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Research Article

Stabilization of Neutral Systems with Saturating Actuators

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A method is proposed for stabilization, using static state feedback, of systems subject to time-varying delays in both the states and their derivatives (i.e., neutral systems), in the presence of saturating actuators. Delay-dependent conditions are given to determine stabilizing state-feedback controllers with large domain of attraction, expressed as linear matrix inequalities, readily implementable using available numerical tools and with tuning parameters that make possible to select the most adequate solution. These conditions are derived by using a Lyapunov-Krasovskii functional on the vertices of the polytopic description of the actuator saturations. Numerical examples demonstrate the effectiveness of the proposed technique.

1. Introduction

Systems with time delays constitute basic mathematical models of many real phenomena in circuits theory, economics, mechanics, and so forth, so they have been extensively studied in the literature: see, for example, [1–4] and references therein for several theoretical studies on this subject. An aspect that has not been frequently taken into account is the fact that, in many of these systems, the actuators have strict limitations. Thus, stabilization of time-delay systems with actuator saturation is an important issue, already addressed by several authors [5–9]. From those, we can emphasize [9], where a delay-independent condition was obtained, but for systems with no uncertainty and with known time-invariant delays, situation which is not frequent in practice. Generally speaking, the delay-independent scheme for control design does not use any information on the magnitude of the delay, whereas the delay-dependent approach employs such information. Moreover, when the delay is not big, delay-independent criteria tend to be conservative, so a delay-dependent approach is considered in this paper. In this context, we can cite [6], where a delay-dependent stabilization problem has been introduced for time-delay systems with actuators constraints and H_∞ control. The methodology followed in this paper follows the frequent approach of using Lyapunov-Krasovskii functionals, providing a set of *LMIs* that can be easily solved using dedicated solvers [10].

This paper concentrates on the specific class of neutral systems, that is, delayed systems in which both the state and its derivative are affected by time delays. Specific examples of these neutral systems appear in population ecology, transmission lines, and other practical systems [11]. These neutral systems are difficult to handle, so although the control design problem has been studied [12], it is not yet completely solved [13]. In particular, neutral systems are particularly sensitive to delays and can be easily destabilized [14]. Stabilization of neutral systems has already been studied in the literature [13, 15–17]. For example, in [18], the stabilizing controller is given for local stability but without giving the domain of initial condition. In any case, stabilization is not yet fully explored for the general class studied in this paper of constrained systems: as it has been mentioned, this situation appears frequently in practical applications, including neutral systems [19].

The proposed methodology is developed in this paper as follows: results are provided that guarantee the local stability of the closed loop system when the initial states are taken within a given region of attraction, by using a method based on the Lyapunov-Krasovskii (L-K) approach. When the proposed stability conditions are derived, a specific

method is followed to avoid cross products of the state and delayed state. Moreover, to reduce conservatism, some free matrices P_i are used (only one of them restricted to be positive definite). This introduction of free matrices can be viewed as an extension of the model description introduced in [6], or similar to the slack variables used in [1, 3], albeit in a different context. Finally, a *LMI* optimization approach is proposed to design the state feedback gain that maximizes the size of the estimated domain of attraction. It is shown for these systems that the proposed results are less conservative than those in the literature.

Notation. The following notations will be used throughout the paper: $\Re$ denotes the set of real numbers, $\Re^n$ denotes the n dimensional Euclidean space, and $\Re^{m\times n}$ denotes the set of all $m \times n$ real matrices. The notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). The symbol $*$ stands for symmetric block in matrix inequalities. $\overline{\lambda}(P)$ and $\underline{\lambda}(P)$ denote, respectively, the maximal and minimal eigenvalues of a matrix P. $\|\cdot\|$ refers to either the Euclidean vector norm, or the induced matrix 2-norm. The symbol $C^1([-d,0],\Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into $\Re^n$. I denotes the identity matrix of appropriate dimensions. For a matrix K, the ith row of K is denoted by k_i. For any vector $u \in \Re^m$, the saturation function is defined by $\mathrm{sat}(u) = [\mathrm{sat}(u_1)\,\mathrm{sat}(u_2)\cdots\mathrm{sat}(u_m)]^T$, where $\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min\{|u_i|,\overline{u}_i\}$, with given bounds $\overline{u}_i > 0$. The convex hull of a set is the minimal convex set containing it. Thus, for a set of points $x_1, x_2, \ldots, x_n \in \Re^n$, its convex hull is $Co\{x_1, x_2, \ldots, x_n\} = \{\sum_{i=1}^n \alpha_i x_i, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0\}$.

