligned} \quad (\text{B.28})$$

(using (B.15) and (B.27)) = $-(-2\lambda\Phi^{-1}(K)) + \lambda\Phi^{-1}(K) = 3\lambda\Phi^{-1}(K)$.

Hence, we obtain

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^4} \Phi^{(3)}\left(\frac{\tau}{\sigma}\right) d\tau = 3\lambda\Phi^{-1}(K). \quad (\text{B.29})$$

Combining the equations (B.23)–(B.29), we obtain recursively (B.30)

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau = \frac{1}{2!} \frac{d^2 \lambda^k}{d\lambda^2} \Phi^{-1}(K). \quad (\text{B.30})$$

Thus, in order to conclude, we should prove

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^{k+2}} \Phi^{(k+1)}\left(\frac{\tau}{\sigma}\right) d\tau = \frac{1}{2!} \frac{d^2 \lambda^{k+1}}{d\lambda^2} \Phi^{-1}(K). \quad (\text{B.31})$$

Hence,

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^{k+2}} \Phi^{(k+1)}\left(\frac{\tau}{\sigma}\right) d\tau \\ &= \lim_{\sigma \rightarrow 0} \frac{1}{2!} \frac{1}{\sigma^{k+1}} \int_{-\infty}^{t/\sigma} z^2 e^{-\lambda\sigma z} \Phi^{(k+1)}(z) dz \\ &= \lim_{\sigma \rightarrow 0} \left(-\int_{-\infty}^t \tau e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \right. \\ & \quad \left. + \lambda \int_{-\infty}^t \frac{\tau^2}{2!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \right) \end{aligned} \quad (\text{B.32})$$

(using (B.16) and (B.30)) = $((k+1)k/2!) \lambda^{k-1} \Phi^{-1}(K) = (1/2!) (d^2 \lambda^{k+1} / d\lambda^2) \Phi^{-1}(K)$.

The proof is concluded by considering. $\square$

Lemma 30. One has

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau = (-1)^\rho \frac{1}{\rho!} \frac{d^\rho \lambda^k}{d\lambda^\rho} \Phi^{-1}(K). \quad (\text{B.33})$$

Proof. For $\rho = 1$, we have proved (B.16).

For $\rho = 2$, we have also proved (B.30).

We assume that (B.33) is true.

Now, we easily have to prove the following expression for $\rho + 1$, that is,

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^{\rho+1}}{(\rho+1)!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \\ &= (-1)^{\rho+1} \frac{1}{(\rho+1)!} \frac{d^{\rho+1} \lambda^k}{d\lambda^{\rho+1}} \Phi^{-1}(K). \end{aligned} \quad (\text{B.34})$$

$\square$

Now, we return to the proof of Lemma 27.

Proof. Consider that

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda\tau} u(\tau) d\tau \\ &= \sum_{k=0}^{n-1} \alpha_k \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau. \end{aligned} \quad (\text{B.35})$$

Then,

$$\begin{aligned} & \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda\tau} u(\tau) d\tau \\ &= \sum_{k=0}^{n-1} \alpha_k \cdot \lim_{\sigma \rightarrow 0} \int_{-\infty}^t \frac{\tau^\rho}{\rho!} e^{-\lambda\tau} \frac{1}{\sigma^{k+1}} \Phi^{(k)}\left(\frac{\tau}{\sigma}\right) d\tau \end{aligned} \quad (\text{B.36})$$

(considering (B.33))

$$\begin{aligned} &= \sum_{k=0}^{n-1} \alpha_k (-1)^\rho \frac{1}{\rho!} \frac{d^\rho \lambda^k}{d\lambda^\rho} \Phi^{-1}(K) \\ &= (-1)^\rho \frac{1}{\rho!} \Phi^{-1}(K) \left(\sum_{k=0}^{n-1} \frac{1}{\rho!} \frac{d^\rho \lambda^k}{d\lambda^\rho} \right). \end{aligned} \quad (\text{B.37})$$

