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Research Article

Robust Stability of a Class of Unstable Systems under Mixed Uncertainty

Rami A. Maher[1] and Raghad Samir[2]

[1] *Electrial Engineering Department, Engineering Faculty, Isra University, Amman, Jordan*
[2] *Computer Engineering Department, IUST University, Daraa, Syria*

Correspondence should be addressed to Rami A. Maher, rami.maher@iu.edu.jo

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For unstable plants, the priority of control goes to the stability of synthesis, which means to find a stabilizer controller. In the case where the plant is subjected to structured and unstructured uncertainties, the stability problem becomes more crucial. The problem was solved by a conservative method based on generalized Kharitonov's theorem and Nevanlinna-Pick's interpolation (NPI) technique. This paper introduces a proposed straightforward numerical approach for loop shaping the unstructured additive or multiplicative maximum uncertainty magnitudes. The approach finds controllers, which are capable of stabilizing the interval system while the uncertainty box is enlarged to its maximum dimensions. To illustrate, we introduce some numerical examples.

1. Introduction

Since most of the theories in control engineering are stated for linear plant modeling, one may wonder whether such modeling is perfectly free from uncertainty. In fact, uncertainty in linear plant models may have several origins. Linearization or order reduction is not the only the reasons but also the measurement errors and the deviation of the operating point. Moreover, at high frequencies, both parameters and structure may change dramatically; uncertainty may exceed 100% at some frequency. Despite the existences of all details, one usually works with a simple law order nominal model and quantities of uncertainty [1]. Since the appearing of the surprisingly simple solution of robust stability with respect to parameter uncertainty given by Kharitonov [2], the initial results have been extended in many directions. One of the earliest most important results is the generalization developed by Chapellat and Bhattacharyya [3]. The applications of the Kharitonov theory to the analysis and synthesis of control systems have been introduced in many literatures such as [4–8].

Uncertainty may be grouped as structured (parametric) and unstructured. The former is formulated by bounding each uncertain parameter p within some region $[p-\varepsilon,\ p+\varepsilon]$, while the latter is more difficult to quantify, and it appears that the frequency domain is well suited for this class. This leads to complex perturbations, which are usually normalized such that the H_∞ norm is less than one.

Additive and multiplicative perturbations are the two classes of unstructured uncertainty usually considered in control systems. In these classes, the $G_0(s)$ and $G^0(s)$ define the nominal transfer function of the plant and the perturbed transfer function, respectively. The transfer function $G^0(s)$ will be in the multiplicative class $M(G_0(s), r(s))$, if $G^0(s)$ has the same number of unstable poles as $G_0(s)$ and $G^0(s) = (1+\delta_m(s))G_0(s)$. The frequency-dependent magnitude constraints are placed on $\delta_m(s)$ by a suitable real rational minimum phase H_∞ function $r(s)$ such that for all real frequencies w the inequality $0 < |\delta_m(jw)| < |r(jw)|$ hold. Similarly, the transfer function $G^0(s)$ will be in the additive class $A(G_0(s), r(s))$, where $G^0(s) = G_0(s) + \delta_m(s)$.

The necessary and sufficient conditions for robust stability in the $M(G_0(s), r(s))$ and $A(G_0(s), r(s))$ classes are, respectively,

$$\begin{gathered}\left\|G_0(s)C(s)(1+G_0(s)C(s))^{-1}r(s)\right\|_\infty < 1,\\ \left\|C(s)(1+G_0(s)C(s))^{-1}r(s)\right\|_\infty < 1.\end{gathered} \tag{1}$$

In this paper, a proposed numerical method for loop shaping the unstructured additive and multiplicative maximum uncertainty magnitudes is introduced. The considered class of systems is assumed to have given parameter uncertainties. The method is based on a rally use of both generalized Kharitonov's theorem and the Nevanlinna-Pick interpolation technique. The proposed method determines a different structure controller that stabilizes the interval system for the specific box of parameter uncertainty. The algorithm continues finding these stabilizers while the uncertainty box is enlarged to its maximum dimensions. The algorithm is executed such that the increment in the number of zeros and poles will be in its minimum value.

2. Controller Parameterization Based on NPI Theory [9, 10]

The robustness conditions (1) can be rewritten as

$$\begin{gathered}\|G_0(s)Q(s)r(s)\|_\infty < 1,\\ \|Q(s)r(s)\|_\infty < 1,\end{gathered} \tag{2}$$

where $Q(s)$ is a proper function defined by

$$Q(s) = C(s)[1 + G_0(s)C(s)]^{-1}. \tag{3}$$

Alternatively, the proper function $C(s)$ is given by

$$C(s) = Q(s)[1 - G_0(s)Q(s)]^{-1}. \tag{4}$$

Thus, the search for a proper or strictly proper, rational stabilizing controller $C(s)$ is reduced to finding $Q(s)$. Therefore, one can parameterize all stabilizing controllers by parameterizing all function $Q(s)$. For unstable SISO systems, the Nevanlinna-Pick interpolation theory [11] can be implemented to determine the function $Q(s)$ through finding a strictly bounded real function $u(s)$ (Schur function). The function $u(s)$ must satisfy the following interpolation and norm conditions:

$$\begin{gathered}u(\alpha_i) = \beta_i, \qquad \mathrm{Re}(\alpha_i) > 0, \quad |\beta_i| < 1,\ i = 1, 2, \ldots, b\\ \|u(s)\|_\infty < 1,\end{gathered} \tag{5}$$

where α_i are the RHS poles of the nominal plant, and β_i are the values given

$$\beta_i = \frac{r(\alpha_i)}{G_0(\alpha_i)B(\alpha_i)}, \tag{6}$$

where $B(s)$ is a Blaschke product defined as

$$B(s) = \frac{(\alpha_1 - s)\cdots(\alpha_b - s)}{(\alpha_1 + s)\cdots(\alpha_b + s)}. \tag{7}$$