2. Problem Formulation and Definitions

Consider the following state-space linear system, with time-varying delays in the state and its derivative:

$$\begin{aligned}\dot{x}(t) - C\dot{x}(t-\tau(t)) &= A_0x(t) + A_1x(t-h(t)) \\ &\quad + B\,\mathrm{sat}(u(t)),\end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input, C, A_0, A_1, and B are known real constant matrices.

It must be noticed that throughout the paper, following [15–17], the delays $\tau(t)$ and $h(t)$ are assumed to be unknown but bounded functions of time, continuously differentiable, with their respective rates of change bounded as follows:

$$\begin{aligned}0 \leq h(t) \leq h_m, &\qquad 0 \leq \tau(t) < \infty, \\ \dot{h}(t) \leq d_1, &\qquad \dot{\tau}(t) \leq d_2,\end{aligned} \tag{2}$$

where $h_m > 0$, $d_1 < 1$, and $d_2 < 1$ are given positive constants (these bounds are strictly smaller than one to ensure causality: see [20]). The initial condition of system (1) is given by

$$x(t_0+\theta) = \phi(\theta), \quad \theta \in \left[-\overline{h}, 0\right], \tag{3}$$

where $\overline{h} = \max_{t\geq 0}\{\tau(t), h(t)\}$ and $\phi(\cdot)$ is a vector of differentiable functions of initial values (i.e., $\phi \in C^1[-\overline{h},0]$).

Now, suppose that the solution $x(t) = 0$ is asymptotically stable, for all delays satisfying (2), then the domain of attraction of the origin is

$$\Psi = \left\{\phi \in C^1\left[-\overline{h},0\right] : \lim_{t\to\infty} x(t) = 0\right\}. \tag{4}$$

The exact determination of Ψ is generally difficult. Consequently, it is useful to search for an estimate $\Xi_\delta \subset \Psi$ of the domain of attraction, where

$$\Xi_\delta = \left\{\phi \in C^1\left[-\overline{h},0\right] : \max_{[-\overline{h},\,0]} \|\phi\| \leq \delta\right\} \tag{5}$$

and the *stability radius* $\delta > 0$ is a scalar to be determined.

Throughout this paper, we assume the following.

A1. All the eigenvalues of matrix C are inside the unit circle.

Controllers in this paper are linear state feedback of the form

$$u(t) = Kx(t). \tag{6}$$

For a given gain matrix K, we define the polyhedron of states that do not cause saturation as follows:

$$D(K,\overline{u}) = \{x \in \Re^n;\ |k_ix| \leq \overline{u}_i,\ i = 1,\ldots,m\}. \tag{7}$$

A similar approach as the one proposed in [5] is used to represent the saturated system by a polytopic model. Denote by Θ the set of all diagonal matrices in $\Re^{m\times n}$ with diagonal elements that are 1 or 0. Thus, there are 2^m elements D_i in Θ, and the matrix $D_i^- = I - D_i$ is also an element of Θ.

Lemma 1 (see [5]). *Given K and H in $\Re^{m\times n}$, then*

$$\mathrm{sat}(Kx,\overline{u}) \in Co\{D_iKx + D_i^-Hx,\ i = 1,\ldots,2^m\} \tag{8}$$

for all $x \in \Re^n$ that satisfy $|h_ix| \leq \overline{u}_i$, $i = 1,\ldots,m$.

Therefore, if we consider any compact set $S_c \subset \Re^n$, for any $x \in S_c$ and H in $\Re^{m\times n}$ such that $|h_ix| \leq \overline{u}_i$, then the closed loop system of (1) and (6) may be written as follows:

$$\dot{x}(t) - C\dot{x}(t-\tau(t)) = \sum_{j=1}^{2^m} \lambda_j\widehat{A}_jx(t) + A_1x(t-h(t)), \tag{9}$$

where $\widehat{A}_j = B(D_jK + D_j^-H) + A_0$, $\sum_{j=1}^{2^m}\lambda_j = 1$, and $\lambda_j \geq 0$.

The following L-K functional candidate will be used throughout the paper:

$$\begin{aligned}V(t) &= x^T(t)P_1x(t) + \int_{t-h(t)}^{t} x^T(s)Qx(s)ds \\ &\quad + \int_{-h_m}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)ds\,d\theta \\ &\quad + \int_{t-\tau(t)}^{t} \dot{x}^T(s)W\dot{x}(s)ds,\end{aligned} \tag{10}$$

where $P_1 = P_1^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$ and $W = W^T > 0$.

