Research Article

H_∞ Observers Design for a Class of Continuous Time Nonlinear Singular Systems

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This paper considers the problem of H_∞ observers design for a class of Lipschitz continuous nonlinear singular systems. The method is based on the parameterization of the solution of the generalized Sylvester equations obtained from the estimation errors. Sufficient conditions for the existence of the observers which guarantee stability and the worst case observers error energy over all bounded energy disturbances is minimized are given. The approach also unifies the full-order, the reduced-order, and the minimal-order observers design. The solutions are obtained through linear matrix inequalities (LMIs) formulation. A numerical example is given to illustrate our results.

1. Introduction

Observers design for nonlinear systems has been a very active field during the last two decades. This is due to the fact that a state estimation is generally required for the control when all states of the system are not available. The observers are also used in the monitoring and fault diagnosis. For standard nonlinear systems, there exist several approaches for the observers design including one based on coordinate transformations which lead to a linear error dynamics [1–3] and one where the problem of the observers design can be treated without the need of these transformations [4]. An important class of standard nonlinear systems, the global Lipschitz, was considered by the authors in [5, 6], where the existence conditions for the observers are presented and constructive design methods were given for full-order and reduced-order cases.

On the other hand, singular systems (known as generalized, descriptor, or differential algebraic (DA) systems) describe a large class of systems. They are encountered in chemical and mineral industries; for example, the dynamic balances of mass and energy are described by differential equations, while thermodynamic equilibrium relations constitute additional algebraic constraints. The problem of the

state estimation for these practical applications arises in data reconciliation, for example, [7]. Singular systems are also frequently encountered in electronic and economics [8]. In recent years a great deal of work has been devoted to the analysis and design techniques for singular systems [9–11]. On the other hand, the problem of observer design for linear systems has been greatly treated for the standard and singular systems with or without unknown inputs (see [12–14] and references therein). In [15], extension to observers design for Lipschitz singular systems has been presented; however, the observer considered has a singular system form. Recently a new method for the observers design is presented for a class of singular systems, where the nonlinearity is assumed to be composed of a Lipschitz one and an arbitrary one; the latter can be considered as an unknown disturbance. The approach is based on the parameterization of the generalized Sylvester equations solutions and unifies the design of full-, reduced-, and minimal-order observers the observer presented is causal and has a standard system form [16]. However, only the case where the model and the measurement are free from noises was considered.

The state estimation problem for linear singular systems in presence of noises has been the subject of several studies

in the past decades. We can distinguish two approaches, the Kalman observing approach and H_∞ approach. In the Kalman observing, the system and the measurement noises are assumed to be Gaussian with known statistics [17–19]. When the noises are arbitrary signals with bounded energy, the H_∞ observing permits to guarantee a noise attenuation level [20]. Recently, a number of papers have appeared that deal with the H_∞ observing for singular systems; see, for example, [21–23] and references therein. In all these works only full- or reduced-order observers were presented for the square singular systems.

In this paper, we consider the H_∞ observers design problem for a class of Lipschitz nonlinear singular systems. The approach extends the work [16] to the case where the model and the measurement are affected by noises. Sufficient conditions in terms of LMIs are given for this H_∞ problem. The method is more general than the one considered in [16] since it assumes only the impulse observability of the linear part. It also unifies the design for full-order, reduced-order, and minimal-order observers. A numerical example is given to illustrate our results.

2. Problem Formulation

Consider the following nonlinear singular system:

$$E\dot{x}(t) = Ax(t) + Bu(t) + Df(t, F_L x, u) + D_1 w(t), \quad (1)$$

$$y(t) = Cx(t) + D_2 w(t), \quad (2)$$

with the initial state $x(0) = x_0$, where $x(t) \in \mathbb{R}^n$ is the semi state vector, $u(t) \in \mathbb{R}^m$ is the known input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector containing both system and measurement noises, and $y(t) \in \mathbb{R}^p$ is the measurement output. Matrix $E \in \mathbb{R}^{n_1 \times n}$, and, when $n_1 = n$, matrix E is singular. Matrices $A \in \mathbb{R}^{n_1 \times n}$, $B \in \mathbb{R}^{n_1 \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{n_1 \times n_f}$, $D_1 \in \mathbb{R}^{n_1 \times n_w}$ and $D_2 \in \mathbb{R}^{p \times n_w}$. The nonlinearity $f(t, F_L x, u)$ verifies the Lipschitz constraints:

$$\|f(t, F_L x_1, u) - f(t, F_L x_2, u)\| \leq \lambda(\|F_L(x_1 - x_2)\|), \quad (3)$$

where λ is a known Lipschitz constant and matrix F_L is real with appropriate dimension.