Finally, the required $Q(s)$ is given by

$$Q(s) = \frac{B(s)u(s)}{r(s)}. \tag{8}$$

Additional interpolation conditions depend on the relative degree of $r(s)$. For proper controller where the relative degree of $r(s)$ is one, the additional condition is $u(\infty) = 0$, while for strictly proper controller where the relative degree of $r(s)$ is greater than one additional conditions must be incorporated.

3. Summary of Robust Stability Synthesis

In this section, a proposed approach to design a robust stabilizer controller for a class of SISO plants under mixed uncertainty is introduced [4]. The objective is to show that the frequency domain uncertainty induced by parametric uncertainty can be covered (loop shaped) by overbounding with suitable bounding function $r(s)$. NPI theory is applied to obtain the $Q(s)$ function, which parameterized all stable controllers. Once this is accomplished, the obtained robust controller, under the norm-bounded perturbation, successfully stabilizes the system subjected to mixed uncertainty.

Assume the nominal unstable plant and the perturbed plant are defined, respectively as,

$$\begin{gathered}G_0(s) = \frac{N_o(s)}{D_o(s)},\\ G^0(s) = \frac{n_0^o + n_1^o s + n_2^o s^2 + \cdots + n_p^o s^p}{d_0^o + d_1^o s + d_2^o s^2 + \cdots + d_q^o s^q},\end{gathered} \tag{9}$$

where N and D are fixed polynomials. The perturbation about the nominal mode is parameterized by ε as it is defined by the expressions (10):

$$\begin{gathered}n_i \in [n_i^o - \varepsilon, n_i^o + \varepsilon]; \quad i = 1, 2, \ldots, p,\\ d_j \in \left[d_j^o - \varepsilon, d_j^o + \varepsilon\right]; \quad j = 1, 2, \ldots, q;\ q \geq p.\end{gathered} \tag{10}$$

For each value of ε, a family of interval system $G(s, \varepsilon)$ and its associated set of Kharitonov's systems G_K (s,ε) can then be invoked. Therefore, an upper bound $\varepsilon_{\max}$ is easily found by letting ε be the smallest number such that the interval family (11) contains an unstable polynomial:

$$\left\{D(s) = d_o + \cdots + d_q s^q\right\} : d_j \in \left[d_j^o - \varepsilon_1, d_j^o + \varepsilon_1\right]. \tag{11}$$

In other words, since it is assumed that the number of unstable poles in the plant should remain unchanged, the maximum allowable value of the parameter change in the plant model is found. It is then required that ε be less than some $\varepsilon_{\max}$ such that the entire family of plants is stable. This value of $\varepsilon_{\max}$ can then be found by checking the Hurwitz stability of the denominator Kharitonov's polynomials in the Kharitonov system:

$$\mathrm{G}_K(s) := \left\{\frac{K_N^i}{K_D^j} : i, j \in \{1, 2, 3, 4\}\right\}. \tag{12}$$

Once $\varepsilon_{\max}$ is determined, a stabilizer controller can be synthesized for any $\varepsilon \leq \varepsilon_{\max}$. This can be done by defining the extremal segments of $G(s)$, say,

$$G_E(\varepsilon, s) := \left\{\frac{K_N^i(\varepsilon, s)}{S_D^j(\varepsilon, s)}; \frac{S_N^i(\varepsilon, s)}{K_D^j(\varepsilon, s)} : i, j \in \{1, 2, 3, 4\}\right\}. \tag{13}$$

Next, the boundary function $r(s)$, which bounds the frequency domain uncertainty induced by the parametric uncertainty at each frequency, is determined. This is accomplished by calculating the maximum magnitude of the extremal

segments of the plant. There, the difference between the extremal set and the nominal model set will represent the maximum additive unstructured perturbation magnitude, $\delta(\omega)$, at each frequency:

$$|G(\varepsilon, jw) - G_\circ(jw)| = |\Delta G(\varepsilon, jw)| = \delta(w). \quad (14)$$

The multiplicative unstructured uncertainty can be defined as

$$\frac{|G^o(\varepsilon, jw) - G_\circ(jw)|}{G_\circ(jw)} = |\Delta G(\varepsilon, jw)| = \delta(w). \quad (15)$$

Hence the maximum perturbation $\delta(\varepsilon, \omega)$ induced at each frequency is

$$\delta(\varepsilon, w) = \max_{G \in G^\varepsilon}\{|\Delta G(\varepsilon, jw)|\}. \quad (16)$$