Finally, for a positive scalar β and a positive definite symmetric matrix P_1, the ellipsoid D_e is defined as follows:

$$D_e \equiv \left\{x(t) \in \Re^n;\ x^T(t)P_1x(t) \le \beta^{-1}\right\}. \tag{11}$$

3. Main Results

This section presents sufficient conditions that guarantee the convergence to the origin of all the trajectories of system (1), starting from the domain Ξ_δ, that is included in the ellipsoid (11). First the main results are derived, which are later extended to neutral systems, and a practical algorithm is presented to design controllers that enlarge the size of the domain of initial conditions.

3.1. Neutral Systems with Time-Varying Delays

Theorem 2. *The system described by* (9) *is asymptotically stable if there exist* $P_1 = P_1^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $W = W^T > 0$, *and appropriately dimensioned matrices* P_i, $i = 2,\dots,6$ *such that the following condition holds:*

$$\Gamma_j = \begin{pmatrix} \Gamma_{11(j)} & \Gamma_{21(j)}^T & \Gamma_{31}^T & -P_4^T & P_2^TC \\ \Gamma_{21(j)} & \Gamma_{22} & \Gamma_{32}^T & -P_5^T & P_3^TC \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & -P_6^T & 0 \\ -P_4 & -P_5 & -P_6 & -\dfrac{1}{h_m}R & 0 \\ C^TP_2 & C^TP_3 & 0 & 0 & (d_2-1)W \end{pmatrix} < 0, \quad j = 1,\dots,2^m, \tag{12}$$

where

$$|h_ix| \le \overline{u}_i, \quad \forall x \in D_e, \tag{13}$$

$$\begin{aligned} \Gamma_{11(j)} &= P_2^T\widehat{A}_j + \widehat{A}_j^TP_2 + P_4 + P_4^T + Q, \\ \Gamma_{21(j)} &= P_1 + P_3^T\widehat{A}_j + P_5^T - P_2, \\ \Gamma_{22} &= h_mR + W - P_3 - P_3^T, \\ \Gamma_{31} &= A_1^TP_2 - P_4 + P_6^T, \\ \Gamma_{32} &= A_1^TP_3 - P_5, \\ \Gamma_{33} &= (d_1-1)Q - P_6 - P_6^T. \end{aligned} \tag{14}$$

Proof of Theorem 2. Calculating the time derivative of the proposed Lyapunov function (10) along the trajectory of the system (9) gives

$$\begin{aligned} \dot V(t) &= 2x^T(t)P_1\dot x(t) + x^T(t)Qx(t) \\ &\quad - \left(1-\dot h(t)\right)x^T(t-h(t))Qx(t-h(t)) \\ &\quad + h_m\dot x^T(t)R\dot x(t) \\ &\quad - \int_{t-h_m}^{t}\dot x^T(s)R\dot x(s)ds + \dot x^T(t)W\dot x(t) \\ &\quad - (1-\dot\tau(t))\dot x^T(t-\tau(t))W\dot x(t-\tau(t)). \end{aligned} \tag{15}$$

From (2), it is clear that the following is true:

$$-\int_{t-h_m}^{t}\dot x^T(s)R\dot x(s)ds \le -\int_{t-h(t)}^{t}\dot x^T(s)R\dot x(s)ds. \tag{16}$$

Also, from the Leibniz-Newton formula $0 = x(t) - x(t-h(t)) - \int_{t-h(t)}^{t}\dot x(s)ds$ and using (9), we can write the first term of (15) as follows:

$$\begin{aligned} &2x^T(t)P_1\dot x(t) \\ &= 2\widetilde{x}^T(t)P^T\begin{pmatrix}\dot x(t)\\ 0\\ 0\end{pmatrix} \\ &= 2\widetilde{x}^T(t)P^T \\ &\quad\times\begin{pmatrix} \dot x(t) \\ -\dot x(t) + C\dot x(t-\tau(t)) + \displaystyle\sum_{j=1}^{2^m}\lambda_j\widehat{A}_jx(t) + A_1x(t-h(t)) \\ x(t) - x(t-h(t)) - \displaystyle\int_{t-h(t)}^{t}\dot x(s)ds \end{pmatrix}, \end{aligned} \tag{17}$$

where $\widetilde{x}(t) = \left(x^T(t)\ \dot x^T(t)\ x^T(t-h(t))\right)^T$ and $P = \begin{pmatrix} P_1 & 0 & 0 \\ P_2 & P_3 & 0 \\ P_4 & P_5 & P_6 \end{pmatrix}$.