Let r be the rank of the matrix E , and let $\Phi \in \mathbb{R}^{r_1 \times m_1}$ be a full row rank matrix such that $\Phi E = 0$, with $r_1 = n_1 - r$. Then, from (1), we obtain

$$\Phi A x(t) + \Phi D_1 w(t) + \Phi D f(t, F_L x, u) = -\Phi B u(t). \quad (4)$$

In the sequel we assume the following.

Assumption 1. Consider $\text{rank} \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix} = n$.

Before presenting our main results, we can make the following remarks.

Remark 2. When $n_1 = n$, Assumption 1 is exactly the impulse observability of the linear singular system (E, A, B, C) .

Remark 3. Condition $\Phi E = 0$ is more general than condition $\Phi \begin{bmatrix} E & D \end{bmatrix} = 0$ considered in [16]. In fact when the matrix $\begin{bmatrix} E & D \end{bmatrix}$ is of full row rank, matrix $\Phi = 0$ and Assumption 1 becomes $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$ which is more restrictive than the impulse observability condition.

Now, let us consider the following reduced-order observer for system (1):

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Jy(t) + Hu(t) + TDf(t, F_L \hat{x}, u), \\ \hat{x}(t) &= P\zeta(t) - Q\Phi Bu(t) + Fy(t) - Q\Phi Df(t, F_L \hat{x}, u), \end{aligned} \quad (5)$$

with the initial condition $\zeta(0) = \zeta_0$. Vector $\zeta(t) \in \mathbb{R}^q$ represents the state vector of the observer, and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. Matrices N, J, T, H, P, Q , and F are unknown matrices of appropriate dimensions, which must be determined such that,

- (1) for $w(t) = 0$, the error $e(t) = \hat{x}(t) - x(t)$ asymptotically converges to 0,
- (2) for $w(t) \neq 0$, we solve the $\min \sup_{\omega \in L_2 - \{0\}} (\|e\|_{L_2} / \|\omega\|_{L_2})$.

Let the error between $\zeta(t)$ and $TEx(t)$ be

$$\varepsilon(t) = \zeta(t) - TEx(t), \quad (6)$$

then we obtain the following dynamics of $\varepsilon(t)$:

$$\begin{aligned} \dot{\varepsilon}(t) &= \dot{\zeta}(t) - TE\dot{x}(t) \\ &= N\varepsilon + (NTE - TA + JC)x(t) \\ &\quad + (H - TB)u(t) + TD\Delta f + (JD_2 - TD_1)w(t), \\ \hat{x}(t) &= P\varepsilon(t) + \begin{bmatrix} P & Q & F \end{bmatrix} \begin{bmatrix} TE \\ \Phi A \\ C \end{bmatrix} x(t) \\ &\quad - Q\Phi D\Delta f + (Q\Phi D_1 + FD_2)w(t), \end{aligned} \quad (7)$$

where $\Delta f = f(t, F_L \hat{x}(t), u) - f(t, F_L x(t), u)$.

Under Assumption 1, if there exists a matrix T such that

$$NTE - TA + JC = 0, \quad (8)$$

$$H = TB,$$

$$\begin{bmatrix} P & Q & F \end{bmatrix} \begin{bmatrix} TE \\ \Phi A \\ C \end{bmatrix} = I, \quad (9)$$

then (7) becomes

$$\dot{\varepsilon}(t) = N\varepsilon(t) + TD\Delta f + \varphi_1 w(t), \quad (10)$$

$$e(t) = P\varepsilon(t) - Q\Phi D\Delta f + \varphi_2 w(t), \quad (11)$$

where $\varphi_1 = JD_2 - TD_1$ and $\varphi_2 = Q\Phi D_1 + FD_2$.

Now the problem of the H_∞ observer design is reduced to find the matrices N, J, H, P, Q, F , and T such that (8) is satisfied and the worst case observers error energy over all bounded energy disturbances $w(t)$ is minimized.

Remark 4. The observer given in (5) is more general than that presented in [16]; this can be seen from the fact that, for $\Phi D = 0$, and $w(t) = 0$, we obtain the observer given in [16]. As in [16] this observer is general, and its design unifies the full-order ($q = n$), the reduced-order ($q = n - p$), and the minimal-order observers design.

