Research Article

Transformation of CLF to ISS-CLF for Nonlinear Systems with Disturbance and Construction of Nonlinear Robust Controller with $\mathcal{L}_2$ Gain Performance

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A new nonlinear control law for a class of nonlinear systems with disturbance is proposed. A control law is designed by transforming control Lyapunov function (CLF) to input-to-state stability control Lyapunov function (ISS-CLF). The transformed CLF satisfies a Hamilton-Jacobi-Isaacs (HJI) equation. The feedback system by the proposed control law has characteristics of $\mathcal{L}_2$ gain. Finally, it is shown by a numerical example that the proposed control law makes a controller by feedback linearization robust against disturbance.

1. Introduction

It is difficult to build strict mathematical models of actual systems and there may exist disturbance such as modelling errors and parameter variations of systems. It is one of the most important problems in control theory to construct a control law considering disturbance. If an input-to-state stability control Lyapunov function (ISS-CLF) or $\mathcal{L}_2$ gain exists, then the stability of a nonlinear system with disturbance can be assured.

If an ISS-CLF can be found, it is possible to construct a control law considering disturbance which depends on the state of a system. If disturbance is lower than a certain value, then a system may be asymptotically stable to the origin. But the construction way of an ISS-CLF is provided only for particular systems and a complicated procedure is required [1–3].

On the other hand, if the existence of $\mathcal{L}_2$ gain can be shown, then it is assured that state and control input remain in $\mathcal{L}_2$ set and a system is stable even if there exists disturbance belonging to $\mathcal{L}_2$ set [4, 5]. It is possible to construct $\mathcal{L}_2$ gain if a Hamilton-Jacobi-Isaacs (HJI) equation can be solved [6]. But the general solution to a HJI equation is not found and

the solution to a HJI equation may not exist for a specified $\mathcal{L}_2$ gain. Though the solution to a HJI equation by numerical calculation is also provided, it is complicated and there are some constraints [7].

It is also possible to construct the solution which satisfies a HJI equation by applying an inverse optimal method [8]. A control law provided by an inverse optimal method can optimize a certain meaningful objective function but an inverse optimal method can be applied only if ISS-CLF can be found. In addition, a certain meaningful objective function and $\mathcal{L}_2$ gain may not be the desired ones.

The main purpose of this paper is to propose a transformation method from CLF to ISS-CLF and to provide a lower bound condition of $\mathcal{L}_2$ gain. A CLF is transformed to an ISS-CLF so that an ISS-CLF satisfies a HJI equation by using a proper transformation coefficient. At the same time, a lower bound condition of $\mathcal{L}_2$ gain which is a function of CLF is provided.

The characteristics of this paper are as follows. It is easily possible to construct an ISS-CLF which satisfies a HJI equation and to design a control law which minimizes a desired objective function specified by a designer in the set of control laws based on a CLF if a CLF is found for a

nominal system, namely, for a system without disturbance. In addition, $\mathcal{L}_2$ gain can be designed if it satisfies the lower bound condition.

This paper is organized as follows. In Section 2, $\mathcal{L}_2$ gain which is an important concept of input output stability is defined and the relation between $\mathcal{L}_2$ gain and a HJI equation is shown. In Section 3, an ISS-CLF satisfying a HJI equation and a stabilizing controller are constructed under the assumption that a CLF can be constructed for a nominal system without disturbance. Also, it is shown that the closed system by the proposed controller has $\mathcal{L}_2$ gain and the proposed control law minimizes a specified objective function in the set of control laws based on a CLF. In Section 4, a CLF is constructed by strict feedback linearization for a nominal system without disturbance. In Section 5, effectiveness of the proposed control law based on a CLF which is constructed by feedback linearization is shown by a numerical simulation.

The following notations are used in this paper. $\mathbb{R}$ denotes the set of real value. $\mathbb{R}^+$ denotes the set of positive real value. For a matrix $\mathcal{A}$, $\mathcal{A} > 0$ (≥ 0) denotes that a matrix $\mathcal{A}$ is positive definite matrix (semipositive definite matrix). $\mathcal{A}^T$ denotes the transposed matrix. For a vector b , $\|b\|$ denotes $\sqrt{b^T b}$. $\mathcal{V}_x(x) = [\partial \mathcal{V} / \partial x_1, \dots, \partial \mathcal{V} / \partial x_n]$ denotes the derivative of $\mathcal{V}(x) \in \mathbb{R}$ with respect to x .

Also, Lie derivative is defined as $L_f \mathcal{V}(x) = \mathcal{V}_x(x) f(x)$. Lie bracket is defined as $[f, g](x) = g_x(x) f(x) - f_x(x) g(x)$. A vector field by repeating Lie bracket is defined as $ad_f^{i+1} g(x) = [f, ad_f^i g(x)]$ where $ad_f^0 g(x) = g(x)$.