□

Remark 7. In all the proofs of Lemma 27–30, we have used

$$\lim_{\sigma \rightarrow 0} \left[z^\rho e^{-\lambda \sigma z} \Phi^{(k)}(z) \right]^{t/\sigma} = 0 \quad (\text{B.38})$$

for every κ and every ρ . In this part of Appendix B, we return to expression (B.2) and we calculate

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_{jz_j} \tau} \underline{\mathbf{b}}_{jz_j} u(\tau) d\tau = \quad (\text{B.39})$$

(using (A.23) and (B.3), and calculating the limit as $\sigma \rightarrow 0$, we obtain what follows)

$$\begin{aligned} &\lim_{\sigma \rightarrow 0} \begin{bmatrix} \int_{-\infty}^t e^{-\lambda_j \tau} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-1} \frac{\tau^{c-1}}{(c-1)!} u(\tau) d\tau \\ \int_{-\infty}^t e^{-\lambda_j \tau} \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-2} \frac{\tau^{c-2}}{(c-2)!} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-\lambda_j \tau} \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-\mu_{z_j}} \frac{\tau^{c-\mu_{z_j}}}{(c-\mu_{z_j})!} u(\tau) d\tau \end{bmatrix} \\ &= \begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-1} \cdot \lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-\lambda_j \tau} \frac{\tau^{c-1}}{(c-1)!} u(\tau) d\tau \\ \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-2} \cdot \lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-\lambda_j \tau} \frac{\tau^{c-2}}{(c-2)!} u(\tau) d\tau \\ \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{c-\mu_{z_j}} \cdot \lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-\lambda_j \tau} \frac{\tau^{c-\mu_{z_j}}}{(c-\mu_{z_j})!} u(\tau) d\tau \end{bmatrix} \\ &= \begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{2(c-1)} \cdot \frac{1}{(c-1)!} \left(\sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d\lambda_j^{c-1}} \right) \Phi^{-1}(K) \\ \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{2(c-2)} \cdot \frac{1}{(c-2)!} \left(\sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d\lambda_j^{c-2}} \right) \Phi^{-1}(K) \\ \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c (-1)^{2(c-\mu_{z_j})} \cdot \frac{1}{(c-\mu_{z_j})!} \left(\sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_{z_j}} \lambda_j^k}{d\lambda_j^{c-\mu_{z_j}}} \right) \Phi^{-1}(K) \end{bmatrix}. \end{aligned} \quad (\text{B.40})$$

Consequently, we obtain

$$\begin{aligned} &\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_{jz_j} \tau} \underline{\mathbf{b}}_{jz_j} u(\tau) d\tau \\ &= \Phi^{-1}(K) \begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d\lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d\lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-\mu_{z_j})!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_{z_j}} \lambda_j^k}{d\lambda_j^{c-\mu_{z_j}}} \end{bmatrix}. \end{aligned} \quad (\text{B.41})$$

Combining (B.1) and (B.41), we obtain (B.42) for every $j = l+1, l+2, \dots, k$, that is,

$$\begin{aligned} &\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_{jt} \tau} \underline{\mathbf{b}}_{jt} u(\tau) d\tau \\ &= \begin{bmatrix} \int_{-\infty}^t e^{-J_{j1} \tau} \underline{\mathbf{b}}_{j1} u(\tau) d\tau \\ \int_{-\infty}^t e^{-J_{j2} \tau} \underline{\mathbf{b}}_{j2} u(\tau) d\tau \\ \vdots \\ \int_{-\infty}^t e^{-J_{jd_j} \tau} \underline{\mathbf{b}}_{jd_j} u(\tau) d\tau \end{bmatrix} \\ &= \Phi^{-1}(K) \begin{bmatrix} \sum_{c=1}^{\mu_1} \mathbf{b}_{j1}^c \frac{1}{(c-1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d\lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_1} \mathbf{b}_{j1}^c \frac{1}{(c-2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d\lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_1}^{\mu_1} \mathbf{b}_{j1}^c \frac{1}{(c-\mu_1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_1} \lambda_j^k}{d\lambda_j^{c-\mu_1}} \\ \sum_{c=1}^{\mu_2} \mathbf{b}_{j2}^c \frac{1}{(c-1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d\lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_2} \mathbf{b}_{j2}^c \frac{1}{(c-2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d\lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_2}^{\mu_2} \mathbf{b}_{j2}^c \frac{1}{(c-\mu_2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_2} \lambda_j^k}{d\lambda_j^{c-\mu_2}} \\ \vdots \\ \sum_{c=1}^{\mu_{d_j}} \mathbf{b}_{jd_j}^c \frac{1}{(c-1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d\lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_{d_j}} \mathbf{b}_{jd_j}^c \frac{1}{(c-2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d\lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_{d_j}}^{\mu_{d_j}} \mathbf{b}_{jd_j}^c \frac{1}{(c-\mu_{d_j})!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_{d_j}} \lambda_j^k}{d\lambda_j^{c-\mu_{d_j}}} \end{bmatrix} \begin{matrix} \mu_1 \times 1 \\ \mu_2 \times 1 \\ \vdots \\ \mu_{d_j} \times 1 \end{matrix}, \end{aligned} \quad (\text{B.42})$$

