Research Article

Delay-Dependent Stability Criteria for Systems with Interval Time-Varying Delay

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This paper is concerned with robust stability of uncertain linear systems with interval time-varying delay. The time-varying delay is assumed to belong to an interval, which means that the derivative of the time-varying delay has an upper bound or a restriction. On other occasions, if we do not take restriction on the derivative of the time-varying delay into consideration, it allows the delay to be a fast time-varying function. The uncertainty under consideration includes a polytopic-type uncertainty and a linear fractional norm-bounded uncertainty. In order to obtain much less conservative results, a new Lyapunov-Krasovskii functional, which makes use of the information of both the lower and upper bounds of the interval time-varying delay, is proposed to derive some new stability criteria. Numerical examples are given to demonstrate the effectiveness of the proposed stability criteria.

1. Introduction

Time delays are frequently encountered in many fields of science and engineering, including communication network, manufacturing systems, biology, economy, and other areas. During the last two decades, the problem of stability of linear time-delay systems has been the subject of considerable research efforts. Many significant results have been reported in the literature. For the recent progress, the reader is referred to Gu et al. [1] and the references therein.

With the development of networked control technology, increasing attention has been paid to the study of stability analysis and controller design of networked control systems (NCSs) due to their low cost, simple installation and maintenance, and high reliability. For the NCSs, the sampling data and controller signals are transmitted through a network. As a result, it leads to a network-induced delay in a networked control closed-loop system. The existence of such kind of delay in a network-based control loop may induce instability or poor performance of NCSs. As pointed out by Yue et al. [2], NCSs are typical systems with interval time-varying delay. In fact, we consider the following system controlled through a network

$$\dot{x}(t) = Ax(t) + \tilde{B}u(t), \quad (1)$$

where $x(t) \in R^n$ is the state vector, and $u(t) \in R^p$ is the input vector. In the presence of the control network, which is shown in Figure 1, data transfer between the controller and the remote system, for example, sensors and actuators in a distributed control system will induce network delay in addition to the controller proceeding delay. First, since there exists the communication delay τ^{sc} between the sensor and the controller and computational delay τ^c in the controller, which is shown in Figure 1, the following control law is employed for the system (1)

$$u(t^+) = K_p x(t - \tau_k^{\text{sc}} - \tau_k^c), \quad (2)$$

$$t \in \{kh + \tau_k^{\text{sc}} + \tau_k^c\}, \quad \forall k \in N,$$

where h is the sampling period, N is the set of nonnegative integers, and K is the controller gain to be determined. Second, substituting (2) into (1) yields the closed-loop system by considering the communication delay τ^{ca} between the controller and the actuator

$$\dot{x}(t) = Ax(t) + \tilde{B}K_p x(kh), \quad (3)$$

$$t \in [kh + \tau_k, (k+1)h + \tau_{k+1}), \quad \forall k \in N,$$

where the time-delay $\tau_k = \tau_k^{\text{sc}} + \tau_k^c + \tau_k^{\text{ca}} > 0$ denotes the time from the instant kh when sensor nodes sample sensor data

FIGURE 1: A networked control system.

from a plant to the instant when actuators transfer data to the plant. Since $x(kh) = x(t - (t - kh))$, defining $\tau(t) = t - kh$, $t \in [kh + \tau_k, (k+1)h + \tau_{k+1}]$, for all $k \in N$, rewrite (3) as

$$\dot{x}(t) = Ax(t) + \tilde{B}K_p x(t - \tau(t)), \quad (4)$$

where $\tau(t)$ is piecewise linear with derivative $\dot{\tau}(t) = 1$ for $t \neq kh + \tau_k$, and $\tau(t)$ is discontinuous at the points $t = kh + \tau_k$, for all $k \in N$. It is clear that $\tau_k \leq \tau(t) \leq h + \tau_{k+1}$ for $t \in [kh + \tau_k, (k+1)h + \tau_{k+1}]$, for all $k \in N$.

Let $\tau_1 := \min_{k \in N} \tau_k > 0$ and $\tau_2 := \max_{k \in N} \{h + \tau_{k+1}\}$. Then we have

$$0 < \tau_1 \leq \tau(t) \leq \tau_2. \quad (5)$$

The system (3) is equivalent to the linear system (4) with interval time-varying delay described by (5). It should be pointed out that $\tau_1 > 0$ is essential for NCSs.

To cover the routine case of $0 \leq \tau(t) \leq \tau_2$, we consider $\tau(t)$ as a uniformly continuous time-varying function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$.

Throughout this paper, we will analyze the following two scenarios of the time-varying delay $\tau(t)$.

Case 1. $\tau(t)$ is a differentiable function satisfying

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad |\dot{\tau}(t)| \leq \mu, \quad \forall t \geq 0. \quad (6)$$

Case 2. $\tau(t)$ is a continuous function satisfying

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \forall t \geq 0, \quad (7)$$

where τ_1 and τ_2 are the lower and upper delay bounds, respectively, and τ_1, τ_2 and μ are constants.