2. Preliminaries

The following nonlinear system is considered in this paper:

$$\dot{x}(t) = f(x) + g_1(x) w(t) + g_2(x) u(t), \quad (1)$$

$$z(t) = h(x), \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^q$ are the state vector, the disturbance vector, the control input vector, and the controlled output vector, respectively. It is assumed that all information on the state $x(t)$ can be used and $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g_1(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$, $g_2(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, and $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}^q$, where $f(0) = 0$, $g_2(x) \neq 0$ for all x and $h(x) \neq 0$ for all $x \neq 0$ and $h(0) = 0$.

A CLF and an ISS-CLF are defined in the following definition. A CLF is defined for $w(t) = 0$ ($0 \leq t$), namely, a nominal system. On the other hand, an ISS-CLF is defined for the system with the disturbance $w(t)$ which depends on the state $x(t)$ [8].

Definition 1. A smooth positive definite radially unbounded function $\mathcal{V}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a CLF for the system (1) without the disturbance $w(t)$ if it satisfies (3) for all $x \neq 0$:

$$\inf_{u \in \mathbb{R}^m} (L_f \mathcal{V}(x) + L_{g_2} \mathcal{V}(x) u(t)) < 0. \quad (3)$$

Definition 2. A smooth positive definite radially unbounded function $\mathcal{V}^*(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is an ISS-CLF for the system

(1) if there exists a function $\rho \in \mathcal{K}_\infty$ such that $\rho^{-1}(|w(t)|) \leq |x(t)|$ holds and if $\mathcal{V}^*(x)$ satisfies (4) for all $x \neq 0$:

$$\inf_{u \in \mathbb{R}^m} (L_f \mathcal{V}^*(x) + L_{g_1} \mathcal{V}^*(x) w(t) + L_{g_2} \mathcal{V}^*(x) u(t)) < 0. \quad (4)$$

Next, $\mathcal{L}_2$ gain is defined as follows.

Definition 3. The system (1) and (2) has $\mathcal{L}_2$ gain less than or equal to γ if there exist the positive constants γ and γ_0 satisfying the following equation:

$$\int_0^\infty (\|z(t)\|^2 + \|u(t)\|^2) dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 dt + \gamma_0. \quad (5)$$

Existence of $\mathcal{L}_2$ gain means that the controlled output $z(t)$ and the control input $u(t)$ are belonging to $\mathcal{L}_2$ set if the disturbance $w(t)$ belongs to $\mathcal{L}_2$ set. Also, the influence of the disturbance $w(t)$ to the controlled output $z(t)$ and the control input $u(t)$ can be decreased if we construct the smaller $\mathcal{L}_2$ gain γ .

The $\mathcal{L}_2$ gain γ can be constructed if it is possible to find a semipositive function $\overline{\mathcal{V}}(x)$ satisfying the following HJI equation:

$$\begin{aligned} \dot{\overline{\mathcal{V}}}(x) &= \dot{\overline{\mathcal{V}}}(x) + s(t) \\ &= L_f \overline{\mathcal{V}}(x) + \frac{1}{4\gamma^2} \left\| (L_{g_1} \overline{\mathcal{V}}(x))^T \right\|^2 \\ &\quad - \frac{1}{4} \left\| (L_{g_2} \overline{\mathcal{V}}(x))^T \right\|^2 + \|z(t)\|^2 = 0, \end{aligned} \quad (6)$$

where

$$s(t) = \|z(t)\|^2 + \|u(t)\|^2 - \gamma^2 \|w(t)\|^2. \quad (7)$$

If the semipositive function $\overline{\mathcal{V}}(x)$ exists, then the time derivative of $\overline{\mathcal{V}}(x)$ becomes (8) by using the HJI equation (6):

$$\begin{aligned} \dot{\overline{\mathcal{V}}}(x) &= -\frac{1}{4\gamma^2} \left\| (L_{g_1} \overline{\mathcal{V}}(x))^T \right\|^2 + \frac{1}{4} \left\| (L_{g_2} \overline{\mathcal{V}}(x))^T \right\|^2 \\ &\quad - \|z(t)\|^2 + L_{g_1} \overline{\mathcal{V}}(x) \overline{w}(t) + L_{g_2} \overline{\mathcal{V}}(x) \overline{u}(t) \\ &= -\|z(t)\|^2 + \gamma^2 \|\overline{w}(t)\|^2 - \|\overline{u}(t)\|^2 \\ &\quad - \gamma^2 \left\| \overline{w}(t) - \frac{1}{2\gamma^2} (L_{g_1} \overline{\mathcal{V}}(x))^T \right\|^2 \\ &\quad + \left\| \overline{u}(t) + \frac{1}{2} (L_{g_2} \overline{\mathcal{V}}(x))^T \right\|^2 \\ &\leq -\|z(t)\|^2 + \gamma^2 \|\overline{w}(t)\|^2 - \|\overline{u}(t)\|^2, \end{aligned} \quad (8)$$

where the control input $\overline{u}(t)$ is given by

$$\overline{u}(t) = -\frac{1}{2} (L_{g_2} \overline{\mathcal{V}}(x))^T. \quad (9)$$

By integrating of (8) from 0 to ∞ with respect to t and setting $\overline{\mathcal{V}}(x(0)) = \gamma_0$, we see that the system has $\mathcal{L}_2$ gain less than or equal to γ .