Since it is required that the function $r(s)$ should be stable, proper (or strictly proper), real, rational, and minimum phase, then the maximum perturbation should satisfy

$$\begin{aligned} &\delta(\varepsilon, w) = \max_{G \in G^\varepsilon}\{|\Delta G(\varepsilon, jw)|\} < |r(jw)|, \\ &\text{that is, } |r(jw)| > \delta(\varepsilon, w), \quad w \in \text{IR}, \\ &\text{or } |\Delta G(\varepsilon, jw)| < |r(jw)|, \quad w \in \text{IR}. \end{aligned} \quad (17)$$

The main important problem is the choice of $r(s)$ function, which has the following features: $r(s)$ should be stable, proper (or strictly proper), real, rational, and minimum phase:

(i) $|r(jw)| > \delta(\varepsilon, w), w \in \text{IR}$ and for any ε;

(ii) $r(s)$ should satisfy the NPI conditions such as $|\beta| < 1, \|u(s)\|_\infty < 1$;

(iii) $r(s)$ should not cause any pole-zero cancellations between the model and the controller.

To illustrate, let us first consider a system of one unstable pole $s = \alpha$. In this case, the NPI theory will hold for some $\varepsilon \le \varepsilon_{\max}$, and there will be one interpolation condition given by

$$\beta = \frac{r(\alpha)}{G_0(\alpha)(\alpha - s)(\alpha + s)^{-1}}. \quad (18)$$

In this case, it is sufficient to let $r(s)$ be equal to a constant value $r = \delta(w)$. To complete the design, a relationship between the unstructured uncertainty $\delta(w)$ and parametric uncertainty values has to be determined over the range of frequency of interest. This can be performed by generating the extremal segments and searching for the largest perturbation at each frequency. Then the obtained $\delta(w)-\varepsilon$ graph serves for specifying the constant, r, corresponding to a specified value of ε. Based on the NPI theorem, the Schur function $u(s)$, the $Q(s)$, and the stabilized controller $C(s)$ can be found. However, when a large amount of parametric uncertainty has to be tolerated for the same value of r, the NPI constraints may be no longer satisfied. In [4] it is recommended to take the frequency information into account and attempt to design rational function $r(s)$ that loop-shaped the function $\delta(\varepsilon, w)$.

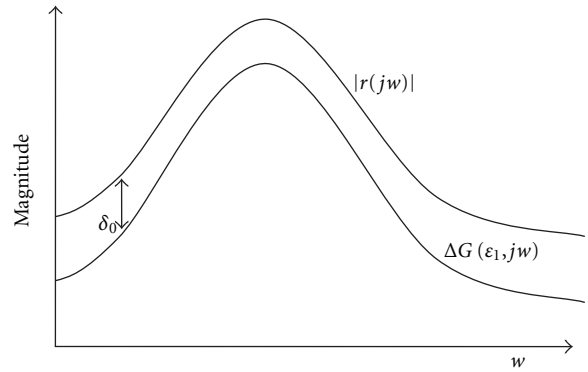

Figure 1: Definition of the norm distance δ_0.

4. Proposed Approach for Mixed Uncertainty Loop Shaping

As mentioned earlier, when it is required to tolerate larger value of parameter variation, the frequency response of the maximum unstructured uncertainty can be used to generate the $r(s)$ function, which in turn can be used through the NPI algorithm to construct a stabilizer controller. Only in simple cases, such a loop shaping can be performed by certain adhoc procedure. However, when the system has many unstable poles, such an adhoc procedure cannot be performed easily. Moreover, the problem becomes more complicated because the function $r(s)$ should itself satisfy specific properties and should be suitable for implementing the NPI technique.

For specific value say ε_1, a parameter δ_0 is defined as a norm distance between the maximum magnitude curve of the unstructured uncertainty $\Delta G(\varepsilon_1, jw)$ and the magnitude curve of $r(s)$ function $|r(jw)|$, as shown in Figure 1.

Hence, the inequality (13) can be transformed to equality written as

$$|r(jw)| = \max\{|\Delta G(\varepsilon, j\omega)|\} + \delta_0; \quad w \in \Re, \left[0, w_f\right], \quad (19)$$

where w_f is the maximum considered frequency.

Since the maximum magnitude of the unstructured uncertainty is known over the whole frequency range, then for specific value of δ_0, the magnitude of the $r(s)$ function is calculated. Let us denote this magnitude by z, and so we write

$$z^2 = |r(jw)|^2 = [\max\{|\Delta G(\varepsilon_1, j\omega)|\} + \delta_0]^2. \quad (20)$$

A general proper rational structure of $r(s)$ can be put in the form:

$$r(s) = \frac{\prod_{i=1}^{n}(a_i s + b_i)}{\prod_{i=1}^{m}(s + c_i)}, \quad n \le m, \quad (21)$$

where a_i, b_i, and c_i are real (can pose also complex values provided that $r(s)$ remains rational) unknown parameters; n and m are assumed orders. Thus,

$$|r(jw)| = \frac{\prod_{i=1}^{n}\sqrt{a_i{}^2w^2+b_i{}^2}}{\prod_{i=1}^{m}\sqrt{w^2+c_i{}^2}},$$
$$\text{or } z^2\prod_{i=1}^{m}(w^2+c_i{}^2) = \prod_{i=1}^{n}\left(a_i{}^2w^2+b_i{}^2\right). \tag{22}$$