Since $\Sigma_{j=1}^{2^m}\lambda_j = 1$, substituting (17) into (15) gives

$$\begin{aligned} \dot V(t) = \sum_{j=1}^{2^m}\lambda_j\Bigg\{ &\widetilde{x}^T(t)\Xi_j\widetilde{x}(t) + 2\widetilde{x}^T(t)P^T\begin{pmatrix}0\\0\\-I\end{pmatrix}\int_{t-h(t)}^{t}\dot x(s)ds \\ &+ x^T(t)Qx(t) \\ &- \left(1-\dot h(t)\right)x^T(t-h(t))Qx(t-h(t)) \\ &- (1-\dot\tau(t))\dot x^T(t-\tau(t))W\dot x(t-\tau(t)) \\ &+ h_m\dot x^T(t)R\dot x(t) + \dot x^T(t)W\dot x(t) \\ &+ 2\widetilde{x}^T(t)P^T\begin{pmatrix}0\\C\\0\end{pmatrix}\dot x(t-\tau(t)) \\ &- \int_{t-h(t)}^{t}\dot x^T(s)R\dot x(s)ds\Bigg\}, \end{aligned} \tag{18}$$

where $\Xi_j = P^T\begin{pmatrix} 0 & I & 0 \\ \widehat{A}_j & -I & A_1 \\ I & 0 & -I \end{pmatrix} + \begin{pmatrix} 0 & \widehat{A}_j^T & I \\ I & -I & 0 \\ 0 & A_1^T & -I \end{pmatrix}P$.

Using Jensen's inequality [21], the last term in (18) can be bounded as follows:

$$-\int_{t-h(t)}^{t}\dot x^T(s)R\dot x(s)ds \le -\frac{1}{h_m}\int_{t-h(t)}^{t}\dot x^T(s)dsR\int_{t-h(t)}^{t}\dot x(s)ds. \tag{19}$$

Therefore, we get

$$\begin{aligned}\dot{V}(t) \le \sum_{j=1}^{2^m}\lambda_j\Bigg\{&\tilde{x}^T(t)\Xi_j\tilde{x}(t) + 2\tilde{x}^T(t)P^T\begin{pmatrix}0\\0\\-I\end{pmatrix}\int_{t-h(t)}^{t}\dot{x}(s)ds\\&+x^T(t)Qx(t)\\&-(1-d_1)x^T(1-h(t))Qx(1-h(t))\\&-(1-d_2)\dot{x}^T(1-\tau(t))W\dot{x}(1-\tau(t))\\&+h_m\dot{x}^T(t)R\dot{x}(t)\\&+\dot{x}^T(t)W\dot{x}(t)+2\tilde{x}^T(t)P^T\begin{pmatrix}0\\C\\0\end{pmatrix}\dot{x}(t-\tau(t))\\&-\frac{1}{h_m}\int_{t-h(t)}^{t}\dot{x}^T(s)dsR\int_{t-h(t)}^{t}\dot{x}(s)ds\Bigg\}.\end{aligned} \tag{20}$$

By simple manipulation, the inequality (20) can be rewritten as:

$$\dot{V}(t) \le \sum_{j=1}^{2^m}\lambda_j\Omega^T(t,s)\Gamma_j\Omega(t,s) < 0,$$
$$\text{with } \Omega(t,s) = \begin{pmatrix}\tilde{x}(t)\\ \displaystyle\int_{t-h(t)}^{t}\dot{x}(s)ds\\ \dot{x}(t-\tau(t))\end{pmatrix}, \tag{21}$$

and Γ_j defined in (12). Therefore, if the condition (13) holds, then $\dot{V}(t)$ is negative definite, which ensures the asymptotic stability of the polytopic system (9) [22].

This result gives a general solution for testing stability. We present now the following result that permits to calculate a stabilizing controller. □

Theorem 3. *The system (1)–(3) is asymptotically stabilized by feedback law $u(t) = Kx(t)$, with Ξ_δ inside the domain of attraction if there exist $\overline{Q} = \overline{Q}^T > 0$, $\overline{R} = \overline{R}^T > 0$, $\overline{W} = \overline{W}^T > 0$, $X_1 = X_1^T > 0$, X_2, $X_3 \in \Re^{n\times n}$, $U, G \in \Re^{m\times n}$, $\varepsilon_1, \varepsilon_2 \in \Re$, and positive scalars β and δ, satisfying the following conditions:*

$$\sum_{(j)} = \begin{pmatrix}\sum_{11} & * & * & * & * & * & * & *\\ \sum_{21(j)} & \sum_{22} & * & * & * & * & * & *\\ -\varepsilon_1\overline{Q}A_1^T & (1-\varepsilon_2)\overline{Q}A_1^T & (d_1-1)\overline{Q} & * & * & * & * & *\\ -\varepsilon_1\overline{R}A_1^T & -\varepsilon_2\overline{R}A_1^T & 0 & -\frac{1}{h_m}\overline{R} & * & * & * & *\\ 0 & \overline{W}C^T & 0 & 0 & (d_2-1)\overline{W} & * & * & *\\ h_mX_2 & h_mX_3 & 0 & 0 & 0 & -h_m\overline{R} & * & *\\ X_2 & X_3 & 0 & 0 & 0 & 0 & -\overline{W} & *\\ X_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{Q}\end{pmatrix} < 0, \quad j = 1,\dots,2^m, \tag{22}$$