Moreover, if we can find the solution $\bar{\mathcal{V}}(x)$, then the objective function (10) can be optimized as follows:

$$\mathcal{J} = \int_0^\infty s(t) dt, \quad (10)$$

where the objective function (10) is derived from (5). If the control input is (9) and the disturbance is

$$\bar{w}(t) = \frac{1}{2\gamma^2} (L_{g_1} \bar{\mathcal{V}}(x))^T \quad (11)$$

then the objective function becomes

$$\begin{aligned} \mathcal{J} &= \int_0^\infty \left(-L_f \bar{\mathcal{V}}(x) - \frac{1}{4\gamma^2} \left\| (L_{g_1} \bar{\mathcal{V}}(x))^T \right\|^2 \right. \\ &\quad \left. + \frac{1}{4} \left\| (L_{g_2} \bar{\mathcal{V}}(x))^T \right\|^2 + \|\bar{u}(t)\|^2 - \gamma^2 \|\bar{w}(t)\|^2 \right) dt \\ &= \int_0^\infty \left(-L_f \bar{\mathcal{V}}(x) - \frac{1}{\gamma^2} L_{g_1} \bar{\mathcal{V}}(x) \bar{w}(t) \right. \\ &\quad \left. - L_{g_2} \bar{\mathcal{V}}(x) \bar{u}(t) - \gamma^2 \left\| \bar{w}(t) - \frac{1}{2\gamma^2} (L_{g_1} \bar{\mathcal{V}}(x))^T \right\|^2 \right. \\ &\quad \left. + \left\| \bar{u}(t) + \frac{1}{2} (L_{g_2} \bar{\mathcal{V}}(x))^T \right\|^2 \right) dt \\ &= - \int_0^\infty \dot{\bar{\mathcal{V}}}(x) dt = \bar{\mathcal{V}}(x(0)). \end{aligned} \quad (12)$$

Therefore, the objective function is minimized by the control input (9) and maximized by the disturbance (11). However, it is generally difficult to solve the HJI equation and obtain the solution $\bar{\mathcal{V}}(x)$. Besides, there may not exist the global solution satisfying the HJI equation for the specified $\mathcal{L}_2$ gain.

In this paper, a new idea to transform the CLF to the ISS-CLF and to construct a robust stabilizing control law is proposed. The proposed approach satisfies the HJI equation and the objective function is minimized by control input and maximized by disturbance in the set of control laws based on the CLF. The proposed approach also provides a lower bound condition which consists of the CLF. If $\mathcal{L}_2$ gain holds a lower bound condition, then the system has $\mathcal{L}_2$ gain.

3. Main Results

The following theorem is the main result and gives a new idea to transform the CLF to the ISS-CLF by using a transformation coefficient, a new robust control law based on the CLF and a lower bound condition of $\mathcal{L}_2$ gain.

Theorem 4. *It is assumed that the CLF $\mathcal{V}(x)$ exists for the system (1) without the disturbance $w(t)$ and γ satisfies the following equations:*

$$\gamma^2 > \frac{\left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2}{\left\| (L_{g_2} \mathcal{V}(x))^T \right\|^2} \quad (L_{g_2} \mathcal{V}(x) \neq 0), \quad (13a)$$

$$\gamma^2 > \frac{\left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2 \left\| z(t) \right\|^2}{\left(L_f \mathcal{V}(x) \right)^2} \quad (13b)$$

$$(L_{g_1} \mathcal{V}(x) \neq 0, L_{g_2} \mathcal{V}(x) = 0),$$

$$\gamma^2 > 0 \quad (L_{g_1} \mathcal{V}(x) = 0, L_{g_2} \mathcal{V}(x) = 0), \quad (13c)$$

for all x . We define $\tilde{\mathcal{V}}(x)$ satisfying the following equation:

$$\dot{\tilde{\mathcal{V}}}(x) = k(x) \dot{\mathcal{V}}(x), \quad (14)$$

where the transformation coefficient $k(x)$ is given by the following equations:

$$k(x) = 2 \frac{L_f \mathcal{V}(x) + \sqrt{\left(L_f \mathcal{V}(x) \right)^2 + \sigma(x) \left\| z(t) \right\|^2}}{\sigma(x)} \quad (\sigma(x) \neq 0), \quad (15a)$$

$$k(x) = -\frac{\left\| z(t) \right\|^2}{L_f \mathcal{V}(x)} \quad (\sigma(x) = 0), \quad (15b)$$

$$k(x) = 0 \quad (x = 0), \quad (15c)$$

$$\sigma(x) = -\frac{1}{\gamma^2} \left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2 + \left\| (L_{g_2} \mathcal{V}(x))^T \right\|^2. \quad (16)$$