3. Main Results

In this section we will present the H_∞ observers design. The H_∞ observer design problem can be formulated as follows: given the nonlinear singular system (1)-(2) and a prescribed level of noise $\gamma > 0$, find a suitable observer of the form (5), such that,

- (1) for $w(t) = 0$, estimation error (11) converges asymptotically to 0, that is, $e(t) \rightarrow 0$ as $t \rightarrow \infty$,
- (2) under the zero initial condition, the error $e(t)$ satisfies $\|e(t)\|_{L^2} < \gamma \|w(t)\|_{L^2}$ for any $w(t) \in L_2 - \{0\}$, where $\gamma > 0$ is the prescribed constant.

Now, the dynamics error (10)-(11) can be written in a singular form as

$$\mathbb{E}\dot{\xi}(t) = \mathbb{A}\xi(t) + \mathbb{B}\Delta f + \mathbb{D}w(t), \quad (12)$$

where $\xi(t) = \begin{bmatrix} \epsilon \\ e \end{bmatrix}$, $\mathbb{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbb{A} = \begin{bmatrix} N & 0 \\ P & -I \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} TD \\ -Q\Phi D \end{bmatrix}$, and $D = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$. Then, the error $e(t) \rightarrow 0$ as $t \rightarrow \infty$, for $w(t) = 0$ if system (12) is asymptotically stable.

3.1. Stability Analysis. Before giving the H_∞ observers design method for system (1)-(2), let us deal with the stability analysis problem and derive a sufficient condition, in a strict LMI form, for system (12) to be asymptotically stable for $w(t) = 0$. The following lemma gives this condition.

Lemma 5. For $w(t) = 0$, system (12) is asymptotically stable, if there exists a matrix Y , such that the following LMIs are satisfied:

$$\mathbb{E}^T Y = Y^T \mathbb{E} \geq 0, \quad (13)$$

$$\begin{bmatrix} \mathbb{A}^T Y + Y^T \mathbb{A} + \mu \rho & Y^T \mathbb{B} \\ \mathbb{B}^T Y & -\mu I \end{bmatrix} < 0 \quad (14)$$

with $\rho = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_1^2 F_L^T F_L \end{bmatrix}$.

Proof. For stability analysis we construct the following Lyapunov candidate function:

$$V(t) = \xi^T(t) \mathbb{E}^T Y \xi(t) = \xi^T(t) Y^T \mathbb{E} \xi(t) \quad (15)$$

with $\mathbb{E}^T Y = Y^T \mathbb{E} \geq 0$. The derivative of $V(t)$ along the solution of (12), for $w(t) = 0$, is given by

$$\begin{aligned} \dot{V}(t) &= \dot{\xi}^T(t) \mathbb{E}^T Y \xi(t) + \xi^T(t) Y^T \mathbb{E} \dot{\xi}(t) \\ &= (\mathbb{A}\xi(t) + \mathbb{B}\Delta f)^T Y \xi(t) + \xi^T(t) Y^T (\mathbb{A}\xi(t) + \mathbb{B}\Delta f) \\ &= \xi^T(t) (\mathbb{A}^T Y + Y^T \mathbb{A}) \xi(t) \\ &\quad + \Delta f^T \mathbb{B}^T Y \xi(t) + \xi^T(t) Y^T \mathbb{B} \Delta f. \end{aligned} \quad (16)$$

Let u and v be two vectors of appropriate dimensions, then for all scalar $\mu > 0$ the following inequality holds:

$$u^T v + v^T u \leq \mu u^T u + \frac{1}{\mu} v^T v. \quad (17)$$

By using (17), it is not difficult to check that

$$\begin{aligned} &\Delta f^T (\mathbb{B}^T)^T Y \xi(t) + \xi^T(t) Y^T (\mathbb{B}) \Delta f \\ &\leq \frac{1}{\mu} \xi^T(t) Y^T \mathbb{B} \mathbb{B}^T Y \xi(t) + \mu \Delta f^T \Delta f. \end{aligned} \quad (18)$$

In this case we have $\dot{V}(t) \leq \xi^T(t) (\mathbb{A}^T Y + Y^T \mathbb{A}) \xi(t) + \mu \Delta f^T \Delta f + (1/\mu) \xi^T(t) Y^T \mathbb{B} \mathbb{B}^T Y \xi(t)$. On the other hand, by using the Lipschitz conditions and the fact that $e(t) = [0 \ 1] \xi(t)$, we have $\mu \Delta f^T \Delta f \leq \mu \xi^T(t) \rho \xi(t)$, where $\rho = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_1^2 F_L^T F_L \end{bmatrix}$. Then we obtain the following inequality:

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t) (\mathbb{A}^T Y + Y^T \mathbb{A}) \xi(t) + \xi^T(t) \mu \rho \xi(t) \\ &\quad + \frac{1}{\mu} \xi^T(t) Y^T \mathbb{B} \mathbb{B}^T Y \xi(t) \\ &= \xi^T(t) \left(\mathbb{A}^T Y + Y^T \mathbb{A} + \frac{1}{\mu} Y^T \mathbb{B} \mathbb{B}^T Y + \mu \rho \right) \xi(t) \end{aligned} \quad (19)$$

and $\dot{V}(t) < 0$, if $\mathbb{A}^T Y + Y^T \mathbb{A} + (1/\mu) Y^T \mathbb{B} \mathbb{B}^T Y + \mu \rho < 0$.

Using the Schur complement we obtain (14), then Lemma 5 is proved. $\square$

From this lemma, one can see that the stability conditions (13) and (14) are nonstrict LMIs, which contain equality constraints; this may result in numerical problems when checking such conditions. Therefore, strict LMIs conditions are more desirable than nonstrict ones from the numerical point of view. The following lemma presents the stability conditions in a strict LMI formulation.

Lemma 6. For $w(t) = 0$, system (12) is asymptotically stable if there exist a positive definite matrix X_1 and matrices X_2, Q_1 , and Q_2 and a scalar μ such that the following LMI is satisfied:

$$\begin{bmatrix} N^T X_1 + X_1^T N + X_2 P + P^T X_2^T + P^T Q_1 + Q_1^T P & -X_2 - Q_1^T + P^T Q_2 & X_1^T T D - Q_1^T Q \Phi D - X_2 Q \Phi D \\ * & -Q_2 - Q_2^T + \mu \lambda^2 F_L^T F_L & -Q_2^T Q \Phi D \\ * & * & -\mu I \end{bmatrix} < 0. \quad (20)$$

Proof. Let $Y = X\mathbb{E} + \mathbb{E}^{\perp T} \bar{Q}$, where $\mathbb{E}^{\perp}$ is the orthogonal matrix of $\mathbb{E}$ satisfying $\mathbb{E}^{\perp} \mathbb{E} = 0$ and $\mathbb{E}^{\perp} \mathbb{E}^{\perp T} > 0$, and

$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$, and $\bar{Q} = [\bar{Q}_1 \ \bar{Q}_2]$. It is easy to see that $\mathbb{E}^T Y = Y^T E \geq 0$, since $X_1 > 0$. In this case (13)-(14) reduce to

$$\begin{bmatrix} \mathbb{A}^T X \mathbb{E} + \mathbb{A}^T \mathbb{E}^{\perp T} \bar{Q} + \mathbb{E}^T X^T \mathbb{A} + \bar{Q}^T \mathbb{E}^{\perp} \mathbb{A} & \mathbb{E}^T X^T \mathbb{B} + \bar{Q}^T \mathbb{E}^{\perp} \mathbb{B} \\ \mathbb{B}^T X \mathbb{E} + \mathbb{B}^T \mathbb{E}^{\perp T} \bar{Q} & -\mu I \end{bmatrix} < 0. \quad (21)$$

On the other hand by inserting the values of $\mathbb{E}, \mathbb{A}, \mathbb{B}$, and ρ into (21) we obtain (20). $\square$

Remark 7. For $\Phi D = 0$, inequality (20) becomes

$$\begin{bmatrix} N^T X_1 + X_1^T N + X_2 P + P^T X_2^T + P^T Q_1 + Q_1^T P & -X_2 - Q_1^T + P^T Q_2 & X_1^T T D \\ -X_2^T - Q_1 + Q_2^T P & -Q_2 - Q_2^T + \mu \lambda_1^2 F_L^T F_L & 0 \\ (TD)^T X_1 & 0 & -\mu I \end{bmatrix} < 0. \quad (22)$$

By premultiplying (22) by $\begin{bmatrix} I & P^T & 0 \\ 0 & 0 & I \end{bmatrix}$ and postmultiplying it by $\begin{bmatrix} I & 0 \\ P & 0 \\ 0 & I \end{bmatrix}$, we obtain $\begin{bmatrix} N^T X_1 + X_1^T N + \mu \lambda_1 F_L^T F_L & X_1^T T D \\ (TD)^T X_1 & -\mu I \end{bmatrix} < 0$, which is exactly the inequality (15) of [16].