$$\begin{pmatrix}\beta & *\\ g_i^T & \overline{u}_i^2X_1\end{pmatrix} \ge 0, \quad i = 1,\dots,m, \tag{23}$$

$$\begin{aligned}\delta^2\max\Big\{&\overline{\lambda}(X_1^{-1}) + 2\frac{h_m}{1-d_1}\overline{\lambda}\left(\overline{Q}^{-1}\right);\ 2h_m^2\overline{\lambda}\left(\overline{Q}^{-1}\right)\\&+\frac{1}{1-d_2}\overline{\lambda}\left(\overline{W}^{-1}\right) + h_m\overline{\lambda}\left(\overline{R}^{-1}\right)\Big\} \le \beta^{-1},\end{aligned} \tag{24}$$

where

$$\begin{aligned}\sum_{11} &= X_2 + X_2^T + \varepsilon_1\left(X_1A_1^T + A_1X_1\right),\\ \sum_{21(j)} &= X_3^T - X_2 + (A_0 + \varepsilon_2A_1)X_1 + B\left(D_jU + D_j^-G\right),\\ \sum_{22} &= -X_3^T - X_3,\end{aligned} \tag{25}$$

and $\overline{\lambda}$ denotes the maximum eigenvalue of the corresponding matrix. The corresponding gain matrix that stabilizes the system is given by

$$K = UX_1^{-1}. \tag{26}$$

Proof of Theorem 3. From the requirement that $P_1 = P_1^T > 0$, if condition (12) is satisfied, then $-P_3 - P_3^T$ must be negative definite. Thus, it follows that $\tilde{P}$ is nonsingular, where

$$\tilde{P}^{-1} = X = \begin{pmatrix}P_1 & 0\\ P_2 & P_3\end{pmatrix}^{-1} = \begin{pmatrix}X_1 & 0\\ X_2 & X_3\end{pmatrix}. \tag{27}$$

Then, multiplying (12) on the left by $\mathrm{diag}\{X^T, I, I, I\}$, on the right by $\mathrm{diag}\{X, I, I, I\}$, introducing the following changes of variables:

$$X_1 = P_1^{-1}, \qquad \overline{Q} = Q^{-1}, \qquad \overline{R} = R^{-1},$$

$$\overline{W} = W^{-1}, \qquad U = KX_1, \qquad G = HX_1,$$

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} X_1 P_4^T + X_2^T P_5^T \\ X_3^T P_5^T \end{pmatrix} X_1, \tag{28}$$

and then using the Schur complement [10], some conditions are obtained that are bilinear due to cross products of P_6 with P_1, P_2, and P_3. To avoid such terms, first we select $P_6 = 0$, which leads to

$$\begin{pmatrix} X_2 + X_2^T + N_1 + N_1^T & * & * & * & * & * & * & * \\ \prod_{21(j)} & -X_3 - X_3^T & * & * & * & * & * & * \\ -X_1^{-1}N_1^T & A_1^T - X_1^{-1}N_2^T & (d_1 - 1)\overline{Q}^{-1} & * & * & * & * & * \\ -X_1^{-1}N_1^T & -X_1^{-1}N_2^T & 0 & -\frac{1}{h_m}\overline{R}^{-1} & * & * & * & * \\ 0 & \overline{W}C^T & 0 & 0 & (d_2 - 1)\overline{W} & * & * & * \\ h_m X_2 & h_m X_3 & 0 & 0 & 0 & -h_m\overline{R} & * & * \\ X_2 & X_3 & 0 & 0 & 0 & 0 & -\overline{W} & * \\ X_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{Q} \end{pmatrix} < 0, \qquad j = 1, \ldots, 2^m, \tag{29}$$

with $\Pi_{21(j)} = X_3^T - X_2 + N_2 + A_0 X_1 + B(D_j U + D_j^- G)$.

This condition (29) cannot be solved directly, due to the presence of the cross products in $X_1^{-1}N_1^T$ and $X_1^{-1}N_2^T$. To overcome this, we select

$$N_1 = \varepsilon_1 A_1 X_1, \qquad N_2 = \varepsilon_2 A_1 X_1, \tag{30}$$

where ε_1 and ε_2 are decision variables. Substituting (30) into (29), the condition in (22) is obtained (which can be solved using the procedure presented in Remark 3).

