Research Article

Slave System Dimension Expansion Approach for Robust Synchronization of Chaotic Systems with Unknown Phase Difference

Huanhuan Mai,1 Weiwei Zhang,1 and Yapeng Zhao2

1 Department of Computer Science and Engineering, Chongqing University, Chongqing 400030, China
2 Department of Logistics Engineering, Wuhan University of Technology, Hubei 430070, China

Correspondence should be addressed to Huanhuan Mai, maihuan123@126.com

Received 18 September 2010; Revised 21 January 2011; Accepted 25 February 2011

1. Introduction

Chaos control and synchronization has been attracting more and more interest since the pioneering work of Pecora and Carroll due to its many potential applications in secure communication, information processing, power converters, chemical reactions, artificial neural networks, and so forth, [1].

The research on synchronization has intensively focused on typical autonomous chaotic systems, for example, Chua’s circuit, Chen systems, and Lorenz oscillator [2–5]. Recently, more and more nonautonomous oscillators have become interesting topics, following their wild existence in engineering and science. Many nonautonomous chaotic systems always include the external sinusoidal force terms, such as Duffing oscillators [6], Ueda equations [7], a two-degrees-of-freedom twin-tail system [8], the generalized van der Pol system of nonautonomous form, [9] and so on. In addition, the use of periodically forced chaotic systems in synchronization has advantages of being far less sensitive to noise than autonomous systems [10].

For synchronization of nonautonomous chaotic oscillators with sinusoidal forcing term, the phase and frequency of two systems are often not in coincidence and desynchronize the systems [6].

Recently, the role of phase difference of the externally driven forces has received increasing attention. Yin et al. considered the phase effect of the two mutually coupled Duffing oscillators [6]. Kyprianidia and Stouboulous studied the dynamics of nonlinear system which consists of three identical resistively coupled nonlinear and nonautonomous electric circuits forming a ring in a neural-type connection [11]. Kusumoto and Ohtsubo [12] numerically investigated regions of chaos synchronization in the phase space of the frequency detuning between transmitter and receiver lasers and the optical injection rate in the systems of semiconductor with optical feedback. Nayak and Kuriakose considered the effect of the phase difference of applied fields on the dynamics of mutually coupled Josephson junctions [13].

To our best knowledge, there are two methods of correcting phase difference. The first one only relies on the effect of various kinds of resistive coupling, and investigated the synchronization effects with assumed phase difference and the assumed amplitude of the sinusoidal forcing term. Most literatures used this method. In some instances, unknown
At first, generalized nonautonomous systems are introduced as follows:

\[ x_n = f(x_1, x_2, \ldots, x_n) + h \sin(wt + \phi), \]  

where \( x_1, x_2, \ldots, x_n \) is the state-space vector, \( f(\cdot) \) is the function space, \( h \sin(wt + \phi) \) is the externally driven force, and \( h, \phi \) are unknown parameters.

A traditional slave system is given by

\[ y_n = f(y_1, y_2, \ldots, y_n) + u_n + h' \sin(wt + \phi'), \]

where \( u_n = k(y_n - x_n) \) and \( h', \phi' \) are any given parameters to estimate the \( h, \phi \).

The frequency detuning in the response system would be regarded as the special case of phase difference. It is not difficult to find that the sinusoidal forcing term has the following portraits:

\[ \frac{d^2[h' \times \sin(wt + \phi')]}{dt^2} = -w^2 \times [h' \times \sin(wt + \phi')]. \]

Inspired by this portraits, we can replace \( my_{n+1} \) with \( h' \times \sin(wt + \phi') \). The value of \( m \) would be any given scalar and regarded as flexible control parameter in the slave system. In other words, taking advantage of the portraits of sinusoidal forcing term that the triangular function is proportional to the second derivative, the increased item \( my_{n+1} \) is used to estimate the whole of the sinusoidal forcing term.