Remark 1. When $\tau_1 = 0$, the interval delay becomes routine delay. When μ is zero, that is, $\tau_1 = \tau_2$, the time-varying delay becomes constant delay, Case 1 is a special case of the Case 2. We may obtain a less conservative result using Case 1 than that using Case 2. However, if the time-varying delay is not differentiable, or information about derivative of time-varying delay is absent, only Case 2 can be used to handle the situation.

Notation 1. R^n denotes the n -dimensional Euclidean space, $R^{n \times m}$ is the set of $n \times m$ real matrices, I is the identity matrix of appropriate dimensions, the notation $X > 0$

(resp., $X \geq 0$), for $X \in R^{n \times n}$ means that the matrix X is real positive definite (resp., positive semidefinite). For an arbitrary matrix B and two symmetric matrices A and C , $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes a symmetric matrix, where $*$ denotes a block matrix entry implied by symmetry.

2. System Descriptions and Preliminaries

Let $B = \tilde{B}K_p$ in (4), we have the following linear system with interval time-varying delay

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t - \tau(t)), \\ x(t) &= \phi(t), \quad t \in [-\tau_2, -\tau_1], \end{aligned} \quad (8)$$

where A and B are the constant matrices with appropriate dimensions, $\phi(t)$ is the initial condition of the system.

In this paper, we will study stability criteria of system described by (8) satisfying (6) or (7), employing the following new Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t, x_t) &= x^T(t)Px(t) + \int_{t-\tau(t)}^t x^T(\theta)Zx(\theta)d\theta \\ &+ \sum_{i=1}^N \int_{t-i\delta}^{t-(i-1)\delta} x^T(\theta)Q_i x(\theta)d\theta \\ &+ \sum_{i=1}^K \int_{t-\tau_1-i\rho}^{t-\tau_1-(i-1)\rho} x^T(\theta)G_i x(\theta)d\theta \\ &+ \sum_{i=1}^N \int_{-i\delta}^{-(i-1)\delta} \int_{t+\theta}^t \dot{x}^T(v)(\delta R_i)\dot{x}(v)dv d\theta \\ &+ \sum_{i=1}^K \int_{-\tau_1-i\rho}^{-\tau_1-(i-1)\rho} \int_{t+\theta}^t \dot{x}^T(v)(\rho H_i)\dot{x}(v)dv d\theta \\ &+ \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}^T(v)(\gamma W)\dot{x}(v)dv d\theta, \end{aligned} \quad (9)$$

where $P > 0$, $Z \geq 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, \dots, N$), $G_i > 0$, $H_i > 0$ ($i = 1, \dots, K$), $W > 0$ of appropriate dimensions, and $\tau_1 = N \times \delta$, $\tau_2 - \tau_1 = K \times \rho$, $\gamma = \tau_2 - \tau_1$, N, K are positive integer numbers of divisions of intervals $[-\tau_1, 0]$ and $[-\tau_2, -\tau_1]$, respectively, and δ, ρ are the length of each division. When $\tau_1 = 0$ and $\tau_1 = \tau_2$, we assume that $N = 0$ and $K = 0$, respectively. Moreover, we will consider a polytopic uncertainty and a linear fractional norm-bounded uncertainty which includes a routine norm-bounded uncertainty as a special case. Some numerical examples will be given to show the improvement over some previous results.

For any delay satisfying Case 1 or Case 2, our objective of this study is to develop new stability criteria which guarantee that system (8) is asymptotically stable and the system (8) subject to some uncertainties is robustly stable. For this purpose, the following lemmas are introduced.

Lemma 1 (see [3]). For any constant matrix $U \in R^{n \times n}$, $U = U^T > 0$, scalar $\sigma > 0$, and vector function $\dot{x} : [-\sigma, 0] \rightarrow R^n$ such that the following integration is well defined, then it holds that

$$\begin{aligned} & -\sigma \int_{t-\sigma}^t \dot{x}^T(w) U \dot{x}(w) dw \\ & \leq \begin{bmatrix} x(t) \\ x(t-\sigma) \end{bmatrix}^T \begin{bmatrix} -U & U \\ * & -U \end{bmatrix} \\ & \times \begin{bmatrix} x(t) \\ x(t-\sigma) \end{bmatrix}. \end{aligned} \quad (10)$$

Applying the Lemma 1 yields the following new integral inequality for cross-product term.

Lemma 2 (see [4]). For any constant matrix $U \in R^{n \times n}$, $U = U^T > 0$, scalars $\sigma_1 \leq \tau(t) \leq \sigma_2$ and vector function $\dot{x} : [-\sigma_2, -\sigma_1] \rightarrow R^n$ such that the following integration is well defined, then it holds that

$$-(\sigma_2 - \sigma_1) \int_{t-\sigma_2}^{t-\sigma_1} \dot{x}^T(w) U \dot{x}(w) dw \leq \eta^T(t) \Omega \eta(t), \quad (11)$$

where

$$\eta(t) = \begin{bmatrix} x(t - \sigma_1) \\ x(t - \tau(t)) \\ x(t - \sigma_2) \end{bmatrix}, \quad \Omega = \begin{bmatrix} -U & U & 0 \\ * & -2U & U \\ * & * & -U \end{bmatrix}. \quad (12)$$

3. New Stability Criteria

We first consider asymptotic stability for the nominal system (8). Employing Lyapunov-Krasovskii functional (9), we have the following result.