Then $k(x) > 0$ for all $x \neq 0$ and $\tilde{\mathcal{V}}(x)$ of (14) becomes an ISS-CLF with the control input (18) against the disturbance (17):

$$w(t) = \frac{1}{2\gamma^2} k(x) (L_{g_1} \mathcal{V}(x))^T, \quad (17)$$

$$u(t) = -\frac{1}{2} k(x) (L_{g_2} \mathcal{V}(x))^T, \quad (18)$$

and $\tilde{\mathcal{V}}(x)$ satisfies the HJI equation. Besides, the system is asymptotically stable to the origin and has $\mathcal{L}_2$ gain less than or equal to γ .

Proof. First of all, we prove $k(x) > 0$ for all $x \neq 0$. In the case of (15a), we separately prove it in the case that $L_{g_2} \mathcal{V}(x) \neq 0$ and $L_{g_2} \mathcal{V}(x) = 0$. In the case that $L_{g_2} \mathcal{V}(x) \neq 0$, since $\sigma(x) > 0$ can be shown by (13a) for all $x \neq 0$, we obtain $k(x) > 0$. In the case that $L_{g_2} \mathcal{V}(x) = 0$, the denominator and the numerator of (15a) become negative from (3) and (13b); therefore, $k(x) > 0$. In the case of (15b), since $L_{g_2} \mathcal{V}(x) = 0$ when $\sigma(x) = 0$ and $L_f \mathcal{V}(x) < 0$ from (3), we obtain $k(x) > 0$. Therefore, we obtain $k(x) > 0$ for all $x \neq 0$.

Next, we prove that $\tilde{\mathcal{V}}(x)$ is a positive definite function. We separately prove it in the case that $\dot{\mathcal{V}}(x) > 0$, $\dot{\mathcal{V}}(x) < 0$, and $\dot{\mathcal{V}}(x) = 0$ because $\dot{\mathcal{V}}(x)$ varies depending on the disturbance $w(t)$. It is assumed that the initial value of $\tilde{\mathcal{V}}(x)$ is positive; that is, $\tilde{\mathcal{V}}(x(0)) > 0$. If $\dot{\mathcal{V}}(x) > 0$, then $\dot{\tilde{\mathcal{V}}}(x) = k(x) \dot{\mathcal{V}}(x) > 0$. Since $\tilde{\mathcal{V}}(x)$ is a monotonous increasing

function with respect to t , the value of $\tilde{\mathcal{V}}(x)$ keeps positive when $\dot{\mathcal{V}}(x) > 0$. If $\dot{\mathcal{V}}(x) < 0$, then $\tilde{\mathcal{V}}(x) = k(x)\dot{\mathcal{V}}(x) < 0$. Therefore, $\tilde{\mathcal{V}}(x)$ becomes a monotonous decreasing function with respect to t . Also, since $\rho^{-1}(|w(t)|) \leq |x(t)|$ holds in Definition 2, $w(t) = 0$ and $\dot{\mathcal{V}}(x) = 0$ when $x(t) = 0$. Therefore, $\tilde{\mathcal{V}}(x) > 0$ when $\dot{\mathcal{V}}(x) < 0$ and it is possible to set $\tilde{\mathcal{V}}(x) = 0$ when $x(t) = 0$. As a result, $\tilde{\mathcal{V}}(x)$ is a positive definite function.

Now, we prove the stability of the closed system with the disturbance (17) by the control law (18). The time derivative of the CLF $\mathcal{V}(x)$ along the trajectory of the system (1) with the disturbance (17) and the control input (18) is given by

$$\begin{aligned}\dot{\mathcal{V}}(x) &= L_f \mathcal{V}(x) + L_{g_1} \mathcal{V}(x) w(t) + L_{g_2} \mathcal{V}(x) u(t) \\ &\leq L_f \mathcal{V}(x) + \frac{1}{2\gamma^2} k(x) \left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2 \\ &\quad - \frac{1}{2} k(x) \left\| (L_{g_2} \mathcal{V}(x))^T \right\|^2\end{aligned}\quad (19)$$

$$= -\sqrt{(L_f \mathcal{V}(x))^2 + \sigma(x) \|z(t)\|^2} \quad (\sigma(x) \neq 0) \quad (20a)$$