3.2. H_{∞} Observes Design. In this section we shall present the H_{∞} observer design. The following lemma gives the sufficient

conditions for system (12) to be stable for $w(t) = 0$ and $\|e(t)\|_{L_2} < \gamma \|w(t)\|_{L_2}$ for $w(t) \neq 0$.

Lemma 8. *The error $e(t)$ given by (11) is asymptotically stable for $w(t) = 0$ and $\|e(t)\|_{L_2} < \gamma \|w(t)\|_{L_2}$ for $w(t) \neq 0$, if there exist a positive definite matrix X_1 and matrices X_2, Q_1, Q_2 and scalars μ and γ such that the following LMI is satisfied:*

$$\begin{bmatrix} N^T X_1 + X_1^T N + X_2 P + P^T X_2^T + P^T Q_1 + Q_1^T P & -X_2 - Q_1^T + P^T Q_2 & (X_1^T T D - Q_1^T Q \Phi D - X_2 Q \Phi D) & (X_1^T \varphi_1 + X_2 \varphi_2 + Q_1^T \varphi_2) \\ * & -Q_2 - Q_2^T + I + \mu \lambda^2 F_L^T F_L & -Q_2^T Q \Phi D & Q_2^T \varphi_2 \\ * & * & -\mu I & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0. \quad (23)$$

Proof. Consider the following Lyapunov candidate function $V(t) = \xi^T \mathbb{E}^T Y \xi = \xi^T(t) Y^T \mathbb{E} \xi(t)$, with $\mathbb{E}^T Y = Y^T \mathbb{E}$. Then we have

$$\begin{aligned} \dot{V}(t) &= \dot{\xi}^T(t) \mathbb{E}^T Y \xi(t) + \xi^T(t) Y^T \mathbb{E} \dot{\xi}(t) \\ &= (\mathbb{A} \xi(t) + \mathbb{B} \Delta f + \mathbb{D} w(t))^T Y \xi(t) \\ &\quad + \xi(t)^T Y^T (\mathbb{A} \xi(t) + \mathbb{B} \Delta f + \mathbb{D} w(t)) \\ &= \xi^T(t) (\mathbb{A}^T \xi + Y^T \mathbb{A}) \xi(t) + \Delta f^T \mathbb{B}^T Y \xi(t) \\ &\quad + \xi(t)^T Y^T \mathbb{B} \Delta f + w^T(t) \mathbb{D}^T Y \xi(t) + \xi(t)^T Y^T \mathbb{D} w(t). \end{aligned} \quad (24)$$

From (17) we have the following inequality:

$$\begin{aligned} &\Delta f^T(\mathbb{B})^T Y \varepsilon(t) + \varepsilon^T(t) Y^T(\mathbb{B}) \Delta f \\ &\leq \frac{1}{\mu} \varepsilon^T(t) Y^T \mathbb{B} \mathbb{B}^T Y \varepsilon(t) + \mu \Delta f^T \Delta f. \end{aligned} \quad (25)$$

Then $\dot{V}(t)$ becomes

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t) (\mathbb{A}^T Y + Y^T \mathbb{A}) \xi(t) \\ &\quad + \mu \Delta f^T \Delta f + \frac{1}{\mu} \xi^T(t) Y^T \mathbb{B} \mathbb{B}^T Y \xi(t) \\ &\quad + w^T(t) \mathbb{D}^T Y \xi(t) + \xi^T(t) Y^T \mathbb{D} w(t). \end{aligned} \quad (26)$$

On the other hand, we have $\Delta f^T \Delta f \leq \lambda^2 e^T(t) F_L^T F_L e(t)$, and we know that $e(t) = \begin{bmatrix} 0 & I \end{bmatrix} \xi(t)$ then $\Delta f^T \Delta f \leq \xi^T(t) \rho \xi(t)$, where $\rho = \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2 F_L^T F_L \end{bmatrix}$. Then

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t) (\mathbb{A}^T Y + Y^T \mathbb{A}) \xi(t) \\ &\quad + \mu \lambda^2 \xi^T(t) \rho \xi(t) + \frac{1}{\mu} \xi^T v Y^T \mathbb{B} \mathbb{B}^T Y \xi(t) \\ &\quad + w^T \mathbb{D}^T Y \xi(t) + \xi^T(t) Y^T \mathbb{D} w(t). \end{aligned} \quad (27)$$