Theorem 1. For some given scalars $0 \leq \tau_1 \leq \tau_2$ and μ , the nominal system (8) satisfying (6) is asymptotically stable if there exist real symmetric matrices $P > 0$, $Z \geq 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, \dots, N$), $G_i > 0$, $H_i > 0$ ($i = 1, \dots, K$), $W > 0$, and any matrix $S = [S_1^T S_2^T \dots S_{N+K+3}^T]^T$ of appropriate dimensions such that the following LMI holds:

$$\Xi = \Xi_1 + SA_d + (SA_d)^T < 0, \quad (13)$$

where

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & 0 & R_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & P \\ * & \Xi_{22} & 0 & 0 & \cdots & 0 & W & 0 & 0 & \cdots & 0 & W & 0 \\ * & * & \Phi_1 & R_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & \Phi_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & \Phi_{N-1} & R_N & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & * & \cdots & * & \Phi_N & H_1 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & * & \cdots & * & * & \Psi_1 & H_2 & \cdots & 0 & 0 & 0 \\ * & * & * & * & \cdots & * & * & * & \Psi_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & * & \cdots & * & * & * & * & \cdots & \Psi_{K-1} & H_K & 0 \\ * & * & * & * & \cdots & * & * & * & * & \cdots & * & \Psi_K & 0 \\ * & * & * & * & \cdots & * & * & * & * & \cdots & * & * & Y \end{bmatrix}, \quad (14)$$

$$\Xi_{11} = Z + Q_1 - R_1, \quad \Xi_{22} = -(1 - \mu)Z - 2W, \quad A_d = \begin{bmatrix} A & B & \underbrace{0 \ 0 \ \cdots \ 0}_{N+K} & -I \end{bmatrix},$$

$$\begin{aligned} \Phi_i &= Q_{i+1} - Q_i - R_i - R_{i+1} \quad (i = 1, 2, \dots, N-1), & \Phi_N &= -Q_N - R_N - W + G_1 - H_1, \\ \Psi_i &= G_{i+1} - G_i - H_i - H_{i+1} \quad (i = 1, 2, \dots, K-1), & \Psi_K &= -G_K - H_K - W, \end{aligned} \quad (15)$$

$$Y = \sum_{i=1}^N \delta^2 R_i + \sum_{i=1}^K \rho^2 H_i + \gamma^2 W.$$

Proof. Taking the time derivative of $V(t, x_t)$ with respect to t along the trajectory of (8) yields

$$\begin{aligned}
\dot{V}(t, x_t) &\leq 2x^T(t)P\dot{x}(t) + x^T(t)Zx(t) \\
&\quad - (1 - \mu)x^T(t - \tau(t))Zx(t - \tau(t)) \\
&\quad + \sum_{i=1}^N x^T(t - (i-1)\delta)Q_i x(t - (i-1)\delta) \\
&\quad - \sum_{i=1}^N x^T(t - i\delta)Q_i x(t - i\delta) \\
&\quad + \sum_{i=1}^K x^T(t - \tau_1 - (i-1)\rho)G_i x(t - \tau_1 - (i-1)\rho) \\
&\quad - \sum_{i=1}^K x^T(t - \tau_1 - i\rho)G_i x(t - \tau_1 - i\rho) \\
&\quad + \dot{x}^T(t) \left(\sum_{i=1}^N \delta^2 R_i + \sum_{i=1}^K \rho^2 H_i + \gamma^2 W \right) \dot{x}(t) \\
&\quad - \sum_{i=1}^N \int_{t-i\delta}^{t-(i-1)\delta} \dot{x}^T(\theta)(\delta R_i)\dot{x}(\theta)d\theta \\
&\quad - \sum_{i=1}^K \int_{t-\tau_1-i\rho}^{t-\tau_1-(i-1)\rho} \dot{x}^T(\theta)(\rho H_i)\dot{x}(\theta)d\theta \\
&\quad - \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(\theta)(\gamma W)\dot{x}(\theta)d\theta.
\end{aligned} \tag{16}$$

Applying the Lemma 1, the following inequalities hold:

$$\begin{aligned}
& - \sum_{i=1}^N \int_{t-i\delta}^{t-(i-1)\delta} \dot{x}^T(\theta)(\delta R_i)\dot{x}(\theta)d\theta \\
& \leq \sum_{i=1}^N \begin{bmatrix} x(t - (i-1)\delta) \\ x(t - i\delta) \end{bmatrix}^T \begin{bmatrix} -R_i & R_i \\ * & -R_i \end{bmatrix} \begin{bmatrix} x(t - (i-1)\delta) \\ x(t - i\delta) \end{bmatrix}, \\
& - \sum_{i=1}^K \int_{t-\tau_1-i\rho}^{t-\tau_1-(i-1)\rho} \dot{x}^T(\theta)(\rho H_i)\dot{x}(\theta)d\theta \\
& \leq \sum_{i=1}^K \begin{bmatrix} x(t - \tau_1 - (i-1)\rho) \\ x(t - \tau_1 - i\rho) \end{bmatrix}^T \begin{bmatrix} -H_i & H_i \\ * & -H_i \end{bmatrix} \\
& \quad \times \begin{bmatrix} x(t - \tau_1 - (i-1)\rho) \\ x(t - \tau_1 - i\rho) \end{bmatrix}.
\end{aligned} \tag{17}$$