Equation (22) can be written in the form:

$$\begin{aligned}&(A_1w^{2n}+A_2w^{2n-2}+\cdots+A_nw^2+A_{n+1})\\&\quad-(B_1w^{2m-2}+B_2w^{2m-4}+\cdots+B_{m-1}w^2+B_m)z^2\\&\quad=z^2w^{2m},\end{aligned} \tag{23}$$

where the coefficients A's and B's are related to the $r(s)$ parameters through the following two quadratic functions:

$$\begin{aligned}A_k &= f_k(a_1^2,a_2^2,\ldots,a_n^2,b_1^2,b_2^2,\ldots,b_n^2), \quad k=1,\ldots,n+1,\\ B_j &= g_j(c_1^2,c_2^2,\ldots,c_m^2), \quad j=1,\ldots,m.\end{aligned} \tag{24}$$

Obviously, these two quadratic functions can be assigned exactly for specific orders n and m. A necessary condition for the required loop shaping is that (23) should be valid for each frequency in the range $[0, w_f]$. Although it is sufficient to divide the frequency range to $(m+n+1)$ points to solve linearly for A's and B's coefficients, but such a small number of points will not certainly ensure the $r(s)$ bounding of the maximum unstructured uncertainty curve, especially in between these points. Therefore, a sufficiently high number of points (may be equal or greater than ten-time w_f) should be taken. Let $l \gg (m+n+1)$ be such a number, then an overdetermined linear system is obtained as

$$\begin{bmatrix}X_1 & \cdots & X_n & X_{n+1} & X_{n+2} & \cdots & X_{n+m+1}\end{bmatrix}\begin{bmatrix}A_1\\A_2\\\vdots\\A_{n+1}\\B_1\\\vdots\\B_m\end{bmatrix} = Y, \tag{25}$$

where the column vectors X_i and Y are of order l and are calculated from

$$\begin{aligned}X_i &= \left[w_1^{2(n-i+1)}\cdots w_l^{2(n-i+1)}\right]^T, \quad i=1,2,\ldots,n,\\ X_{n+1} &= [1 \quad 1\cdots,1]^T,\\ X_{n+1+h} &= \left[w_1^{2(m-h)}\cdots w_l^{2(m-h)}\right]^T, \quad h=1,2,\ldots,m,\\ Y &= \left[z_1^2w_1^{2m}\cdots z_l^2w_l^{2m}\right]^T,\end{aligned} \tag{26}$$

where w_i and z_i; $i=1,2,\ldots,l$ are the values of the selective frequencies and the corresponding magnitudes, respectively.

The least mean square linear regression is then used to obtain the values of the coefficients A's and B's. The parameters of the $r(s)$ function can be now calculated by reassignment of the nonlinear functions stated in (23); numerical methods may be required. The strictly proper minimum phase stable function $r(s)$ has the following rational form:

$$r(s) = \frac{\sum_{i=0}^{n}\alpha_{i+1}s^{n-i}}{\sum_{j=0}^{m}\alpha_{n+j+2}s^{m-j}}, \tag{27}$$

where α_i, $i=1,2,\ldots,m+n+1$ are the required parameters. With such reassignment, no loss of generality takes place, but a simplification of algebraic manipulation is rather gained. In the next sections, two considered cases will be shown for illustrating the proposed approach.

From mathematical point of view, the solution vector $[A_i\ B_i]$ can possess positive real or positive-negative real values (at least one coefficient is negative). This, consequently, causes complex parameters to appear in $r(s)$, and so the rationality property is not satisfied. Moreover, even with positive real parameters (to have stable minimum phase $r(s)$), the NPI algorithm may not work due to the failure of satisfying the condition $\beta_i < 1$, which related causally to α_i parameters.

To release the solution from such cases, the solution is parameterized by the norm distance δ_0. A small value of δ_0 may cause either complex parameters to appear or non-complete bounding. On the other hand, large values may cause the failure of the NPI conditions. Therefore, a search for a suitable value must be performed. A bisection searching can be carried out in the range $0 < \delta_0 < \delta_{\max}$, where $\delta_{\max}$ is any large value, which causes the NPI algorithm failure. If no such δ_0 exists, one can conclude that the assumed $r(s)$ structure is not correctly chosen with respect to the structured uncertainty parameter ε_1. However, this in turn completes the procedure to find a stabilizer controller while enlarging the box of parameter uncertainty up to its maximum. For example, for $\varepsilon_1 \le \varepsilon \le \varepsilon_2$, if one zero-two poles $r(s)$ is a workable structure, then a structure of two zeros-three poles is workable with $\varepsilon_2 \le \varepsilon \le \varepsilon_3$, where $\varepsilon_3 > \varepsilon_2$. Obviously, this enlargement is at the expense of a higher-order controller.