$$= L_f \mathcal{V}(x) \quad (\sigma = 0) \quad (20b)$$

$$= 0 \quad (x = 0). \quad (20c)$$

In the case that $\sigma(x) \neq 0$, γ satisfies (13a) for all $x \neq 0$. Therefore, $\sigma(x) > 0$. Since $z(t) \neq 0$ for all $x \neq 0$, (20a) becomes $\dot{\mathcal{V}}(x) < 0$ when $\sigma(x) \neq 0$. For (20b), since $\mathcal{V}(x)$ is a CLF and satisfies (3), (20b) also becomes $\dot{\mathcal{V}}(x) < 0$. From $k(x) > 0$ and $\dot{\mathcal{V}}(x) < 0$ for all $x \neq 0$, $k(0) = 0$, $\mathcal{V}(0) = 0$, and (14), we obtain $\tilde{\mathcal{V}}(x) < 0$ for all $x \neq 0$ and $\tilde{\mathcal{V}}(0) = 0$. From the above results, the solution of the closed system (1) by the control law (18) converges to the maximum invariant sets $\mathcal{S}$ included in the sets $\mathcal{X} = \{x \mid \tilde{\mathcal{V}}(x) = 0\}$ by Lasalle's invariant principle [9]. Since the invariant sets $\mathcal{S}$ consist of the origin only, the closed system is asymptotically stable to the origin. Therefore, $\tilde{\mathcal{V}}(x)$ is an ISS-CLF.

Next, we prove that $\tilde{\mathcal{V}}(x)$ satisfies the HJI equation. In (6), by substituting $\tilde{\mathcal{V}}(x)$ for $\mathcal{V}(x)$ and using the disturbance (17), the control input (18), and transformation coefficient (15a), (15b), and (15c), we obtain

$$\begin{aligned}\mathcal{H}(x) &= \tilde{\mathcal{V}}(x) + s(t) \\ &= k(x) L_f \mathcal{V}(x) + \frac{1}{4\gamma^2} k^2(x) \left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2 \\ &\quad - \frac{1}{4} k^2(x) \left\| (L_{g_2} \mathcal{V}(x))^T \right\|^2 + \|z(t)\|^2 \\ &= k(x) L_f \mathcal{V}(x) - \frac{1}{4} k^2(x) \sigma(x) + \|z(t)\|^2 \\ &= 0.\end{aligned}\quad (21)$$

Therefore, $\tilde{\mathcal{V}}(x)$ satisfies the HJI equation for all x .

Finally, we prove that the system has $\mathcal{L}_2$ gain. Equation (22) is provided from (18) and (21):

$$\begin{aligned}k(x) \dot{\mathcal{V}}(x) &= k(x) \left(L_f \mathcal{V}(x) + L_{g_1} \mathcal{V}(x) w(t) \right. \\ &\quad \left. + L_{g_2} \mathcal{V}(x) u(t) \right) \\ &= \frac{1}{4} k^2(x) \sigma(x) - \|z(t)\|^2 \\ &\quad + k(x) \left(L_{g_1} \mathcal{V}(x) w(t) + L_{g_2} \mathcal{V}(x) u(t) \right) \\ &= -\|z(t)\|^2 - \|u(t)\|^2 + \gamma^2 \|w(t)\|^2 \\ &\quad - \gamma^2 \left\| w(t) - \frac{1}{2\gamma^2} k(x) (L_{g_1} \mathcal{V}(x))^T \right\|^2 \\ &\quad + \left\| u(t) + \frac{1}{2} k(x) (L_{g_2} \mathcal{V}(x))^T \right\|^2 \\ &\leq -\|z(t)\|^2 - \|u(t)\|^2 + \gamma^2 \|w(t)\|^2.\end{aligned}\quad (22)$$

By integrating (22) from 0 to t , we obtain

$$\begin{aligned}\int_0^t \dot{\tilde{\mathcal{V}}}(x) dt &= \int_0^t k(x) \dot{\mathcal{V}}(x) dt \\ &\leq \int_0^t \left(-\|z(t)\|^2 - \|u(t)\|^2 + \gamma^2 \|w(t)\|^2 \right) dt.\end{aligned}\quad (23)$$

Since (23) is equivalent to (5) by setting $\tilde{\mathcal{V}}(x(0))$ for γ_0 , the system has $\mathcal{L}_2$ gain less than or equal to γ . $\square$

Remark 5. The $\mathcal{L}_2$ gain γ is constant in Theorem 4. Even if γ is a function of x and holds the lower bound condition (13a), (13b), and (13c) then the system is still stable and has $\mathcal{L}_2$ gain. Since the lower bound condition (13a), (13b), and (13c) is the function of x , we can easily construct the $\mathcal{L}_2$ gain $\gamma(x)$ and make the objective function small by using the function $\gamma(x)$. The function $\gamma(x)$ is used in the numerical example in Section 5.

In the next theorem, it is shown that the disturbance (17) and the control law (18) give the saddle point of the objective function.

Theorem 6. The objective function $\mathcal{J}$ of (10) is maximized by the disturbance (17) and minimized by the control law (18) in the set of control laws using the CLF $\mathcal{V}(x)$.