where μ_{z_j} are the Weyr characteristics via Ferrer diagrams, for $j = l+1, l+2, \dots, \kappa$ and $z_j = 1, 2, \dots, d_j$. Note that $\rho_j = \max_{z_j=1,2,\dots,d_j} \mu_{z_j}$ is the index of annihila-

tion for the eigenvalue λ_j . Note that matrix (B.42) has d_j -blocks. Each one of the d_j blocks can be written as follows:

$$\begin{aligned} & \Phi^{-1}(K) \begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-1)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-1} \lambda_j^k}{d \lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-2)!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-2} \lambda_j^k}{d \lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-\mu_{z_j})!} \sum_{k=0}^{n-1} \alpha_k \frac{d^{c-\mu_{z_j}} \lambda_j^k}{d \lambda_j^{c-\mu_{z_j}}} \end{bmatrix} \\ &= \Phi^{-1}(K) \underbrace{\begin{bmatrix} \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j^0}{d \lambda_j^{c-1}}, \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j}{d \lambda_j^{c-1}}, \dots, \sum_{c=1}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-1)!} \cdot \frac{d^{c-1} \lambda_j^{n-1}}{d \lambda_j^{c-1}} \\ \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j^0}{d \lambda_j^{c-2}}, \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j}{d \lambda_j^{c-2}}, \dots, \sum_{c=2}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-2)!} \cdot \frac{d^{c-2} \lambda_j^{n-1}}{d \lambda_j^{c-2}} \\ \vdots \\ \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j^0}{d \lambda_j^{c-\mu_{z_j}}}, \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j}{d \lambda_j^{c-\mu_{z_j}}}, \dots, \sum_{c=\mu_{z_j}}^{\mu_{z_j}} \mathbf{b}_{jz_j}^c \frac{1}{(c-\mu_{z_j})!} \cdot \frac{d^{c-\mu_{z_j}} \lambda_j^{n-1}}{d \lambda_j^{c-\mu_{z_j}}} \end{bmatrix}}_{\mathbf{V}_{jz_j}} \cdot \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}}_{\mathbf{a}} \\ &= \Phi^{-1}(K) \mathbf{V}_{jz_j} \mathbf{a}, \end{aligned} \quad (\text{B.43})$$

for every $j = l+1, l+2, \dots, \kappa$, and $z_j = 1, 2, \dots, d_j$.

Consequently, the matrix (B.42), by using (B.43), can be written as follows

$$\lim_{\sigma \rightarrow 0} \int_{-\infty}^t e^{-J_j t} \underline{\mathbf{b}}_{\tau_j} u(\tau) d\tau = \Phi^{-1}(K) \begin{bmatrix} \mathbf{V}_{j1} \\ \mathbf{V}_{j2} \\ \vdots \\ \mathbf{V}_{jz_j} \\ \vdots \\ \mathbf{V}_{jd_j} \end{bmatrix} \cdot \mathbf{a}. \quad (\text{B.44})$$

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