An iterative bisection algorithm can carry out the proposed approach of determining $r(s)$. The magnitude z is either increased or decreased by a magnitude Δz, depending on satisfying both the interpolation and other NPI conditions or on the closest bounding, respectively. Furthermore, the algorithm checks the necessary conditions that $r(s)$ function is a rational, minimum phase, and stable. When neither increasing nor decreasing Δz solves the problem, then the assumed structure of $r(s)$ is unsuitable to loop-shaping the maximum perturbation curve of the required parametric uncertainty ε_1. The assumed structure should be further upgraded as necessary. The numerical solution of the nonlinear functions (23) is completed in each iteration.

5. Numerical Examples and Simulation

Let us first illustrate the proposed approach for loop shaping, that is, finding the $r(s)$ structure for two cases as follows.

(1) Assume that the $r(s)$ is a one zero-two pole of the form

$$r(s) = \frac{\alpha_1 s + \alpha_2}{s^2 + \alpha_3 s + \alpha_4}, \quad n = 1,\ m = 2. \tag{28}$$

Then the coefficient vector is assigned to the $r(s)$ parameters as follows:

$$\begin{bmatrix} A_1 & A_2 & B_1 & B_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^2 & \alpha_2^2 & \alpha_3^2 - 2\alpha_4 & \alpha_4^2 \end{bmatrix}. \tag{29}$$

Hence, it easy to compute the $r(s)$ parameters as

$$\begin{aligned} \alpha_1 &= \sqrt{A_1}, \qquad \alpha_2 = \sqrt{A_2}, \\ \alpha_3 &= \sqrt{B_1 + 2\sqrt{B_2}}, \qquad \alpha_4 = \sqrt{B_2}, \end{aligned} \tag{30}$$

which should be all real.

(2) Assume that the $r(s)$ is two zeros-three poles of the form

$$r(s) = \frac{\alpha_1 s^2 + \alpha_2 s + \alpha_3}{s^3 + \alpha_4 s^2 + \alpha_5 s + \alpha_6}; \quad n = 2,\ m = 3 \tag{31}$$

In this case, (23) has the form:

$$A_1 w^4 + A_2 w^2 + A_3 - [B_1 w^4 + B_2 w^2 + B_3] z^2 = z^2 w^6. \tag{32}$$

The $r(s)$ parameters are related to the linear regression coefficients as follows:

$$\begin{aligned} \alpha_1 &= \sqrt{A_1}, \qquad \alpha_2 = \sqrt{A_2 + 2\sqrt{A_1 A_3}}, \\ \alpha_3 &= \sqrt{A_3}, \qquad \alpha_4 = \sqrt{B_1 + 2\alpha_5}, \\ \alpha_5 &= \sqrt{B_2 + 2\alpha_4\sqrt{B_3}}, \qquad \alpha_6 = \sqrt{B_3}. \end{aligned} \tag{33}$$

Note that the coefficients, α_4 and α_5, are coupled in the equations. Therefore, to solve, a numerical method has to be utilized.

Consider the unstable plant defined by a nominal transfer function [12]:

$$\begin{aligned} G_0(s) &= \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \\ &= \frac{30s + 10}{s^3 - 3.52s^2 - 3.59s + 14.9}. \end{aligned} \tag{34}$$

The poles are $\{-2, 2.35, 3.17\}$, that is, two unstable poles. The plant is subjected to parameter variations as follows:

$$\begin{aligned} b_1 &\in [30, 30], \quad b_0 \in [10 - \varepsilon, 10 + \varepsilon], \\ a_2 &\in [-3.52 - \varepsilon, -3.52 + \varepsilon], \\ a_1 &\in [-3.59 - \varepsilon, -3.95 + \varepsilon], \\ a_0 &\in [14.9 - \varepsilon, 14.9 + \varepsilon]. \end{aligned} \tag{35}$$

(I) Multiplicative Uncertainty. To start the proposed approach, it is required first to evaluate the maximum structured uncertainty, $\varepsilon_{\max}$, such that there is no change in the number of unstable poles ($\varepsilon_{\max} = 6.5$ is found). To illustrate the loop-shaping approach details, a parameter uncertainty value of $\varepsilon_1 = 1 < \varepsilon_{\max}$ is selected. Since the constant value case of $r(s)$ limits seriously the enlargement of ε_1, we exclude it here.

For this example, the extremal segments joining both Kharitonov's vertices and segments for numerator and denominator are

$$\begin{aligned} K_{n_1} &= b_1 s + (b_0 - \varepsilon_1), \\ K_{n_2} &= K_{n_1}, \\ K_{n_3} &= b_1 s + (b_0 + \varepsilon_1), \\ K_{n_4} &= K_{n_3}, \\ K_{d_1} &= s^3 + (a_2 + \varepsilon_1)s^2 + (a_1 - \varepsilon_1)s + (a_0 - \varepsilon_1), \\ K_{d_2} &= s^3 + (a_2 + \varepsilon_1)s^2 + (a_1 + \varepsilon_1)s + (a_0 - \varepsilon_1), \\ K_{d_3} &= s^3 + (a_2 - \varepsilon_1)s^2 + (a_1 - \varepsilon_1)s + (a_0 + \varepsilon_1), \\ K_{d_4} &= s^3 + (a_2 - \varepsilon_1)s^2 + (a_1 + \varepsilon_1)s + (a_0 + \varepsilon_1), \\ S_n &= (1 - \lambda)K_{n_2} + \lambda K_{n_3}, \\ S_{d_1} &= (1 - \lambda)K_{d_2} + \lambda K_{d_1}, \\ S_{d_2} &= (1 - \lambda)K_{d_1} + \lambda K_{d_3}, \\ S_{d_3} &= (1 - \lambda)K_{d_2} + \lambda K_{d_4}, \\ S_{d_4} &= (1 - \lambda)K_{d_3} + \lambda K_{d_4}, \quad 0 < \lambda < 1. \end{aligned} \tag{36}$$