Proof. If the disturbance (17) and the control law (18) are applied to the system (1), then the objective function $\mathcal{J}$ of (10) becomes (24) by using (21):

$$\begin{aligned}\mathcal{J} &= \int_0^\infty \left(-k(x) L_f \mathcal{V}(x) - \frac{1}{4\gamma^2} k(x) \left\| (L_{g_1} \mathcal{V}(x))^T \right\|^2 \right. \\ &\quad \left. + \frac{1}{4} k(x) \left\| (L_{g_2} \mathcal{V}(x))^T \right\|^2 \right. \\ &\quad \left. + \|u(t)\|^2 - \gamma^2 \|w(t)\|^2 \right) dt\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \left(-k(x) L_f \mathcal{V}(x) \right. \\
&\quad - \frac{1}{\gamma^2} k(x) L_{g_1} \mathcal{V}(x) w(t) - k(x) L_{g_2} \mathcal{V}(x) u(t) \\
&\quad - \gamma^2 \left\| w(t) - \frac{1}{2\gamma^2} k(x) (L_{g_1} \mathcal{V}(x))^T \right\|^2 \\
&\quad \left. + \left\| u(t) + \frac{1}{2} k(x) (L_{g_2} \mathcal{V}(x))^T \right\|^2 \right) dt \\
&= - \int_0^\infty \dot{\tilde{V}}(x) dt = \tilde{V}(x(0)), \tag{24}
\end{aligned}$$

where $\tilde{V}(x(\infty)) = 0$ because the system is asymptotically stable to the origin. $\square$

Remark 7. When the disturbance is (17) and the control law is (18), the value of the objective function $\mathcal{J}$ becomes the function of the initial value $x(0)$ and CLF $\mathcal{V}(x)$, namely, $\mathcal{J}(x(0), \mathcal{V}(x))$. In the case of $\bar{\mathcal{V}}(x)$, the value of the objective function also becomes the function $\mathcal{J}(x(0), \bar{\mathcal{V}}(x))$ of initial value $x(0)$ and $\bar{\mathcal{V}}(x)$ if the disturbance is (11) and the control law is (9). The proposed approach can construct the control law which satisfies the HJI equation and gives the saddle point of the objective function $\mathcal{J}$. But even if both of the $\mathcal{L}_2$ gains are the same, the values of the objective function are different, namely, $\mathcal{J}(x(0), \bar{\mathcal{V}}(x)) \leq \mathcal{J}(x(0), \mathcal{V}(x))$. Also $\mathcal{J}(x(0), \bar{\mathcal{V}}(x)) = \mathcal{J}(x(0), \mathcal{V}(x))$ if and only if $\bar{\mathcal{V}}(x) = \mathcal{V}(x)$.

The next theorem provides the condition under which the control input $u(t)$ and the transformation coefficient $k(x)$ are continuous.

Theorem 8. If $f(x)$, $g_1(x)$, $g_2(x)$, and $z(t)$ are uniformly continuous for all x and CLF $\mathcal{V}$ is belonging to C^1 then the transformation coefficient $k(x)$ is uniformly continuous for all $x \neq 0$. If the following equation holds, then the control input (18) is uniformly continuous for all x :

$$\lim_{x \rightarrow 0} \frac{\|z(t)\|^2}{L_f \mathcal{V}(x)} < \infty. \tag{25}$$

Proof. We prove that the transformation coefficient $k(x)$ is uniformly continuous for all $x \neq 0$. Each of (15a) and (15b) is uniformly continuous from the assumption. We prove that (15a) converges to (15b) when $\sigma(x) \rightarrow 0$ as follows. Since $\mathcal{V}(x)$ is a CLF, $L_f \mathcal{V}(x) < 0$ when $L_{g_2} \mathcal{V}(x) = 0$. Therefore, (15a) becomes an indeterminate form of 0/0 when $\sigma(x) \rightarrow 0$ and we can apply l'Hospital's theorem to this case. The partial derivative of the denominator and the numerator of (15a) with respect to σ becomes the following equation:

$$\lim_{\sigma \rightarrow 0} \frac{\|z(t)\|^2}{\sqrt{(L_f \mathcal{V}(x))^2 + \sigma \|z(t)\|^2}} = \frac{\|z(t)\|^2}{\sqrt{(L_f \mathcal{V}(x))^2}}. \tag{26}$$

Since (26) is equivalent to (15b), (15a) and (15b) are continuous for all $x \neq 0$. Therefore, the transformation coefficient $k(x)$ is uniformly continuous for all $x \neq 0$.