Applying the Lemma 2, the following inequality holds:

$$\begin{aligned}
& - \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(\theta)(\gamma W)\dot{x}(\theta)d\theta \\
& \leq \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau(t)) \\ x(t - \tau_2) \end{bmatrix}^T \begin{bmatrix} -W & W & 0 \\ * & -2W & W \\ * & * & -W \end{bmatrix} \\
& \quad \times \begin{bmatrix} x(t - \tau_1) \\ x(t - \tau(t)) \\ x(t - \tau_2) \end{bmatrix}.
\end{aligned} \tag{18}$$

For any matrix $S = [S_1^T \ S_2^T \ \cdots \ S_{N+K+3}^T]^T$ of appropriate dimensions, the following equality holds:

$$2\xi^T(t)S[Ax(t) + Bx(t - \tau(t)) - \dot{x}(t)] = 0, \tag{19}$$

where is defined in the top of next page,

$$\begin{aligned}
\xi^T(t) &= [x^T(t)x^T(t - \tau(t))\xi_1^T(t)\xi_2^T(t)\dot{x}^T(t)], \\
\xi_1^T(t) &= [x^T(t - \delta)x^T(t - 2\delta) \cdots \\
&\quad x^T(t - (N-1)\delta)x^T(t - \tau_1)], \\
\xi_2^T(t) &= [x^T(t - \tau_1 - \rho)x^T(t - \tau_1 - 2\rho) \cdots \\
&\quad x^T(t - \tau_1 - (K-1)\rho)x^T(t - \tau_2)].
\end{aligned} \tag{20}$$

Considering (16)–(19) together, we have $\dot{V}(t, x_t) \leq \xi^T(t)\Xi\xi(t)$. If (13) is satisfied, then $\dot{V}(t, x_t) \leq -\lambda x^T(t)x(t)$ for some scalar $\lambda > 0$, from which we conclude that the nominal system (8) is asymptotically stable. This completes the proof. $\square$

When the restriction on the derivative of the interval time-varying delay is removed, that is, choosing $Z \equiv 0$ in Theorem 1, we can obtain a delay variety rate-independent criterion for a delay that only satisfies (7).

Corollary 1. For some given scalars $0 \leq \tau_1 \leq \tau_2$, the nominal system (8) satisfying (7) is asymptotically stable, if there exist real symmetric matrices $P > 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, \dots, N$), $G_i > 0$, $H_i > 0$ ($i = 1, \dots, K$), $W > 0$ and any matrix S of appropriate dimensions such that the LMI (13) holds with $Z \equiv 0$.

Remark 2. For system (8) with the routine delay case described by $\tau_1 = 0$ and $\tau_2 > 0$, that is, $N = 0$, $K > 0$, the

corresponding Lyapunov-Krasovskii functional reduces to

$$\begin{aligned}
 V(t, x_t) = & x^T(t)Px(t) + \int_{t-\tau(t)}^t x^T(\theta)Zx(\theta)d\theta \\
 & + \sum_{i=1}^K \int_{t-i\rho}^{t-(i-1)\rho} x^T(\theta)G_i x(\theta)d\theta \\
 & + \sum_{i=1}^K \int_{-i\rho}^{-(i-1)\rho} \int_{t+\theta}^t \dot{x}^T(v)(\rho H_i)\dot{x}(v)dv d\theta \\
 & + \int_{-\tau_2}^0 \int_{t+\theta}^t \dot{x}^T(v)(\gamma W)\dot{x}(v)dv d\theta.
 \end{aligned} \quad (21)$$

And for system (8) with the constant delay case described by $\tau_1 = \tau(t) = \tau_2 > 0$, that is, $N > 0$, $K = 0$, the corresponding Lyapunov-Krasovskii functional reduces to

$$\begin{aligned}
 V(t, x_t) = & x^T(t)Px(t) \\
 & + \sum_{i=1}^N \int_{t-i\delta}^{t-(i-1)\delta} x^T(\theta)Q_i x(\theta)d\theta \\
 & + \sum_{i=1}^N \int_{-i\delta}^{-(i-1)\delta} \int_{t+\theta}^t \dot{x}^T(v)(\delta R_i)\dot{x}(v)dv d\theta.
 \end{aligned} \quad (22)$$

Similar to the proof of the Theorem 1, one can easily derive a less conservative results than some exiting ones,

respectively, which will be shown through numerical examples in the next section. For the sake of simplicity, the results are omitted.