The response system could be designed in the following form:

\[ y_n = f(y_1, y_2, \ldots, y_n) + k(y_n - x_n) + my_{n+1} \]

\[ y_{n+1} = y_{n+2}, \]

\[ y_{n+2} = -w^2 y_{n+1}. \]

Looking at the increased-dimension slave system, one could wonder whether it is possible to exhibit the similar chaotic behavior of the master system, what happens when such systems interact. The following section explores the relationship among the dynamical states, different flexible control variable \( m \), and the coupling parameter \( k \) by numerical simulation analysis.

3. Horizontal Platform System

3.1. Description of the Horizontal Platform System Model and Equations. The state equations describe the dynamics of horizontal platform system as follows:

\[ Ax + Dx + Rg \sin x = \frac{3g}{R} (B - C) \cos x = F \cos (wt + \phi), \]

where \( A, B, C \) are the inertia moment of the platform for three axes, respectively, which penetrate the mass center of the platform, \( D \) is the damping coefficient, \( R \) is the acceleration constant of gravity, \( x \) is the rotation of the platform relative to the earth, \( R \) is the radius of the Earth, and \( F \cos wt \) is harmonic torque.

The horizontal platform system can be represented as the nonautonomous form:

\[ x_1 = x_2, \]

\[ x_2 = -ax_2 - bs_1 + l \cos x_1 l \sin x_1 + h \cos (wt + \phi), \]

where \( a = D/A > 0, b = Rg/A > 0, l = (3g/RA)(B - C), h = F/A > 0 \). Equation (6) is considered as a master system.

The traditional slave system is given by

\[ y_1 = y_2 + u_1, \]

\[ y_2 = -ay_2 - bs_1 + l \cos y_1 l \sin y_1 + h' \cos (wt + \phi') + u_2, \]
and the corresponding slave system would be
\[
\begin{align*}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= -ay_2 - bsiny_1 + l\cos y_1 \sin y_1 + my_3 + u_2, \\
\dot{y}_3 &= y_4, \\
\dot{y}_4 &= -w^2 y_3.
\end{align*}
\] (8)

For simulation, the values of the system parameters were fixed as \(a = 4/3\), \(b = 3.776\), \(l = 4.6 \times 10^{-6}\), \(h = 34/3\), \(w = 1.8\), and \(\varphi = 0\). The initial state of the master system is chosen as \((x_1(0), x_2(0)) = (-3.4, 2.1)\), and the initial state of the slave system is chosen as \((y_1(0), y_2(0), y_3(0), y_4(0)) = (0.78, -2.9, 0.1, 0.01)\).

Remark 1. The systems have been carefully studied by the following authors. Ge et al. [14] had studied the phase effect in unidirectional chaos synchronization of them and investigated the bifurcation diagrams and Lyapunov exponent for phase difference between 0 and \(2\pi\). Wu et al. obtained the sufficient synchronization criteria by Lyapunov’s stability theory and estimated the corresponding synchronization error bound [15].

Remark 2. The external forcing item \(h\cos(wt + \varphi)\) could be regarded as \(h\sin(wt + \varphi + \pi/2)\), and \(\varphi + \pi/2\) have nothing to do with the incremental dimension technique. Therefore, (4) are suitable for both sin and cos type of forcing item.

3.2. Synchronization Analysis of the Horizontal Platform System. In order to estimate the control functions, we define the state errors between the nonautonomous system (6) and autonomous system (8) as \(e_1 = y_1 - x_1\), \(e_2 = y_2 - x_2\). We get the error system as follows which helps to determine the stability boundaries of the synchronization process. The phase difference is defined as \(\phi = \varphi - \varphi'\)
\[
\begin{align*}
\dot{e}_1 &= \dot{y}_1 - \dot{x}_1 = e_2 + u_1, \\
\dot{e}_2 &= \dot{y}_2 - \dot{x}_2 = -ae_2 - b(\sin y_1 - \sin x_1) \\
&= \cos y_1 - \cos x_1 + my_3 + u_2 - h\cos wt.
\end{align*}
\] (9)

The average value of Euclidean distance is defined as \(\text{mean}(d(t)) = (\sqrt{\sum_{i=1}^{m} (y_{1m} - x_{1m})^2 + (y_{2m} - x_{2m})^2})/m\) and \(m\) is the length of the state vector.