Next, we prove that the control input (18) is continuous for all x . Since the transformation coefficient $k(x)$ is uniformly continuous for all $x \neq 0$, the control input (18) is continuous for all $x \neq 0$. Since the transformation coefficient $k(x)$ is bounded by (25) when $x \rightarrow 0$, the control input (18) converges to 0 when $x \rightarrow 0$. Therefore, the control input (18) is uniformly continuous for all x . $\square$

Remark 9. We can easily confirm from the degree of the denominator and the numerator with which condition (25) is satisfied. If the degree of the numerator is greater than the degree of the denominator, then the left hand side of (25) converges to 0. If the degree of denominator and numerator is the same, then the left hand side of (25) is bounded. Since the control output $z(t)$ can be designed by an appropriate selection of $h(x)$, it is possible to satisfy condition (25).

4. Derivation of CLF by Feedback Linearization

In order to construct a CLF, an approach by feedback linearization or by backstepping for a strict feedback system is well known. In this paper, we use feedback linearization and apply it to a nominal system without disturbance.

The following system is considered:

$$\dot{x} = f(x) + g_2(x) u(t), \tag{27}$$

where the control input $u(t)$ is first order for simplicity (refer to [10] in the case of multicontrol inputs). Applicability of feedback linearization to the system (27) can be confirmed by the following theorem [10].

Theorem 10. If there exists the neighborhood $\mathcal{U}$ of the origin satisfying the following two conditions, then strict linearization can be applied to the system (27):

- (1) $(ad_f^0 g_2(x), \dots, ad_f^{n-1} g_2(x))$ is linearly independent for all $x \in \mathcal{U}$,
- (2) $(ad_f^0 g_2(x), \dots, ad_f^{n-2} g_2(x))$ is involutive on $\mathcal{U}$.

If the system (27) satisfies Theorem 10, then it is shown from Frobenius' theorem that there exists a continuously differentiable function $\phi(x) \in \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned}
L_{ad_f^i g_2} \phi(x) &= 0, \quad i = 0, \dots, n-2, \\
L_{ad_f^{n-1} g_2} \phi(x) &\neq 0.
\end{aligned} \tag{28}$$

Using $\phi(x)$ which satisfies (28), we obtain the following coordinate transformation and the control law:

$$\begin{aligned}
\xi &= (\phi(x), L_f \phi(x), \dots, L_f^{n-1} \phi(x))^T, \\
u(t) &= -\frac{L_f^n \phi(x)}{L_g L_f^{n-1} \phi(x)} + \frac{1}{L_g L_f^{n-1} \phi(x)} v(t), \tag{29}
\end{aligned}$$

where $v(t)$ is the control law for the linearized system. From (29), the system becomes

$$\dot{\xi}(t) = A\xi(t) + Bv(t), \quad (30)$$

where A and B are (31) and (32), respectively, as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (31)$$

$$B = (0, \dots, 0, 1)^T. \quad (32)$$

It is possible to construct a Lyapunov function for (30) by applying linear optimal control theory and solving a Riccati equation. The Riccati equation for the objective function (33) is given by (34) as follows:

$$\mathcal{J} = \int_0^\infty (\xi^T(t) Q \xi(t) + Rv(t)^2) dt, \quad (33)$$

$$A^T P + PA - \frac{1}{R} P B B^T P + Q = 0, \quad (34)$$

where $Q \geq 0 \in \mathbb{R}^n$ and $R > 0 \in \mathbb{R}$ are weighting matrices and $P > 0 \in \mathbb{R}^{n \times n}$ is the solution to the Riccati equation (34). The Lyapunov function and the control law are given by (35) using the solution P as follows;

$$\begin{aligned} \mathcal{V}(\xi) &= \xi^T(t) P \xi(t), \\ v(t) &= -\frac{1}{2R} B^T P \xi(t). \end{aligned} \quad (35)$$

From the above, a Lyapunov function and an asymptotically stabilizing control law are provided. We adopt $\mathcal{V}(\xi)$ as a CLF.

5. Numerical Example

The proposed approach is verified by a simple numerical example of a system with disturbance.

The numerical simulation is done for the following second order system:

$$\begin{aligned} \dot{x}_1(t) &= \sin(x_1(t)) + x_2(t), \\ \dot{x}_2(t) &= x_2^2(t) w(t) + (x_2^2(t) + 1) u(t). \end{aligned} \quad (36)$$

The controlled output is defined by

$$z(t) = (x_1(t) \ x_2(t)). \quad (37)$$

Frobenius' theorem is applied in order to confirm the applicability of feedback linearization to the system (36). It is shown that Frobenius' theorem is satisfied from

$$\begin{aligned} ad_f^0 g_2(x) &= g_2(x) \\ &= (0 \ 1 + x_2^2(t))^T, \\ ad_f^1 g_2(x) &= \frac{\partial g_2(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g_2(x) \\ &= \left(- (1 + x_2^2(t)) \ 0 \right)^T. \end{aligned} \quad (38)$$