In what follows, we consider robust stability of the system (8) satisfying (6) or (7) subject to a poly-topic uncertainty and a linear fractional norm-bounded uncertainty. For the poly-topic uncertainty, that is, the matrices A and B in (8) can be expressed as

$$[A \ B] = \sum_{j=1}^r \zeta_j [A_j \ B_j], \quad (23)$$

where $\sum_{j=1}^r \zeta_j = 1$, $0 \leq \zeta_j \leq 1$. Based on Theorem 1, we can easily obtain the following result.

Theorem 2. For some given scalars $0 \leq \tau_1 \leq \tau_2$ and μ , the system described by (8) satisfying (6) subject to polytopic uncertainty (23) is robustly stable, if there exist real symmetric matrices $P_j > 0$, $Z_j \geq 0$, $Q_{ij} > 0$, $R_{ij} > 0$ ($i = 1, \dots, N$), $G_{ij} > 0$, $H_{ij} > 0$ ($i = 1, \dots, K$), $W_j > 0$, and any matrix S of appropriate dimensions such that the following LMIs hold for $j = 1, 2, \dots, r$:

$$\Xi^{(j)} = \Xi_1^{(j)} + SA_d^{(j)} + (SA_d^{(j)})^T < 0, \quad (24)$$

where

$$\Xi_1^{(j)} = \begin{bmatrix}
 \Xi_{11}^{(j)} & 0 & R_{1j} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & P_j \\
 * & \Xi_{22}^{(j)} & 0 & 0 & \cdots & 0 & W_j & 0 & 0 & \cdots & 0 & W_j & 0 \\
 * & * & \Phi_1^{(j)} & R_{2j} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 * & * & * & \Phi_2^{(j)} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 * & * & * & * & \cdots & \Phi_{N-1}^{(j)} & R_{Nj} & 0 & 0 & \cdots & 0 & 0 & 0 \\
 * & * & * & * & \cdots & * & \Phi_N^{(j)} & H_{1j} & 0 & \cdots & 0 & 0 & 0 \\
 * & * & * & * & \cdots & * & * & \Psi_1^{(j)} & H_{2j} & \cdots & 0 & 0 & 0 \\
 * & * & * & * & \cdots & * & * & * & \Psi_2^{(j)} & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 * & * & * & * & \cdots & * & * & * & * & \cdots & \Psi_{K-1}^{(j)} & H_{Kj} & 0 \\
 * & * & * & * & \cdots & * & * & * & * & \cdots & * & \Psi_K^{(j)} & 0 \\
 * & * & * & * & \cdots & * & * & * & * & \cdots & * & * & \Upsilon^{(j)}
 \end{bmatrix}, \quad (25)$$

S is defined in Theorem 1, and

$$\begin{aligned}
\Xi_{11}^{(j)} &= Z_j + Q_{1j} - R_{1j}, & \Xi_{22}^{(j)} &= -(1 - \mu)Z_j - 2W_j, \\
A_d^{(j)} &= \begin{bmatrix} A_j & B_j & \underbrace{0 \ 0 \ \cdots \ 0}_{N+K} & -I \end{bmatrix}, \\
\Phi_i^{(j)} &= Q_{i+1,j} - Q_{ij} - R_{ij} - R_{i+1,j} \quad (i = 1, 2, \dots, N-1), \\
\Phi_N^{(j)} &= -Q_{Nj} - R_{Nj} - W_j + G_{1j} - H_{1j}, \\
\Psi_i^{(j)} &= G_{i+1,j} - G_{ij} - H_{ij} - H_{i+1,j} \quad (i = 1, 2, \dots, K-1), \\
\Psi_K^{(j)} &= -G_{Kj} - H_{Kj} - W_j, \\
Y^{(j)} &= \sum_{i=1}^N \delta^2 R_{ij} + \sum_{i=1}^K \rho^2 H_{ij} + \gamma^2 W_j.
\end{aligned} \tag{26}$$

Proof. Construct the following parameter-dependent Lyapunov-Krasovskii functional:

$$\begin{aligned}
V_u(t, x_t) &= \sum_{j=1}^r x^T(t) \zeta_j P_j x(t) \\
&+ \sum_{j=1}^r \int_{t-\tau(t)}^t x^T(\theta) \zeta_j Z_j x(\theta) d\theta \\
&+ \sum_{j=1}^r \sum_{i=1}^N \int_{t-i\delta}^{t-(i-1)\delta} x^T(\theta) \zeta_j Q_{ij} x(\theta) d\theta \\
&+ \sum_{j=1}^r \sum_{i=1}^K \int_{t-\tau_1-i\rho}^{t-\tau_1-(i-1)\rho} x^T(\theta) \zeta_j G_{ij} x(\theta) d\theta \\
&+ \sum_{j=1}^r \sum_{i=1}^N \int_{-i\delta}^{-(i-1)\delta} \int_{t+\theta}^t \dot{x}^T(v) \zeta_j (\delta R_{ij}) \dot{x}(v) dv d\theta \\
&+ \sum_{j=1}^r \sum_{i=1}^K \int_{-\tau_1-i\rho}^{-(i-1)\rho} \int_{t+\theta}^t \dot{x}^T(v) \zeta_j (\rho H_{ij}) \dot{x}(v) dv d\theta \\
&+ \sum_{j=1}^r \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t \dot{x}^T(v) \zeta_j (\gamma W_j) \dot{x}(v) dv d\theta,
\end{aligned} \tag{27}$$