Moreover, the satisfaction of LMIs (23) guarantee that $|h_i x| \le \overline{u}_i, \forall x \in D_e$, $i = 1, \ldots, m$. This can be proven in the same manner as in [5, 9].

Furthermore, following [2], the Lyapunov functional defined in (10) satisfies

$$\pi_1 \|\Delta\phi\|^2 \le V(\phi) \le \pi_2 \max_{[-\overline{h},0]} \|\phi\|^2, \tag{31}$$

with $\pi_1 = \underline{\lambda}(X_1^{-1})$ and

$$\pi_2 = \max\left\{\overline{\lambda}(X_1^{-1}) + 2\frac{h_m}{1-d_1}\overline{\lambda}\left(\overline{Q}^{-1}\right);\ 2h_m^2\overline{\lambda}\left(\overline{Q}^{-1}\right) + \frac{1}{1-d_2}\overline{\lambda}\left(\overline{W}^{-1}\right) + h_m\overline{\lambda}\left(\overline{R}^{-1}\right)\right\}. \tag{32}$$

From $\dot{V}(t) < 0$, it follows that $V(t) < V(\phi)$, and, therefore,

$$x^T(t)X_1^{-1}x(t) \le V(t) < V(\phi) \le \max_{\theta\in[-\overline{h},0]} \|\phi(\theta)\|^2 \pi_2 \le \beta^{-1}. \tag{33}$$

Then, the inequality (25) guarantees that the trajectories of $x(t)$ remain within D_e for all initial functions $\phi \in \Xi_\delta$; moreover, $\dot{V}(t) < 0$ along the trajectories of (9), which implies that $\lim_{t\to\infty} x(t) = 0$, completing the proof. □

Remark 1. The result of Theorem 3 is derived by using Theorem 2, when P_6 is fixed to be zero, in order to simplify the numerical solution: although this makes the solution only slightly more conservative, it reduces significantly the computational cost.

3.2. Practical Algorithms. In this section, we give as remarks some practical procedures to design controllers using the results derived in this paper.

Remark 2. Theorem 3 provides a condition allowing us to compute both a control law and a domain of attraction in which the closed loop neutral system is asymptotically stable. It would be interesting to develop a methodology to estimate the largest possible domain of initial conditions that ensure stability of the system. Unfortunately, this is very difficult, due to the nonlinearity in the system. An interesting solution consists in imposing conditions on the maximal eigenvalues of X_1^{-1}, $\overline{Q}^{-1}$, $\overline{R}^{-1}$, and $\overline{W}^{-1}$ and constructs a feasibility problem, for given h_m, as follows.

$$\begin{aligned} &\text{Find } \overline{Q}, \overline{R}, \overline{W}, X_1, X_2, X_3, U, G, \beta, \varepsilon_1, \varepsilon_2, \delta, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \\ &\text{subject to } X_1 > 0, \quad \overline{Q} > 0, \quad \overline{R} > 0, \quad \overline{W} > 0, \\ &\qquad \beta > 0, \quad \delta > 0, \quad \sigma_1 > 0, \quad \sigma_2 > 0, \quad \sigma_3 > 0, \\ &\qquad \sigma_4 > 0, \quad (22), \quad (23), \\ &\qquad \begin{pmatrix} \sigma_1 I & I \\ I & X_1 \end{pmatrix} \ge 0, \quad \begin{pmatrix} \sigma_2 I & I \\ I & \overline{Q} \end{pmatrix} \ge 0, \end{aligned}$$

$$\begin{pmatrix} \sigma_3 I & I \\ I & \overline{R} \end{pmatrix} \geq 0, \qquad \begin{pmatrix} \sigma_4 I & I \\ I & \overline{W} \end{pmatrix} \geq 0,$$
$$\delta^2 \max\left\{\sigma_1 + 2\frac{h_m}{1-d_1}\sigma_2;\ 2h_m^2\sigma_2 + h_m\sigma_3 + \frac{1}{1-d_2}\sigma_4\right\} \leq \beta^{-1}. \tag{34}$$

If the above problem has a solution for a given h_m, then there exists a controller $u(t) = UX_1^{-1}x(t)$ that guarantees stability of the saturated neutral system (1)–(3).