Therefore, there exists a continuously differentiable function $\phi(x)$. If we select $\phi(x) = x_1(t)$, then the control law and the CLF are given by (39) and (40), respectively, as follows:

$$\begin{aligned} u &= -\frac{\cos(x_1(t)) (\sin(x_1(t)) + x_2(t))}{x_2^2(t) + 1} \\ &\quad - \frac{x_1(t) + \sqrt{3} (\sin(x_1(t)) + x_2(t))}{2(x_2^2(t) + 1)} \end{aligned} \quad (39)$$

$$\begin{aligned} \mathcal{V}(x) &= \xi^T P \xi \\ &= \sqrt{3} x_1^2(t) + 2x_1(t) (\sin(x_1) + x_2(t)) \\ &\quad + \sqrt{3} (\sin(x_1) + x_2(t))^2, \end{aligned} \quad (40)$$

where the weighting matrices Q and R are selected as (41) and (42), respectively, as follows:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (41)$$

$$R = 1. \quad (42)$$

The solution of the Riccati equation is provided by

$$P = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}. \quad (43)$$

Next, the proposed approach is applied to the CLF (40) to construct the ISS-CLF and the control law. $L_f \mathcal{V}(x)$, $L_{g_1} \mathcal{V}(x)$, and $L_{g_2} \mathcal{V}(x)$ become (44), from the CLF (40):

$$\begin{aligned} L_f \mathcal{V}(x) &= 2 \left[\sqrt{3} x_1(t) + \sin(x_1(t)) + x_2(t) \right. \\ &\quad \left. + x_1(t) \cos(x_1(t)) + \sqrt{3} \cos(x_1(t)) \right. \\ &\quad \left. \times (\sin(x_1(t)) + x_2(t)) \right] \\ &\quad \times (\sin(x_1(t)) + x_2(t)), \\ L_{g_1} \mathcal{V}(x) &= 2 \left\{ x_1(t) + \sqrt{3} (\sin(x_1(t)) + x_2(t)) \right\} \\ &\quad \times x_2^2(t), \\ L_{g_2} \mathcal{V}(x) &= 2 \left\{ x_1(t) + \sqrt{3} (\sin(x_1(t)) + x_2(t)) \right\} \\ &\quad \times (1 + x_2^2(t)). \end{aligned} \quad (44)$$

FIGURE 1: Time histories of $x_1(t)$. FIGURE 2: Time histories of $x_2(t)$.

γ is selected as

$$\gamma(x) = \frac{x_2^2(t) + a}{x_2^2(t) + 1} > \frac{x_2^2}{x_2^2(t) + 1}, \quad (45)$$

where the design parameter a is a constant and satisfies $0 < a < 1$. Therefore, $\gamma(x) < 1$. The transformation coefficient $k(x)$ and the control law $u(t)$ become as the following equations:

$$k(x) = 2 \frac{L_f \mathcal{V}(x) + \sqrt{(L_f \mathcal{V}(x))^2 + \sigma(x)(x_1^2(t) + x_2^2(t))}}{\sigma(x)} \quad (\sigma(x) \neq 0), \quad (46a)$$

$$k(x) = -\frac{x_1^2(t) + x_2^2(t)}{L_f \mathcal{V}(x)} \quad (\sigma(x) = 0), \quad (46b)$$

$$k(x) = 0 \quad (x = 0), \quad (46c)$$

$$\sigma(x) = 4 \{x_1(t) + \sqrt{3}(\sin(x_1(t)) + x_2(t))\} \times \frac{(x_2^2(t) + 1)^2 (2x_2^2(t) + a^2)}{(x_2^2(t) + a^2)^2}, \quad (47)$$

$$u(t) = -k(x) \{x_1(t) + \sqrt{3}(\sin(x_1(t)) + x_2(t))\} \times (x_2^2 + 1). \quad (48)$$

The simulation results by the control laws (39) and (48) are shown in Figures 1, 2, 3, 4, and 5. The initial state is $x(0) = [1, -1]$. The disturbance is $w(t) = 0.5 \sin(2t)$. The design parameter is $a = 0.9$. Figures 1–3 show that the influence to $x(t)$ from $w(t)$ is suppressed by the proposed control law (48) and the convergence of the state $x(t)$ to the origin is faster than one by the control law (39).

FIGURE 3: Time histories of $u(t)$.

6. Conclusion

In this paper, a new approach to construct an ISS-CLF satisfying a HJI equation and a nonlinear robust control law for a nonlinear affine system is proposed by introducing a new transformation coefficient and applying it to a CLF. A lower bound condition of $\mathcal{L}_2$ gain is provided and makes it possible to design a control system with $\mathcal{L}_2$ gain performance. The effectiveness of the proposed control law is shown by the simple numerical example.

The characteristic of the proposed approach is that if a CLF can be constructed for a nominal system, then an ISS-CLF which satisfies a HJI equation and the robust control law can be easily constructed. Therefore, the proposed approach is more simple than the inverse optimal control

FIGURE 4: Time history of $k(x)$. FIGURE 5: Time history of $w(t)$.

approach which requires an ISS-CLF and other approximate approaches which require numerical calculation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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