Figure 2: Phase portrait, errors, and similarity of unidirectional coupled systems with \(k = -10\) and \(m = -100\).
The mean \( \langle d(t) \rangle \) is monitored with different values of flexible control variable \( m \) and is shown in Figure 1. It indicated that \( m = 110 \sim 120 \) would be the critical value with which the synchronization error will be converged and stabilized in the smallest area. The phase portrait, the error, and similarity of unidirectional coupled systems are shown in Figures 2, 3, and 4. The parameters are set as follows: \( m = -100, k = -10 \) in Figure 2, \( m = 112, k = -10 \) in Figure 3, \( m = -100 \), and \( k = -50 \) in Figure 4.

From Figure 1, one can find that the mean \( \langle d(t) \rangle \) is small when \( m \) is in the range \( 110 \sim 120 \), which we called critical range for different values of coupling parameter. Moreover, we do similar studies on Duffing system in which the critical range of \( m \) is \( 2.5 \sim 3.5 \). However, we have to find the good value of \( m \) depending on experiments.

From Figures 2 and 3, one can easily find the similar analysis as Figure 1, the choice of the value of the flexible control variable \( m \) is very important. It shows that in one case (Figure 3) with the value of \( m \) in the critical range and the coupling parameter is bigger, the synchronization error bound may be smaller than in other case (Figure 4) with the value of \( m \) far beyond critical range when the coupling parameter is smaller enough.

Both Figures 5 and 6 are the largest Lyapunov exponent diagrams for the error system by the method of small datasets of Rosenstein et al. [16]. The parameters, such as delay times and embedding windows, are needed in the small datasets method which is determined by the C-C method [17]. Figure 5 indicates that the better synchronization results are obtained under \( m = 110 \sim 120 \) when the coupling parameter value is fixed. In Figure 6, the largest Lyapunov exponent transverse the zeros value from positive to negative when \( k = -62 \). It can be seen that the error system approaches to a limit cycle when \( k \leq -62 \).

Remark 3. Zero crossing of Lyapunov exponent is widely used as a criterion of chaos synchronization. The small data method is a popular method to calculate the largest Lyapunov exponent for its speed, ease of implementation, and robustness to changes in the following quantities: embedding dimension, size of data set, reconstruction delay, and noise level. However, the realization of Lyapunov exponent needs numerical calculation for infinite evolution time; therefore, this method is not complete in practice [18]. By the way, it is difficult to estimate the critical value of flexible control \( m \) or \( k \) to use the Lyapunov direct method, since the error
system is not a pure function of state error, especially when the feedback control is a linear states error.

As we known, with $h' = h, \phi = 0$, synchronization of the master system (6) and the slave system (7) is identical except for their initial states. In Figure 7, curve 2 displayed that with $h' = h, \phi = 0$ the complete synchronization of two coupled systems can be easily obtained. All the state variables of synchronization error between system (6) and system (7) (curves 2–4) and between system (6) and system (8) (curve 1) converges on the decreasing coupling parameter $k$. Curve 1 is the nearest one with curve 2 which indicates the effectiveness of our dimension expansion approach.

**Remark 4.** The coupling term does carry enough information to correct the value of phase difference. For autonomous continuous systems and nonautonomous continuous systems without phase difference, the synchronization would be achieved when the coupling parameter is below a certain critical value. Wu et al. introduced a new definition of global synchronization with error bound, and all of their simulations have shown that the master-slave systems with phase difference hardly achieve the complete (zero error) synchronization even though the feedback coupling parameters are chosen to be smaller enough [15].

The level of mismatch of chaotic synchronization can be given quantitatively by taking the similarity function $S(\tau)$ as a time-averaged difference between the variables $x_1$ and $x_3$ taken with the time drift $\tau$ [19]

$$S^2(\tau) = \frac{\{x_1(t + \tau) - x_3(t)\}^2}{\{x_1^2(t)\}^{1/2} \{x_3^2(t)\}^{1/2}}.$$  \hspace{1cm} (10)

In Figure 8, the minimum of $S(\tau_0)$ indicates the existence of some characteristic time shift $\tau_0$ between system (6) and system (8) (curve 1), as same as system (6) and system (7) (curves 2–5). For smaller phase difference, the minimum of $S(\tau_0)$ continuously decreases (curves 2–5). The minimum of $S(\tau_0)$ (curves 1–3) appears to be zero. For more clarity, we enlarged the part of Figure 8 to obtain Figure 9, and get Figures 10 and 11 to correspond to Figures 8 and 9 with different values of coupling parameter $k$.  