Let $\eta(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix}$, then from (27) and (12) we obtain the following inequality:

$$\dot{V} + e^T(t) e(t) - \gamma^2 w^T(t) w(t) \leq \eta^T(t) \Sigma \eta(t) \quad (28)$$

with

$$\Sigma = \begin{bmatrix} \mathbb{A}^T Y + Y^T \mathbb{A} + \frac{1}{\mu} Y^T \mathbb{B} \mathbb{B}^T Y + \bar{\rho} & Y^T \mathbb{D} \\ \mathbb{D}^T Y & -\gamma^2 I \end{bmatrix}, \quad (29)$$

where $\bar{\rho} = \begin{bmatrix} 0 & 0 \\ 0 & I + \mu \lambda^2 F_L^T F_L \end{bmatrix}$.
If $\Sigma < 0$, we obtain

$$\dot{V} < \gamma^2 w^T(t) w - e^T(t) e(t). \quad (30)$$

Integrating the two sides of this inequality gives

$$\int_0^\infty \dot{V}(\tau) d\tau < \int_0^\infty \gamma^2 w^T(\tau) w(\tau) d\tau - \int_0^\infty e^T(\tau) e(\tau) d\tau \quad (31)$$

or equivalently

$$V(\infty) - V(0) < \gamma^2 \|w(t)\|_2^2 - \|e(t)\|_2^2, \quad (32)$$

under zero initial conditions, we obtain

$$V(\infty) < \gamma^2 \|w(t)\|_2^2 - \|e(t)\|_2^2 \quad (33)$$

which leads to

$$\|e(t)\|_2^2 < \gamma^2 \|w(t)\|_2^2. \quad (34)$$

Using Schur complement we obtain

$$\Sigma = \begin{bmatrix} \mathbb{A}^T Y + Y^T \mathbb{A} + \bar{\rho} & Y^T \mathbb{B} & Y^T \mathbb{D} \\ \mathbb{B}^T Y & -\mu I & 0 \\ \mathbb{D}^T Y & 0 & -\gamma^2 I \end{bmatrix} < 0. \quad (35)$$

Let $Y = XE + \mathbb{E}^{\perp T} \bar{Q}$ with

$$\bar{Q} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}, \quad (36)$$

we obtain

$$\mathbb{E}^T Y = Y^T E \geq 0. \quad (37)$$

And by substituting $\mathbb{E}, \mathbb{A}, \mathbb{B}$ and $\mathbb{D}$ by their values we obtain (23), which proves the lemma. $\square$

Remark 9. The results of Lemmas 6 and 8 are independent of the choice of matrix $\mathbb{E}^\perp$; this can be seen from the fact that the general form of $\mathbb{E}^\perp$ is $\mathbb{E}^\perp = \begin{bmatrix} 0 & M \end{bmatrix}$, where M is an arbitrary nonsingular matrix; in this case we have $\mathbb{E}^{\perp T} \bar{Q} = \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix}$, where $\bar{Q}_1 = MQ_1$ and $\bar{Q}_2 = M$, which shows that it suffices to choose $\mathbb{E}^\perp = \begin{bmatrix} 0 & I \end{bmatrix}$.

Before giving the design method for the observer (5), let us consider (8) and let $\tilde{T} = T + \Psi\Phi$, where Ψ is an arbitrary matrix of appropriate dimension; they can be written as

$$\begin{bmatrix} N & \Psi & J \end{bmatrix} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \end{bmatrix} = \tilde{T}A, \quad (38)$$

$$\begin{bmatrix} P & Q & F \end{bmatrix} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \end{bmatrix} = I_n. \quad (39)$$

Equations (38) and (39) have a solution if and only if

$$\text{rank} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \\ \tilde{T}A \\ I_n \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \end{bmatrix} = n. \quad (40)$$

Now, from Assumption 1 and (40), we have

$$\text{rank} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \end{bmatrix} = \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix} = n. \quad (41)$$

Let R be any full row rank matrix such that

$$\text{rank} \begin{bmatrix} R \\ \Phi A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{T}E \\ \Phi A \\ C \end{bmatrix} = n, \quad (42)$$

then there always exist matrices parameter K and $\tilde{T}$ such that

$$\tilde{T}E = R - K \begin{bmatrix} \Phi A \\ C \end{bmatrix} \quad (43)$$

or equivalently

$$\begin{bmatrix} \tilde{T} & K \end{bmatrix} \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix} = R. \quad (44)$$