Remark 3. When the scalar parameters ε_1 and ε_2 are fixed, the condition (22) of Theorem 3 becomes LMI. However, choosing arbitrary ε_1 and ε_2 does not lead to the best result. In the following, a tuning procedure for the parameters ε_1 and ε_2 is proposed to enlarge the bound h_m on the time varying delay. If we select as optimization parameters ε_1 and ε_2 and choose a cost function $t_{\min}$, with $\Sigma_{(j)} \leq t_{\min} I$, where $\Sigma_{(j)}$ is defined in (22), then if there exists a combination of parameters ε_1 and ε_2 that gives a negative $t_{\min}$, these parameters give a feasible solution of the conditions in Theorem 3 (finding this combination can be carried out by solving the corresponding feasibility problem). Finally, applying a numerical optimization algorithm, it is possible to obtain a locally convergent solution to the problem (using, e.g., fminsearch in the Optimization Toolbox [23]). If the resulting minimum value of the cost function is negative, then a combination of tuning parameters that solves the problem is found.

Thus, this procedure to look for a feasible solution of the conditions in Theorem 3 can be summarized as follows.

Algorithm (maximization of $h_m > 0$).

Step 1. Fix initial values $\varepsilon_1 = \varepsilon_{10}$, $\varepsilon_2 = \varepsilon_{20}$, and $h_m = h_{m0}$, where h_{m0} must be small enough to have a feasible solution, and set a step variation $h_{m\text{step}}$.

Step 2. Solve the following problem:

$$\min_{\varepsilon_1,\varepsilon_2} t_{\min} \text{ such that } \sum_{(j)} \leq t_{\min} I, \tag{35}$$

and obtain new values of ε_1 and ε_2.

Step 3. If $t_{\min} > 0$, the previous values of ε_1 and ε_2 give the largest domain of attraction; otherwise ($t_{\min} \leq 0$), to improve the solution, set $h_m = h_m + h_{m\text{step}}$ and repeat from Step 2.

4. Numerical Examples

This section provides some numerical examples to illustrate that the proposed method is less conservative than previous results in the literature.

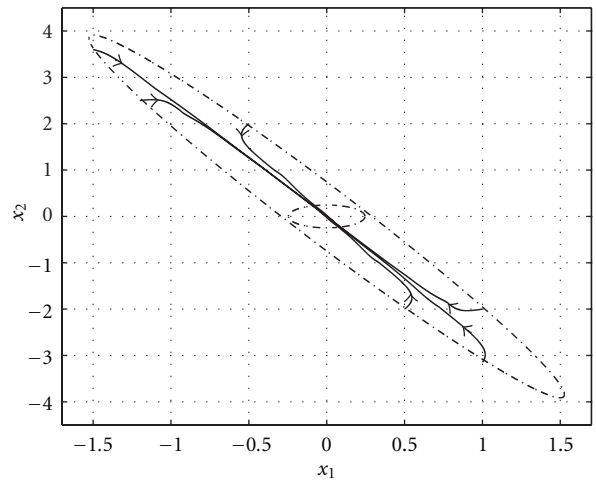

Figure 1: Trajectories and ellipsoid D_e for a maximum delay of $h_m = 1.566$.

Example 1. Consider a neutral system described by (1)–(3), with the following parameters:

$$A_0 = \begin{pmatrix} 0.5 & -1 \\ 0.5 & -0.5 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.2 \end{pmatrix}, \tag{36}$$
$$\overline{u} = 5.$$

It can be seen that this system is unstable, so, for stabilization, a controller was designed based on Theorem 3. In particular, using the algorithm proposed in Section 3.2 to enlarge the bound on the state delay, it was found that, when $d_1 = 0$ and $d_2 = 0$, the system is stabilizable for all state delays $h(t) \leq 1.566$, when the state feedback gain is $K = (-2.5939 \quad 0.0653)$. For this case, the stability radius is $\delta = 0.2472$, obtained when $\varepsilon_1 = 0.1350$ and $\varepsilon_2 = 0.9241$.

To check the stability and the corresponding time responses, the closed-loop system was simulated, starting from different initial values inside the domain of attraction given by (5). To check the closed-loop stability, we show in Figure 1 some trajectories of the saturated closed-loop system, together with the ellipsoid D_e.

Example 2. Consider the system of Example 1, in which we set $C = 0$. This example gives the system studied in [5, 6, 19], where upper bounds h_m and maximum radius δ were calculated for which a state feedback control K stabilizes (36). Their results are listed in Table 1 along with the results obtained by Theorem 3 (with $\varepsilon_1 = 0.4569$ and $\varepsilon_2 = 0.8166$, selected following the algorithm provided in Section 3.2).

Note that, in [19] the search of the values of ε_1 and ε_2 is achieved by using an iterative algorithm, while, in this paper, we adopt an optimization procedure which leads to less conservative results.