where $P_j > 0$, $Z_j \geq 0$, $Q_{ij} > 0$, $R_{ij} > 0$ ($i = 1, \dots, N$), $G_{ij} > 0$, $H_{ij} > 0$ ($i = 1, \dots, K$), $W_j > 0$ are unknown matrices of appropriate dimensions. Then the proof follows a linear similar to the proof of Theorem 1, we can obtain the following inequality:

$$\dot{V}_u(t, x_t) \leq \sum_{j=1}^r \xi^T(t) \zeta_j \Xi^{(j)} \xi(t), \tag{28}$$

where $\xi(t)$ is defined in the formula (19), then $\dot{V}_u(t, x_t) \leq -\lambda x^T(t) x(t)$ for some scalar $\lambda > 0$, from which we conclude that the system described by (8) satisfying (6) subject to

polytopic uncertainty (23) is robustly stable. This completes the proof. $\square$

Similar to Corollary 1, we can easily obtain a delay variety rate-independent criterion for a delay that only satisfies (7).

Corollary 2. For some given scalars $0 \leq \tau_1 \leq \tau_2$, the system described by (8) satisfying (7) subject to polytopic uncertainty (23) is robustly stable if there exist real symmetric matrices $P_j > 0$, $Q_{ij} > 0$, $R_{ij} > 0$ ($i = 1, \dots, N$), $G_{ij} > 0$, $H_{ij} > 0$ ($i = 1, \dots, K$), $W_j > 0$, and any matrix S of appropriate dimensions such that the LMIs (24) hold with $Z_j \equiv 0$ for $j = 1, 2, \dots, r$.

Remark 3. We succeed to separate the system matrices and Lyapunov matrices in Theorem 1, so we can easily use parameter-dependent Lyapunov-Krasovskii functional method. Different Lyapunov matrices are used in Theorem 2 for r different LMIs, which are distinguished with Proposition 7 in Jiang and Han [6], in which used fixed Lyapunov matrices for r different LMIs. In fact parameter-dependent Lyapunov-Krasovskii functional method can reduce stability criteria conservatism significantly. Numerical example will be given to show the improvement with Jiang and Han [6] in the next section.

Next we address the linear fractional norm-bounded uncertainty. Suppose that matrices A and B have parameter perturbations as $\Delta A(t)$ and $\Delta B(t)$, which are in the form of

$$[\Delta A(t) \ \Delta B(t)] = D(I - F(t)M)^{-1}F(t)[E_1 \ E_2], \tag{29}$$

where D, M, E_1 , and E_2 are known real constant matrices of appropriate dimensions, and $M^T M < I$; $F(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying $F^T(t)F(t) \leq I$.

For system (8) with uncertainty (29), we can establish the following result by considering Theorem 1 and applying S procedure [9].

Theorem 3. For some given scalars $0 \leq \tau_1 \leq \tau_2$ and μ , the system described by (8) satisfying (6) subject to the linear fractional norm-bounded uncertainty (29) is robustly stable if there exist real symmetric matrices $P > 0$, $Z \geq 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, \dots, N$), $G_i > 0$, $H_i > 0$ ($i = 1, \dots, K$), $W > 0$, any matrix S of appropriate dimensions, and a scalar $\varepsilon > 0$ such that the following LMI holds:

$$\begin{bmatrix} \Xi & SD & \varepsilon \Gamma \\ * & -\varepsilon I & \varepsilon M^T \\ * & * & -\varepsilon I \end{bmatrix} < 0, \tag{30}$$

where $\Gamma^T = \begin{bmatrix} E_1 & E_2 & \underbrace{0 \ 0 \ \cdots \ 0}_{N+K+1} \end{bmatrix}$, Ξ is defined in the formula (13), and S is defined in Theorem 1.

Proof. Replacing A and B in (13) with $A + D(I - F(t)M)^{-1}F(t)E_1$ and $B + D(I - F(t)M)^{-1}F(t)E_2$, respectively, and multiply both sides of the resulting matrix by vectors x_i for $i = 1, 2, \dots, N + K + 3$. Next, define $p = (I - F(t)M)^{-1}F(t)E_1 x_1$, $q = (I - F(t)M)^{-1}F(t)E_2 x_2$. Then, we

TABLE 1: MAUB of the time-varying delay for different μ .