Then, under Assumption 1, there exists a solution to (43) given by

$$\begin{bmatrix} \tilde{T} & K \end{bmatrix} = R \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix}^\dagger. \quad (45)$$

In this case matrices $\tilde{T}$ and K are given by

$$\begin{aligned} \tilde{T} &= R \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix}^\dagger \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ K &= R \begin{bmatrix} E \\ \Phi A \\ C \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ I \end{bmatrix}. \end{aligned} \quad (46)$$

Also, under Assumption 1, the general solution to (38) is given by

$$\begin{bmatrix} N & \Psi & J \end{bmatrix} = \tilde{T} A \Omega^\dagger - Z_1 (I - \Omega \Omega^\dagger), \quad (47)$$

here $\Omega = \begin{bmatrix} \tilde{T} E \\ \Phi A \\ C \end{bmatrix}$ and Z_1 is an arbitrary matrix of appropriate dimension.

And also under Assumption 1 one solution to (39) is given by

$$\begin{bmatrix} P & Q & F \end{bmatrix} = I_n \Omega^\dagger. \quad (48)$$

Now, define the following matrices:

$$\begin{aligned} \Lambda_P &= \Omega^\dagger \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, & \Lambda_Q &= \Omega^\dagger \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \\ \Lambda_N &= \tilde{T} A \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, & \Lambda_\Psi &= \tilde{T} A \Lambda_Q, & \Lambda_J &= \tilde{T} A \Lambda_F, \\ \Delta_N &= (I - \Omega \Omega^\dagger) \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, & \Delta_\Psi &= (I - \Omega \Omega^\dagger) \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \\ \Delta_J &= (I - \Omega \Omega^\dagger) \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, & \Lambda_F &= \Omega^\dagger \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \end{aligned} \quad (49)$$

then we obtain

$$\begin{aligned} N &= \Lambda_N - Z_1 \Delta_N, & \Psi &= \Lambda_\Psi - Z_1 \Delta_\Psi, \\ J &= \Lambda_J - Z_1 \Delta_J, & P &= \Lambda_P, \\ Q &= \Lambda_Q, & F &= \Lambda_F, \\ \varphi_1 &= \Lambda_{\varphi_1} - Z_1 \Delta_{\varphi_1}, & \varphi_2 &= \Lambda_{\varphi_2}, \\ TD &= \Lambda_{TD} - Z_1 \Delta_{TD}, \end{aligned} \quad (50)$$

where

$$\begin{aligned} \Lambda_{\varphi_1} &= (\Lambda_J D_2 - \tilde{T} D_1 + \Lambda_\Psi \Phi D_1), \\ \Delta_{\varphi_1} &= (\Delta_J D_2 + \Delta_\Psi \Phi D_1), & \Delta_{TD} &= -\Delta_\Psi \Phi D, \end{aligned} \quad (51)$$

$$\Lambda_{\varphi_2} = (\Lambda_Q \Phi D_1 + \Lambda_F D_2), \quad \Lambda_{TD} = \tilde{T} D - \Lambda_\Psi \Phi D.$$

Now, the H_∞ observer design can be obtained from the following theorem.

Theorem 10. *There exists an observer of the form (5) such that the error $e(t)$ given by (10)-(11) is asymptotically stable for $w(t) = 0$ and $\|e(t)\|_{L_2} < \gamma \|w(t)\|_{L_2}$ for $w(t) \neq 0$ if there exist a symmetric positive definite matrix X_1 and matrices X_2, Ω_{X_1} and scalars μ and γ such that the following LMIs are satisfied:*

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (1,2)^T & (2,2) & (2,3) & (2,4) \\ (1,3)^T & (2,3)^T & -\mu I & 0 \\ (1,4)^T & (2,4)^T & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad (52)$$