Therefore, in this example, we have to consider only 12 (out of the theoretical 32 plants) extremal plants defined as

$$\begin{aligned} g_j &= \frac{K_{n_1}}{S_{d_j}}, \quad j = 1, 2, 3, 4, \\ g_{j+4} &= \frac{K_{n_3}}{S_{d_j}}, \quad j = 1, 2, 3, 4, \\ g_{j+8} &= \frac{S_n}{K_{d_j}}, \quad j = 1, 2, 3, 4. \end{aligned} \tag{37}$$

The unstructured uncertainty, $\Delta G(\varepsilon_1, jw)$, magnitudes of the interval system are computed from

$$|\Delta G(\varepsilon_1, jw)| = \left| \frac{g_i(\varepsilon_1, jw) - G_0(jw)}{G_0(jw)} \right|, \quad i = 1, 2, \ldots, 12. \tag{38}$$

The proposed approach of shaping the unstructured uncertainty is performed for assumed one zero-two poles $r(s)$ function. The result is a strictly proper stable function:

$$r(s) = \frac{1.382s + 1.8795}{s^2 + 5.93s + 10.47.} \tag{39}$$

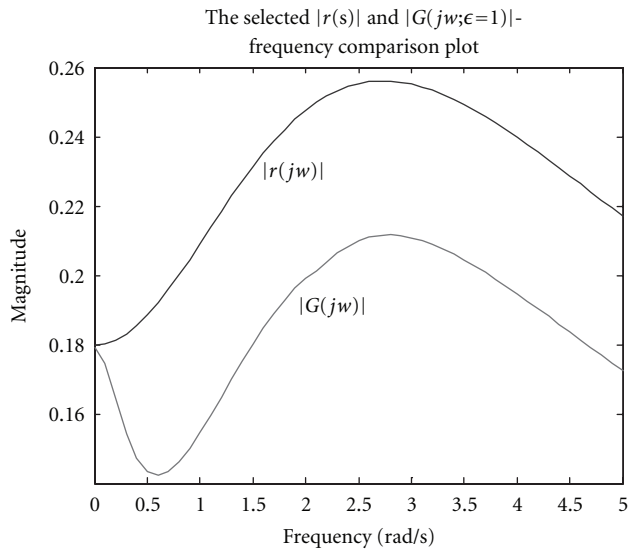

Figure 2: Magnitude of $r(jw)$ and the maximum magnitude of $\Delta G(\varepsilon_1, jw)$.

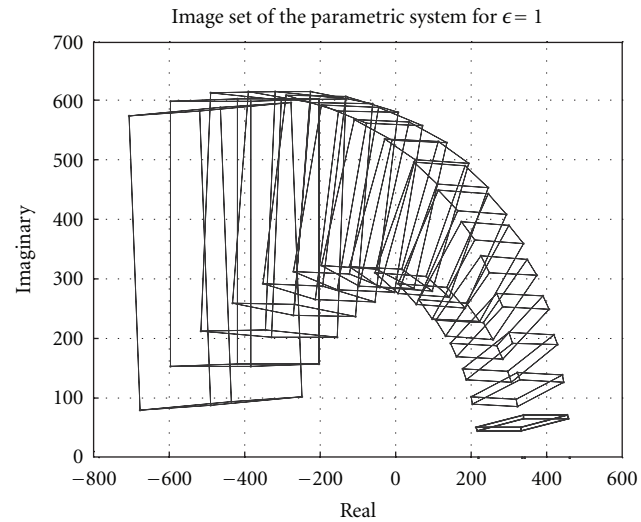

Figure 3: Kharitonov's templates with $\varepsilon_1 = 1$.

Figure 2 shows the plot of the magnitude of both $r(jw)$ and the maximum magnitude of $\Delta G(\varepsilon_1, jw)$. Finally, the NPI algorithm is applied (consisting of three interpolation conditions ($\{\alpha_1 = 2.35,\ \alpha_2 = 3.17,\ \alpha_3 = \infty\}$, $\{\beta_1 = 0.2404,\ \beta_2 = 0.2743,\ \beta_3 = 0\}$) to obtain the Schur function. The result is

$$u(s) = \frac{4.874s - 3.876}{s^2 + 7.498s + 8.398}, \tag{40}$$

which satisfies also the norm condition: $\|u(s)\|_\infty = 0.6795 < 1$.

The stabilizer controller is

$$C(s) = \frac{3.526s^4 + 25.161s^3 + 56.509s^2 + 11.226s - 58.721}{s^4 + 11.893s^3 + 47.996s^2 + 75.113s + 22.718}. \tag{41}$$

To verify that the controller $C(s)$ stabilizes the interval plant with $\varepsilon = \varepsilon_1 = 1$, Kharitonov's templates are plotted in Figure 3. As seen the origin is excluded, which indicates that the controller stabilizes the plant irrespective of structure uncertainty as well as multiplicative unstructured uncertainty.