τ_1	He et al. [5]	Jiang and Han [6]	Shao [7]	This work for $N = 2, K = 2$	This work for $N = 3, K = 3$
0	1.34	1.3453	—	1.3454 ($N = 0$)	1.3454 ($N = 0$)
1	1.74	1.8043	1.8737	1.8043	1.8111
2	2.43	2.5213	2.5049	2.5213	2.5339
3	3.22	3.3311	3.2591	3.3311	3.3475
4	4.07	4.1880	4.0744	4.1880	4.2072
4.4697	4.47	4.6009 ^a	—	4.6007	4.6210
5	—	5.0722	—	5.0722	5.0937
5.71	—	5.7107	—	5.7107	5.7337

have the following condition for the admissible uncertainty $F(t)$:

$$\beta^T \begin{bmatrix} \Xi & SD \\ * & 0 \end{bmatrix} \beta < 0, \quad (31)$$

where $\beta^T = [x_1^T, x_2^T, \dots, x_{N+K+3}^T, (p+q)^T]$. It is easy to know that p and q can be rewritten as $p = F(t)Mp + F(t)E_1x_1$ and $q = F(t)Mq + F(t)E_2x_2$, since $F^T(t)F(t) \leq I$, it is obvious that the following inequality holds:

$$(p+q)^T(p+q) \leq \begin{bmatrix} x_1 \\ x_2 \\ p+q \end{bmatrix}^T \begin{bmatrix} E_1^T E_1 & E_1^T E_2 & E_1^T M \\ * & E_2^T E_2 & E_2^T M \\ * & * & M^T M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p+q \end{bmatrix}. \quad (32)$$

Applying S procedure, both inequalities (31) and (32) are true if and only if there is a $\varepsilon > 0$, promising that the following condition holds:

$$\beta^T \begin{bmatrix} \Xi + \varepsilon \Gamma \Gamma^T & SD + \varepsilon \Gamma M \\ * & \varepsilon M^T M - \varepsilon I \end{bmatrix} \beta < 0. \quad (33)$$

By Schur complement, for any $\beta \neq 0$, (33) is equivalent to (30). This completes the proof. $\square$

Similar to Corollary 1, we can easily obtain a delay variety rate-independent criterion for a delay that only satisfies (7).

Corollary 3. For some given scalars $0 \leq \tau_1 \leq \tau_2$, the system described by (8) satisfying (7) subject to the linear fractional norm-bounded uncertainty (29) is robustly stable if there exist real symmetric matrices $P > 0$, $Q_i > 0$, $R_i > 0$ ($i = 1, \dots, N$), $G_i > 0$, $H_i > 0$ ($i = 1, \dots, K$), $W > 0$, any matrix S of appropriate dimensions, and a scalar $\varepsilon > 0$ such that the LMI (30) holds with $Z \equiv 0$.

Remark 4. It is clear to see that if we set $M = 0$, the linear fractional norm-bounded uncertainty reduces to the routine norm-bounded uncertainty, and we can derive a corresponding results for the routine norm-bounded uncertainty from Theorem 3 and Corollary 3. For the sake of simplicity, the results are omitted.

4. Numerical Examples

In this section, two examples are given to show the effectiveness of the results derived in this paper.

Example 1. Consider the following time-delay system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - \tau(t)). \quad (34)$$

Table 1 lists the maximum allowable upper bound (MAUB) of the time-varying delay for different τ_1 , and those in He et al. [5], Jiang and Han [6], and Shao [7].

- (1) If $\tau_1 = 0$, by using Corollary 1, we have $\tau_2 = 1.3454$ for $N = 0$ and any integer $K \geq 2$, which is larger than $\tau_2 = 1.34$ in derived in He et al. [5] and close to $\tau_2 = 1.3453$ in derived in Jiang and Han [6].
- (2) If $0 < \tau_1 \leq \tau(t) \leq \tau_2$ and $\tau_1 \neq \tau_2$. By using Corollary 1 for $N = 2$, $K = 2$, our results listed in the second last column is same as the results derived in Jiang and Han [6]. Moreover, by using Corollary 1 for $N = 3$, $K = 3$, we have the results listed in the last column, which are larger than the results derived in He et al. [5] and Jiang and Han(2008).
- (3) By using Proposition 8 in Jiang and Han [6], we can have $\tau_2 = 4.6007$ for $\tau_1 = 4.4697$, not have 4.6009^a.
- (4) For the constant delay $\tau_1 = \tau_2$, that is, $K = 0$. By using Corollary 1, we have $\tau_{\max} = 4.4721$ for $N = 1$; 5.7175 for $N = 2$; 5.9677 for $N = 3$; 6.0983 for $N = 5$; 6.1537 for $N = 10$, which is close to analytical delay limit for stability 6.17258.

Example 2. Consider the following time-delay system:

$$\begin{aligned} \dot{x}(t) = & \begin{bmatrix} -2 + \delta_1 & 0 \\ 0 & -1 + \delta_2 \end{bmatrix} x(t) \\ & + \begin{bmatrix} -1 + \rho_1 & 0 \\ -1 & -1 + \rho_2 \end{bmatrix} x(t - \tau(t)), \end{aligned} \quad (35)$$

where $|\delta_1| \leq 1.6$, $|\delta_2| \leq 0.05$, $|\rho_1| \leq 0.1$, $|\rho_2| \leq 0.3$, and suppose that $\tau(t)$ is continuous function. We handle

TABLE 2: MAUB of the time varying delay for different τ_1 .