where

$$\begin{aligned} (1,1) &= \Lambda_N^T X_1 - \Delta_N^T \Omega_{X_1} + X_1 \Lambda_N - \Omega_{X_1}^T \Delta_N \\ &\quad + X_2 \Lambda_P + \Lambda_P^T X_2^T + \Lambda_P^T Q_1 + Q_1^T \Lambda_P, \\ (1,2) &= -X_2 - Q_1^T + \Lambda_P^T Q_2, \\ (1,3) &= X_1 \Lambda_{TD} - \Omega_{X_1}^T \Delta_{TD} - Q_1^T \Lambda_Q \Phi D - X_2 \Lambda_Q \Phi D, \\ (1,4) &= X_1 \Lambda_{\varphi_1} - \Omega_{X_1}^T \Delta_{\varphi_1} + Q_1^T \Lambda_{\varphi_1} + X_2 \Lambda_{\varphi_2}, \\ (2,2) &= -Q_2 - Q_2^T + I + \mu \lambda^2 F_L^T F_L, \\ (2,3) &= -Q_2^T \Lambda_Q \Phi D, \\ (2,4) &= Q_2^T \Lambda_{\varphi_2}. \end{aligned} \quad (53)$$

Proof. Let $\Omega_{X_1} = Z_1^T X_1$, and by substituting N, P, Q, φ_1 , and φ_2 by their values in (23) we obtain (52). $\square$

Procedure for the Observers Design

- (1) Under Assumption 1, compute matrices $\Lambda_N, \Delta_N, \Lambda_J, \Delta_J, \Lambda_P, \Lambda_Q, \Lambda_F, \Lambda_\Psi, \Delta_\Psi, \Lambda_{\varphi_1}, \Delta_{\varphi_1}, \Lambda_{\varphi_2}, \Delta_{\varphi_2}, \Lambda_{TD}, \Delta_{TD}$.
- (2) Solve the LMI (52) to obtain the matrix parameter Z_1 .
- (3) Compute the observers parameters N, J, T, H, P, Q , and F .

FIGURE 1: Perturbation $w(t)$.

4. Numerical Example

Let us consider the following continuous nonlinear singular system of the form (1) with

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad (54)$$

$D_2 = [1 \ 1]$ and $u(t) = \sin(2t)$. The nonlinearity $f(x, u, t) = \sin(x_2(t))$. For this system, the matrix $\Phi = [0 \ 0 \ 1]$. In this case it is easy to see that Assumption 1 is verified. We will design a reduced-order observer of dimension $q = 2$; let $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\text{rank} \begin{bmatrix} R \\ \Phi^A \\ C \end{bmatrix} = 3$. For $\gamma^2 = 9.762$, from Section 3 we obtain the following results:

$$\Omega_{X_1} = 10^2 \begin{bmatrix} -4.183 & 0.5 \\ -0.331 & -0.11 \\ -0.567 & 0.276 \\ -0.045 & 0.627 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 3.919 & -2.628 \\ -2.628 & 5.139 \end{bmatrix}. \quad (55)$$

FIGURE 2: x_i (blue) and $\hat{x}_i$ (red).

The H_∞ observer is given by the following model:

$$\dot{\zeta}(t) = \begin{bmatrix} 0 & -1.996 \\ 0.666 & -2.121 \end{bmatrix} \zeta(t) + \begin{bmatrix} -0.331 \\ -1.414 \end{bmatrix} y(t) + \begin{bmatrix} -0.331 \\ -0.747 \end{bmatrix} u(t) + \begin{bmatrix} -0.331 \\ -0.747 \end{bmatrix} \sin(\hat{x}_2(t)),$$

$$\hat{x}(t) = \begin{bmatrix} 0 & -0.352 \\ 1 & 0 \\ 0 & 0.705 \end{bmatrix} \zeta(t) - \begin{bmatrix} 0.235 \\ 0 \\ 0.529 \end{bmatrix} u(t) + \begin{bmatrix} 0.764 \\ 0 \\ -0.529 \end{bmatrix} y(t) - \begin{bmatrix} 0.235 \\ 0 \\ 0.529 \end{bmatrix} \sin(\hat{x}_2(t)). \quad (56)$$

Simulation results are presented in Figures 1–3. Figure 1 presents the noises $w_1(t)$ and $w_2(t)$. Figures 2 and 3 show the estimation of the states x_1 , x_2 , and x_3 . It can be seen that the observer performs as expected.

FIGURE 3: Errors $e_i = \hat{x}_i - x_i$.

5. Conclusion

In this paper a new method for the H_∞ observers design for a class of Lipschitz nonlinear singular systems has been developed. The obtained results unify the observers design of full, reduced, and minimal orders. Sufficient conditions for the existence of these observers are given in terms of LMIs. The advantage of these LMIs conditions is that they can be performed by using convex optimization techniques available in Matlab LMI toolbox, for example. A numerical example has been presented to show the applicability of our approach. The extension of our work to more general nonlinear singular systems is under study.

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