Up to the value, $\varepsilon_1 = 2.2$, the assumed $r(s)$ structure, but obviously, with different parameters, gives the strictly proper bounded real Schur's function $u(s)$ whose norm is less than one, and, consequently, the corresponding stabilizer controller.

A summary of the results for $\varepsilon_1 = 2.2$ is

$$r(s) = \frac{2.977s + 3.563}{s^2 + 4.58s + 8.197},$$

$$\|u(s)\|_\infty = 0.9974,$$

$$C(s) = \frac{41.93s^4 + 252.55s^3 + 547.22s^2 + 282.6s - 382}{s^4 + 1397s^3 + 700.4s^2 + 1432s + 455.5}. \tag{42}$$

Greater value of ε_1 requires considering upgrading (in the sense of increasing the number of zeros and poles) the $r(s)$ structure. For instant, for $\varepsilon_1 = 2.5$, it is found that the $r(s)$ structure of two zeros-three poles should be adopted. It has the form:

$$r(s) = \frac{2.786\ (s^2 + 1.593s + 1.303)}{(s + 1.364)(s^2 + 3.208s + 5.3)}. \tag{43}$$

The stabilizer controller becomes of fifth order:

$$C(s) = \frac{30.9s^5 + 186.2s^4 + 471s^3 + 504.7s^2 - 0.223s - 243.14}{s^5 + 102.1s^4 + 494.1s^3 + 1174.4s^2 + 1149.4s + 286.4.} \tag{44}$$

The Kharitonov templates are shown in Figure 4. Clearly, even when the structure uncertainty, value is increased to 2.5 with multiplicative unstructured uncertainty, the interval system is stable over the completely considered range of frequency.

Confirmation of the stabilized closed-loop system is illustrated also in the step responses of the nominal and interval systems (of 32 plants) as in Figure 5. It can be noted that even for highly overshoot interval system members the stability is secured.

If it is still required to tolerate larger value of ε_1, then further upgrading of $r(s)$ structure should be assumed. However, it will be on the expense of increasing the controller order.

(II) Additive Uncertainty. In this case, the only difference from what we did above is the computation of the unstructured uncertainty $\Delta G(\varepsilon_1, jw)$ magnitudes for the interval system. It is computed from:

$$|\Delta G(\varepsilon_1, jw)| = |g_i(\varepsilon_1, jw) - G_0(jw)|, \quad i = 1, 2, \ldots, 12. \tag{45}$$

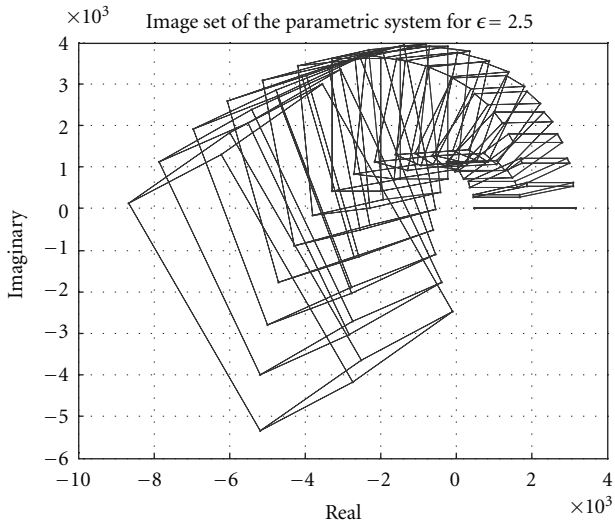

Figure 4: Kharitonov's templates with $\varepsilon_1 = 2.5$.

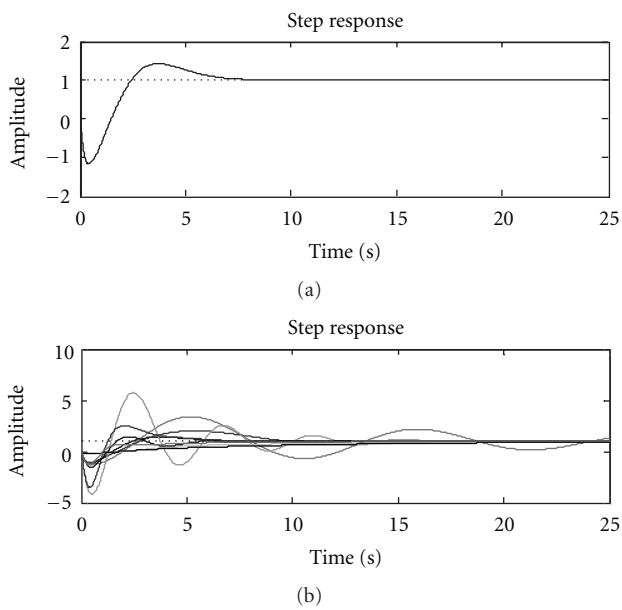

Figure 5: Step response of the stabilized interval system.