**Figure 4:** Phase portrait, errors, and similarity of unidirectional coupled systems with $k = -50$ and $m = -100$.  

![Figure 4](image-url)
The largest Lyapunov exponent

Figure 5: The relationship of the largest Lyapunov exponents (LLEs) and \( m \) under different coupling parameter values. Curve 1 is with \( k = -10 \). Curve 2 is with \( k = -50 \). Curve 3 is with \( k = -70 \). Curve 4 is with \( k = -100 \). Curve 5 is with \( k = -150 \).

Figure 6: The relationship of the largest Lyapunov exponent (LLE) and \( k \) under \( m = 112 \).

To sum up, all the simulations above in different aspects show that the better synchronization results would be obtained when the flexible control variable \( m \) is in the critical range with different values of the coupling parameter.

4. The Ueda Equations with Noise

4.1. Description of Ueda Equations System Model and Equations. The Ueda oscillator is one of the most classical nonautonomous dynamical models and its various features in response, bifurcation, chaos and chaos control have focused considerable attentions, see [20–22]:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -q_1 x_2 - x_1^3 + q_2 \cos (w t + \varphi) + \varepsilon \eta, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -w^2 x_3,
\end{align*}
\]  

(11)

where \( \varepsilon \eta \) is the noise term, \( \varepsilon \) is a parameter specifying the intensity of the noise, and \( \eta \) is a random variable chosen to be uniformly distributed in the interval \([-1, 1]\). For all the simulations, the values of the system parameters were fixed as \( q_1 = 0.1, q_2 = 11.5, w = 1, \varphi = 0, \) and \( \varepsilon = 0.3 \). The initial state of the master system is chosen as \((x_1(0), x_2(0)) = (1, 0)\), and the initial state of the slave system is chosen as \((y_1(0), y_2(0), y_3(0), y_4(0)) = (-0.3, 0.4, 0.01, 0.1)\).

The corresponding response system would be

\[
\begin{align*}
\dot{y}_1 &= y_2 + k(y_1 - x_1), \\
\dot{y}_2 &= -q_1 y_2 - y_1^3 + m y_3 + k(y_2 - x_2), \\
\dot{y}_3 &= y_4, \\
\dot{y}_4 &= -w^2 y_3.
\end{align*}
\]  

(12)

4.2. Synchronization Analysis of Ueda Equations. Figure 12 indicates that better synchronization results are obtained when \( m < 145 \) and the coupling parameter value is fixed at \( k = -10 \). In Figure 13, the largest Lyapunov exponent transverse the zeros value form positive to negative when \( k \) is near zero.

For simplicity, other similar results are omitted here. From the above simulation results, we can see that the critical range of flexible control variable are totally different for
5. Conclusions

The phase of forcing term in the slave system may not be the same as that in the master system. We proposed a method, which uses the properties of the triangular function and increase the number of dimensions in the slave system to estimate the phase of forcing term in the master system, while the value of phase difference was assumed to be known at first and its effect of chaos synchronization was investigated.
in majority of the existing literature. It is noticeable that the flexible control variable we first investigated plays a great role in imitation results. Numerical simulations are performed to support the accuracy of the analytical method. However, the main results of the paper are obtained by numeric simulation analysis, and it lacks strong mathematic theory support. It is an open range for more exploration in the future.

Acknowledgments

This paper has been funded by Project no. CDJXS10181131, supported by the Fundamental Research Funds for the Central Universities. The authors wish to thank the anonymous reviewers for their valuable comments and helpful suggestions which greatly improved the paper’s quality.

References


Submit your manuscripts at http://www.hindawi.com