Table 1: Comparison of bounds on h_m, maximum radius δ, and stabilizing control gains K using different approaches.

	h_m	δ	Control gain K
Cao et al. [5]	0.35	0.9680	$(-2.6383 \quad 0.7204)$
Fridman et al. [6]	1.854	0.091	$(-25.8809 \quad -4.9315)$
El Haoussi and Tissir [19]	2	0.0718	$(-5.7702 \quad -0.9754)$
Theorem 3	2	0.0722	$(-5.6104 \quad -0.9147)$

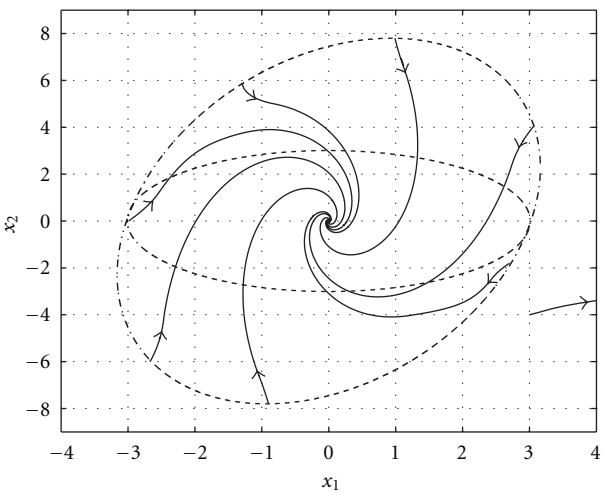

Figure 2: Trajectories and estimated domain of attraction for a maximum delay of $h_m = 0.35$.

In order to compare with the results of [5, 6, 19], we take $h(t) = h_m = 0.35$. Theorem 3 yields the stability radius of $\delta = 3.0092$ (when $\varepsilon_1 = 0.0015$, $\varepsilon_2 = 0.9984$, and $K = (-1.7150 \quad 0.7143)$), whereas the results of [5, 6, 19] give, respectively, $\delta = 0.9680$, $\delta = 2.9089$, and $\delta = 2.852$. It is clear that our method gives the largest stability radius. This confirms that the stabilization conditions in this note are less conservative than those of [5, 6, 19].

Figure 2 shows some trajectories of the closed-loop system and the domain of attraction with this controller. The outer ellipsoid is D_e, and the inner ellipsoid is the stability circle of radius $\delta = 3.0092$.

Example 3. Consider the example studied in [5, 6, 9], where the plant can be described by (1)–(3), where

$$A_0 = \begin{pmatrix} 1 & 1.5 \\ 0.3 & -2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$
$$B = \begin{pmatrix} 10 \\ 1 \end{pmatrix}, \qquad C = 0, \tag{37}$$
$$h(t) = h_m = 1, \qquad d_1 = d_2 = 0, \qquad \overline{u} = 15,$$

(i.e., the system is nonneutral with constant delays). In [9] stabilization via state feedback was achieved for all initial conditions in Ξ_δ with $\delta \leq 42.3308$, when the origin of the saturated system is requested to be asymptotically stable and the unsaturated system be α-stable with $\alpha = 1$. If we only require that the saturated system be asymptotically stable (i.e., $\alpha = 0$), it is found that $\delta \leq 58.395$. In [5], stabilization by a saturated memoryless state feedback law was obtained for all initial conditions in Ξ_δ with a $\delta \leq 67.0618$. Following [6], this domain can be still enlarged, with stability radius $\delta = 79.43$, in [7], is 79.54, and in [8], is 83.55.

The application of Theorem 3 in the present paper gives a larger stability region: when $\varepsilon_1 = -0.3307$, $\varepsilon_2 = 1.3307$, $\beta = 1$, the stability radius is $\delta = 96.1645$ when the state feedback gain is

$$K = \begin{pmatrix} -10.1970 & 0.9550 \end{pmatrix}. \tag{38}$$

It is clear that this estimation is less conservative than previous results in [5–9].

5. Conclusions

This paper has presented a new approach for delay-dependent stabilization of neutral systems with saturating actuators, under time-varying delays. This was accomplished by combining the Lyapunov-Krasovskii technique and the transformation of a system with actuator saturation into a convex polytope of linear systems. An estimation of the domain of attraction is proposed that can be numerically solved using linear matrix inequalities.

The derived conditions depend on the tuning parameters ε_1 and ε_2 that can be used to enlarge the domain of attraction. A simple iterative procedure based on numerical optimization has been proposed to obtain adequate values for these parameters. This procedure has been illustrated using an example. Finally, additional examples have shown that the particularization of the results for standard delayed systems gives less conservative results than previous results proposed in the literature.

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