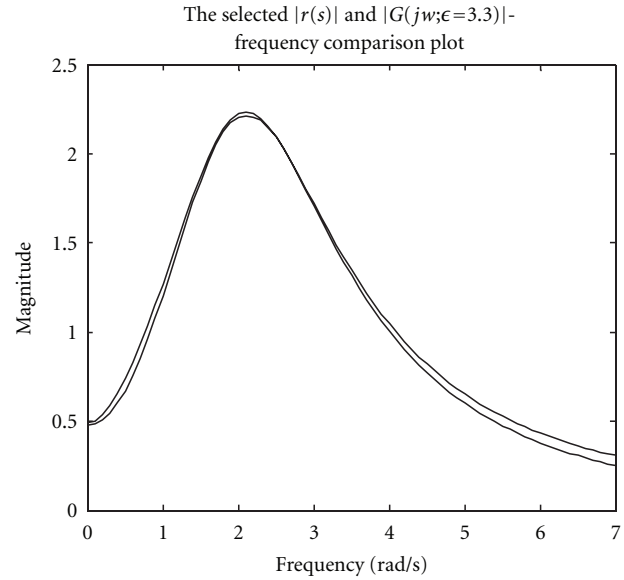

Figure 6: Additive uncertainty case with $\varepsilon_1 = 3.3$.

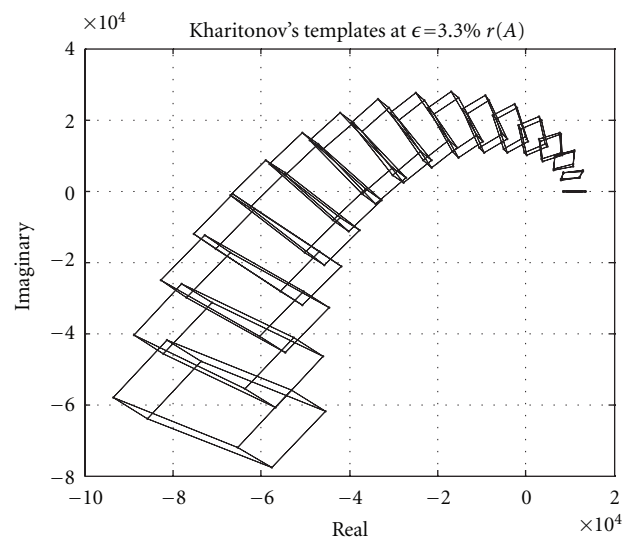

Figure 7: Kharitonov's templates with $\varepsilon_1 = 3.3$ with additive uncertainty.

For $\varepsilon_1 = 3.3$, the results are

$$r(s) = \frac{0.572(s+19.4)(s+5.5)(s+0.464)}{(s^2+2.32s+4.847)(s^2+5s+11.93)},$$

$$u(s) = \frac{28.75s+0.6377}{s^2+29.28s+7.477},$$

$$\|u(s)\|_\infty = 0.983,$$

$$C(s) = \frac{50.23(s+2)(s+0.0222)}{(s+65.29)(s+0.321)(s^2-2.68s+4.847)} \times \frac{(s^2+2.32s+4.85)(s^2+5s+11.93)}{(s^2+4.748s+9.599)}. \tag{46}$$

Figures 6 and 7 are the corresponding results. Clearly, the interval system is stable with respect to the parametric (structured) uncertainty of $\varepsilon_1 = 3.3$ and the additive unstructured uncertainty. It is worth mentioning that, for a specific value of ε_1 as the infinity norm of the function $u(s)$ approaches 1, the coverage of assumed $r(s)$ structure reaches its end.

6. Conclusions

A robust stabilization synthesis for systems under mixed structured (parameters) and unstructured multiplicative or additive uncertainties is considered. The implementation of Kharitonov's theory and the maximum perturbation of the unstructured uncertainty with respect to the range

of frequency of interest propose an approach for manipulating these mixed uncertainties. The NPI theorem is used to parameterize all stabilized controllers. A numerical straightforward approach for bounding (loop shaping) the unstructured uncertainty by a proper stable function is proposed to enlarge the box of parametric uncertainty in tandem with either multiplicative or additive uncertainty. The results show that it is always possible to enlarge the box on the expense of increasing the controller order.

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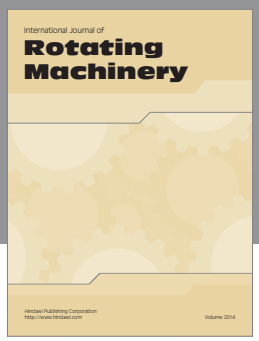

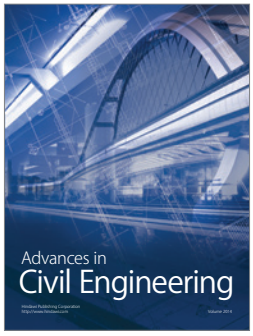

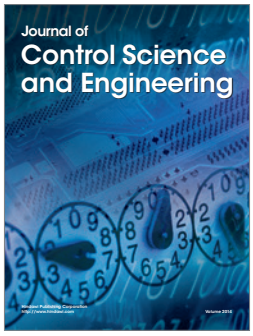

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