τ_1	Jiang and Han [6]	This work for $N = 2, K = 2$
0	0.8748	1.0238 ($N = 0$)
0.5	0.9976	1.1887
1.0	1.1949	1.4511
1.2051	1.2870	1.5361
$\tau_2 = \tau_1$	1.3500	1.8130 ($K = 0$)

TABLE 3: MAUB of the time-varying delay for different μ .

τ_1	Jiang and Han [8]	Jiang and Han [6]	This work for $N = 2, K = 2$	This work for $N = 3, K = 3$
0.0	0.8654	0.9442	0.9442 ($N = 0$)	0.9442 ($N = 0$)
0.2	0.9097	0.9757	0.9859	0.9864
0.4	0.9575	1.0208	1.0280	1.0307
0.6	1.0104	1.0795	1.0829	1.0888
0.8	1.0692	1.1500	1.1506	1.1601
1.0	1.1336	1.2308	1.2302	1.2432
1.2051	—	1.3231	1.3223	1.3389
$\tau_1 = \tau_2$	1.2051	1.4380	1.4368 ($K = 0$)	1.4810 ($K = 0$)

the uncertainty as polytopic uncertainty and norm-bounded uncertainty, respectively. First, the uncertainty is handled as polytopic uncertainty. Then

- (1) if $\tau(t)$ is a constant, applying Corollary 2 in this paper yields the result $\tau_2 = 1.8130$ for $\tau_1 = \tau_2$, which is less conservative than the result $\tau_2 = 1$ given in Fridman and Shaked [10], the result $\tau_2 = 1.02$ given in Han et al. [11] and the result $\tau_2 = 1.3500$ given in Jiang and Han [6];
- (2) if $\tau_1 = 0$, we have $\tau_2 = 1.0238$, which is larger than $\tau_2 = 0.7692$ in derived in Fridman and Shaked [10], $\tau_2 = 0.8654$ in derived in Jiang and Han [8] and $\tau_2 = 0.8748$ in derived in Jiang and Han [6], respectively;
- (3) if $0 < \tau_1 \leq \tau(t) \leq \tau_2$, Table 2 lists the maximum allowable upper bounds of the time-varying delay for different lower bounds the delay by Corollary 2. From Table 2, we can see that our results are larger than the previous ones.

Next, the uncertainty is handled as norm-bounded uncertainty, and we choose D, E_1 and E_2 as

$$\begin{aligned}
 D &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix}, \\
 E_1 &= \begin{bmatrix} 160 & 0 \\ 0 & 1.25 \end{bmatrix}, \\
 E_2 &= \begin{bmatrix} 10 & 0 \\ 0 & 7.5 \end{bmatrix}.
 \end{aligned} \tag{36}$$

Table 3 lists the maximum allowable upper bound (MAUB) of the time-varying delay for different τ_1 , and those in Jiang and Han [6, 8].

- (1) If $\tau_1 = 0$, we have $\tau_2 = 0.9442$ for $N = 0$ and any integer $K \geq 2$, which is larger than $\tau_2 = 0.8654$ in derived in He et.al [5] and same as $\tau_2 = 0.9442$ in derived in Jiang and Han [6].
- (2) If $0 < \tau_1 \leq \tau(t) \leq \tau_2$ and $\tau_1 \neq \tau_2$, By using Corollary 3, our results listed in the second last column for $N = 2, K = 2$ and in the last column for $N = 3, K = 3$, which is larger than those in derived in Jiang and Han [6].
- (3) If $\tau(t)$ is a constant, this is $\tau_1 = \tau_2$, we have $\tau_2 = 1.4368$ for $N = 2, K = 0$; 1.4810 for $N = 3, K = 0$.

It is clear that proposed stability criteria in this paper can significantly improve some exiting results in the literature.

Remark 5. From Tables 2 and 3, we can find that the conservatism using the criterion derived for the poly-topic uncertainty is less than using the criterion derived for norm-bounded uncertainty, which is distinguished to the result derived in Jiang and Han [6], and there are some reverse results, for there are some fixed Lyapunov matrices in Jiang and Han [6] for different LMIs. Moreover, the conservatism can be reduced for $\tau_1 \neq 0$ by increasing divisions of the intervals $[-\tau_1, 0]$ or $[-\tau_2, -\tau_1]$ in this paper.

5. Conclusion

This note presents some new stability criteria for the interval time-varying delay system with a poly-topic uncertainty and a linear fractional norm-bounded uncertainty. We have proposed a new Lyapunov-Krasovskii functional, which is

based on dividing intervals $[-\tau_1, 0]$ and $[-\tau_2, -\tau_1]$ into N and K divisions, respectively, to derive some new stability criteria. Numerical examples show that our criteria would be less conservative along with N , K increased and demonstrate our criteria are less conservative than previous ones. Furthermore, based on time-delay system stability results, some future research can be focused on designing feedback controller, which can promise system robust stabilization and satisfy some system performance.

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